APPLICATIONS OF RIEMANNIAN METRICS TO SYSTEMS OF PLANAR DIFFERENTIAL EQUATIONS

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Abstract. We present a generalization of Bendixson-Dulac criterion and a new method for the study of the stability of periodic orbits for differential equations in the plane. Proofs are based on the consideration of different Riemannian metrics on $\mathbb{R}^2$

1. INTRODUCTION

Consider the C^1-planar differential equation $\dot{\mathrm{x}} = X(\mathrm{x})$, given by

$$\begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y), \end{cases} \tag{1}$$

where $(x, y) \in U$, an open subset of $\mathbb{R}^2$.

Take $\mathbb{R}^2$ endowed with a (Riemannian) metrics g. Many properties of the trajectories of the above equation do not depend on g, because all these metrics induce in $\mathbb{R}^2$ the usual topology. For instance, α and ω-limit sets of the solutions of (1) are independent of g. Usually, it is assumed that in $\mathbb{R}^2$ we have the Euclidean metrics. Here we will take advantage of the consideration of X with a metrics g in $\mathbb{R}^2$. We will represent by (X, g), the pair given by (1) and g.

There is a relation between changes of variables in equation (1) and the consideration of different metrics. More explicitly, an invertible map $H = (F, G) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ gives an isometry between $(\mathbb{R}^2, g_H)$ and $(\mathbb{R}^2, \mathrm{Id})$, where g_H has associated the matrix

$$\begin{pmatrix} {F_x}^2 + {G_x}^2 & F_xF_y + G_xG_y \\ F_xF_y + G_xG_y & {F_y}^2 + {G_y}^2 \end{pmatrix}$$

So, it is equivalent to study (X, g_H) or $(DH \circ X \circ H^{-1}, \mathrm{Id})$. Conversely, given a metrics g it is not true that such an invertible map H always exists. Hence, the study of X as the pair (X, g) can give more information than the study of (1) using changes of variables.

In this paper we will use the above approach in two different subjects. First, a generalization of the Bendixson-Dulac criterion and, second, the study of the stability of periodic orbits of (1).

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