

ON THE LIMIT CYCLES OF SOME PLANAR DISCONTINUOUS PIECEWISE LINEAR DIFFERENTIAL SYSTEMS WITH POLYNOMIAL FIRST INTEGRALS

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ABSTRACT. In this paper we study how many algebraic limit cycles can exhibit the discontinuous piecewise linear differential systems separated by a straight line when the two linear differential systems have polynomial first integrals and at least one of the systems is Hamiltonian. Under these assumptions these piecewise differential systems at most have one limit cycle. We include an example of these systems with one algebraic limit cycle. This study needs to characterize the linear differential systems having polynomial first integrals.

1. INTRODUCTION

Discontinuous piecewise linear differential systems had been deeply studied from their introduction in Andronov, Vitt and Khaikin [1]. Their applications in electrical circuits, genetic networks or economy, for example, motivates the great amount of references. An introduction to and a great compilation of references can be found at the books of di Bernardo et al. [4], and of Simpson [31], and the survey of Makarenkov and Lamb [30].

A discontinuous piecewise linear differential system with two pieces separated by a straight line in the plane \mathbb{R}^2 can be written into the form

$$(1) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11}^- & a_{12}^- \\ a_{21}^- & a_{22}^- \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1^- \\ b_2^- \end{pmatrix},$$

in the half-plane $x \leq 0$, and

$$(2) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11}^+ & a_{12}^+ \\ a_{21}^+ & a_{22}^+ \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1^+ \\ b_2^+ \end{pmatrix},$$

in the half-plane $x \geq 0$. For the definition of a discontinuous piecewise differential system on the straight line $x = 0$ we follow the rules of Filippov [9].

We recall that a *limit cycle* of a planar differential system or of a piecewise differential system is a periodic solution of the system isolated in the set of all periodic solutions of the system. Since the planar linear differential systems have no limit cycles, the limit cycles of the discontinuous piecewise linear differential systems separated by a straight line must cross the straight line in two points,

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because in this paper we do not consider the possible limit cycles which have a segment on the discontinuity straight line, called sliding limit cycles.

The limit cycles of the planar differential systems play a main role for understanding the dynamics of such systems, and the same occurs for the planar discontinuous piecewise linear differential systems. Thus the limit cycles of the piecewise linear differential systems separated by one straight line has been studied intensively during these last twenty years, see for instance the papers [2, 5–8, 11, 13–29, 32].

We recall that there are three classes of linear nodes: nodes with different eigenvalues N, nodes with equal eigenvalues whose linear part does not diagonalize N', and nodes with equal eigenvalues whose linear part diagonalizes, called star nodes. Clearly if we have a star node this prevents the existence of periodic orbits in a discontinuous piecewise linear differential separated by a straight line. The linear differential systems having a node N or N' will be denoted by N and N'. If a linear differential system has a focus, a center or a saddle we denote them by F, C and S, respectively. Then we can consider 15 classes of planar discontinuous piecewise linear differential systems separated by a straight line: FF, FC, FN, FN', FS, CC, CN, CN', CS, NN, NN', NS, N'N', N'S and SS.

Summarizing the results of the previous mentioned articles it follows that the maximum number of known limit cycles that one of these discontinuous piecewise linear differential systems can exhibit is given in the following table.

	F	C	N	N'	S
F	3	2	3	3	3
C	–	0	1	1	1
N	–	–	2	2	2
N'	–	–	–	2	2
S	–	–	–	–	2

But the main open question remains: *Is 3 the maximum number of limit cycles that a discontinuous piecewise linear differential systems with a straight line of separation can have?*

Recently the particular class of algebraic limit cycles in the discontinuous piecewise linear differential systems separated by a straight line started to be analyzed by Buzzi, Gasull and Torregrosa [6]. In this paper a limit cycle is *algebraic* if all its points are contained in the level sets of polynomials. One of the main results of the paper [6] is to show the existence of discontinuous piecewise linear differential systems separated by a straight line having two algebraic limit cycles.

In order to deal with algebraic limit cycles for discontinuous piecewise linear differential systems we must work with linear differential systems having polynomial first integrals at each side of $x = 0$. Therefore we need to identify and classify the planar linear differential systems having a polynomial first integral. As far as we know such classification was not done. In [3] the authors provided a classification of all quadratic polynomial differential systems having polynomial first integrals. But it does not cover our cases because they do not consider the cases where all

the coefficients of the quadratic terms vanish at the same time. In what follows we classify all the linear differential systems having polynomial first integrals.

Theorem 1. *Consider the linear differential system*

$$(3) \quad \begin{aligned} \dot{x} &= a + bx + cy, \\ \dot{y} &= d + ex + fy, \end{aligned}$$

with at most an equilibrium point (it means the associated vector field has not common factors) and satisfying that $b^2 + c^2 + e^2 + f^2 \neq 0$. This system has a polynomial first integral $H(x, y)$ if and only if one of the following conditions hold.

