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LIMIT CYCLES FOR A CLASS OF ABEL EQUATIONS*

A. GASULL† AND J. LLIBRE†

Abstract. The number of solutions of the Abel differential equation $dx(t)/dt = A(t)x(t)^3 + B(t)x(t)^2 + C(t)x(t)$ satisfying the condition $x(0) = x(1)$ is studied, under the hypothesis that either $A(t)$ or $B(t)$ does not change sign for $t \in [0, 1]$. The main result obtained is that there are either infinitely many or at most three such solutions. This result is also applied to control the maximum number of limit cycles for some planar polynomial vector fields with homogeneous nonlinearities.

Key words. Abel differential equation, limit cycle, Riccati equation

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1. Introduction and statement of the main results. A problem proposed by Pugh (see [12]) consists of the following: Let $a_0, a_1, \cdots, a_n : \mathbf{R} \to \mathbf{R}$ be smooth functions and consider the differential equation

$$\frac{dx}{dt} = a_n(t)x^n + a_{n-1}(t)x^{n-1} + \cdots + a_1(t)x + a_0(t), \qquad 0 \leqq t \leqq 1. \tag{1}$$

We will say that a solution $x(t)$ of (1) is a *closed solution* or a *periodic solution* if it is defined in the interval $[0, 1]$ and $x(0) = x(1)$. The adjectives "closed" and "periodic" are motivated by the case where $a_0, a_1, \cdots, a_n$ are 1-periodic, in which (1) can be considered in the cylinder and the "closed" solutions really correspond to periodic orbits in the cylinder. An isolated closed solution in the set of all the closed solutions will be called a *limit cycle.* Then the problem is: Does there exist a bound on the number of limit cycles of (1)?

In the case $n = 2$, (1) is called the *Riccati equation* and the problem of determining the number of limit cycles is already known: there are at most two of them (see, for instance, [12], [14]). When $n = 3$, (1) is called the *Abel equation.* Also in [12] it is proved that there is no upper bound for the number of closed solutions for the Abel equations. Hence a more specific problem arises: Give a bound on the number of limit cycles of Abel equations assuming additional hypotheses on $a_3(t)$, $a_2(t)$, $a_1(t)$, and $a_0(t)$.

A problem that is studied in several papers is Pugh's problem for Abel equations when $a_3(t)$ does not change sign (see [7], [12], [18]). In this case the maximum number of closed solutions is three.

The Ricatti equation acquired importance when it was introduced by Jacopo Francesco, Count Riccati of Venice (1676-1754), who worked in acoustics, to help solve second-order ordinary differential equations. Abel's differential equation arose in the context of the studies of N. H. Abel on the theory of elliptic functions.

The aim of this paper is to study the problem of determining the maximum number of limit cycles of Abel equations when $a_0(t) \equiv 0$ and one of the other three functions that define the differential equation does not change sign. For simplicity we write the

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† Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Spain.