- (i) *If $f = -b$, then $H_1(x, y) = ex^2 - cy^2 - 2bxy + 2dx - 2ay$.*
- (ii) *If $f \neq -b$, $c \neq 0$, $ec \neq fb$ and there exist two positive integers p and q such that $p \neq q$ and $ce = (pb + qf)(pf + qb)/(q - p)^2$. Then*

$$H_2(x, y) = \left(cy + \frac{pf + qb}{q - p}x + \frac{a(pf + qb) + dc(q - p)}{q(f + b)} \right)^p \left(cy - \frac{pb + qf}{q - p}x + \frac{a(pb + qf) - dc(q - p)}{p(f + b)} \right)^q.$$

- (iii) *If $fb \neq 0$, $f^2 \neq b^2$ and $c = 0$ and there exist two positive integers p and q such that $p \neq q$ and $pb + qf = 0$, then*

$$H_3(x, y) = (a + bx)^p (efx + f(f - b)y + d(f - b) + ae)^q.$$

The main goal of this paper is to characterize the maximum number of limit cycles of the discontinuous piecewise linear differential systems separated by a straight line formed by two linear differential systems having polynomial first integrals being at least one of these differential systems a Hamiltonian system. Our main result is the following.

Theorem 2. *The discontinuous piecewise linear differential systems separated by a straight line when both linear differential systems have a polynomial first integral have at most one limit cycle if only one of these two systems is Hamiltonian. Moreover, if both systems are Hamiltonian then the discontinuous piecewise linear differential system has no limit cycles.*

Section 2 is devoted to show a proof of Theorem 1 following arguments related with factorization and divisibility of polynomials. Theorem 2 is proved in section 3 applying the first integrals of Theorem 1. Finally in section 4 we provide a discontinuous piecewise linear differential system separated by a straight line having both linear differential systems a polynomial first integral and having exactly one limit cycle.

2. PROOF OF THEOREM 1

In order to prove Theorem 1 we have to introduce a previous result. It is devoted to the polynomial resolution of polynomial differential equations of the form $NH + UH_y = 0$, where N and U are polynomial and H is a polynomial solution of degree n .

Proposition 3. *We consider the differential equation*

$$(4) \quad NH + UH_y = 0,$$

where N and U are polynomials, non identically zero and coprime. If $U = \prod_{i=1}^r P_i^{r_i}$ where P_i are the irreducible real factors of U , then equation (4) has a polynomial solution H of degree n , different from the trivial $H = 0$, if and only if there exists $n_1, n_2, \dots, n_r \in \mathbb{N}$ such that $\sum_{i=1}^r n_i \deg P_i \leq n$ and

$$N + \sum_{j=1}^r n_j P_{j,y} \frac{U}{P_j} = 0.$$

Moreover when the polynomial H exists then $H = W \prod_{i=1}^r P_i^{n_i}$, where W is a polynomial of degree $k = n - \sum_{i=1}^r n_i \deg P_i$ which does not depend on the variable y . If H is homogeneous then U and W are homogeneous and $W = \gamma x^k$ with $\gamma \in \mathbb{R}$.

Proof. Since N and U are coprime polynomials and H must be also polynomial, it follows that U divides H . So there exists $n_1, n_2, \dots, n_r \in \mathbb{N}$ such that $H = RW$ and $R = \prod_{i=1}^r P_i^{n_i}$ with $n_i \geq r_i$. Furthermore we can assume that R and W are coprime. Taking into account these considerations in (4) we obtain that

$$\begin{aligned} NH + UH_y &= NRW + U \left(\left(\sum_{j=1}^r n_j P_j^{n_j-1} P_{j,y} \prod_{i=1, i \neq j}^r P_i^{n_i} \right) W + RW_y \right) \\ &= NRW + U \left(\left(\sum_{j=1}^r n_j P_{j,y} \frac{R}{P_j} \right) W + RW_y \right) \\ &= NRW + R \left(\left(\sum_{j=1}^r n_j P_{j,y} \frac{U}{P_j} \right) W + UW_y \right) = 0. \end{aligned}$$

Now dividing this equation by R we have

$$\left(N + \sum_{j=1}^r n_j P_{j,y} \frac{U}{P_j} \right) W + UW_y = 0.$$

Since U and W are coprime and $\deg U > \deg \left(N + \sum_{j=1}^r n_j P_{j,y} \frac{U}{P_j} \right)$, we have that $N + \sum_{j=1}^r n_j P_{j,y} \frac{U}{P_j} = 0$ and $W_y = 0$, it means that $W = W(x)$. Finally if H is homogeneous then P_j for all $j = 1, \dots, r$ and W are also homogeneous because all of them are factors of H . Thus U is homogeneous and $W = \gamma x^k$ taking $k = n - \sum_{i=1}^r n_i \deg P_i$. \square

For computing a polynomial first integral H of degree n of system (3) we shall use the decomposition in homogeneous parts of such a polynomial first integral.

Proposition 4. We consider $P(x, y) = \tilde{a} + \tilde{b}x + \tilde{c}y$ and $Q(x, y) = \tilde{d} + \tilde{e}x + \tilde{f}y$ with $P \cdot Q \neq 0$. We suppose that the polynomial differential system

$$(5) \quad \dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

has a polynomial first integral H of degree n . We write $xQ - yP = \sum_{i=1}^2 T_i$, where T_i is the homogeneous part of degree i of the polynomial $xQ - yP$.

If $H = \sum_{i=0}^n H_i$ being H_i the homogeneous part of degree i of H , then the H_i 's verify the following system of equations

$$(6) \quad \begin{aligned} nP_1H_n + T_2H_{n,y} &= 0, \\ (n-1)P_1H_{n-1} + T_2H_{n-1,y} &= -(nP_0H_n + T_1H_{n,y}), \\ &\dots \\ (n-j)P_1H_{n-j} + T_2H_{n-j,y} &= -((n-j+1)P_0H_{n-j+1} + T_1H_{n-j+1,y}), \\ &\dots \\ P_1H_1 + T_2H_{1,y} &= -(2P_0H_2 + T_1H_{2,y}), \\ 0 &= -(P_0H_1 + T_1H_{1,y}), \end{aligned}$$

where $H_{j,y}$ is the partial derivative of H_j with respect to the variable y and P_i is the homogeneous part of the polynomial P of degree i .

Proof. We consider the partial derivatives of H , i.e. H_x and H_y . Then we have that

$$(7) \quad PH_x + QH_y = 0.$$

By the Euler Theorem for homogeneous functions we have that

$$(8) \quad xH_x + yH_y = \sum_{j=1}^n (xH_{j,x} + yH_{j,y}) = \sum_{j=1}^n jH_j.$$

Therefore we obtain that

$$(9) \quad xH_x = \sum_{j=1}^n jH_j - yH_y.$$

Now if we multiply equation (7) by x and substitute xH_x by the value given in (9), we get

$$(10) \quad P \sum_{j=1}^n jH_j + (xQ - yP)H_y = 0.$$

Finally taking into account that $P = P_0 + P_1$ and $xQ - yP = T_1 + T_2$, the homogeneous parts of (10), arranged from the greatest to the lowest degree provide system (6). \square

Remark 5. If system (5) is homogeneous, it means $\tilde{a} = \tilde{d} = 0$, any polynomial first integral H verifies that each homogeneous part of H also is a polynomial first integral of (5). Therefore in homogeneous differential systems it has sense just to consider homogeneous polynomial first integrals.

Corollary 6. If system (5) is homogeneous then any homogeneous polynomial first integral H of degree n satisfies $nPH + (xQ - yP)H_y = 0$, where $xQ - yP = -\tilde{c}y^2 + (\tilde{f} - \tilde{b})xy + \tilde{e}x^2$.

Proof. It is easy to see, since we have that H has only one homogeneous part of degree n and it is H itself. Then system (6) is reduced to the first equation. Since $P_0 = T_1 = 0$, $P_1 = P$ and $T_2 = xQ - yP$, the proof follows. \square

Proof of Theorem 1. We separate the proof in several cases.

Case I: $c \neq 0$. We apply to system (3) the change of variables

$$(11) \quad x = x, \quad Y = \frac{a}{c} + \frac{b}{c}x + y,$$

and we obtain the system

$$(12) \quad \dot{x} = cY, \quad \dot{Y} = \tilde{d} + \tilde{e}x + \tilde{f}Y,$$

where $\tilde{d} = (dc - fa)/c$, $\tilde{e} = (ec - fb)/c$ and $\tilde{f} = f + b$.

We can assume that

$$(13) \quad \tilde{d}^2 + \tilde{e}^2 \neq 0.$$

Otherwise if $\tilde{d} = \tilde{e} = 0$ then $d = fa/c$ and $e = fb/c$. So $Q = fP/c$ with $f \in \mathbb{R}$ and system (3) will have common factors in contradiction with the hypotheses.

Subcase (I.1): $\tilde{f} = 0$. Then $f = -b$, and we can separate the variables in system (12). So we have the first integral

$$H(x, Y) = \tilde{d}x + \frac{\tilde{e}}{2}x^2 - \frac{c}{2}Y^2.$$

Then if we undo the change of variables (11) we obtain the first integral $H_1(x, y)$ equal to

$$\begin{aligned} & \frac{cd + ab}{c}x + \frac{ce + b^2}{2c}x^2 - \frac{c}{2} \frac{a^2}{c^2} - \frac{c}{2} \frac{2ab}{c^2}x - \frac{c}{2} \frac{2a}{c}y - \frac{c}{2} \frac{2b}{c}xy - \frac{c}{2} \frac{b^2}{c^2}x^2 - \frac{c}{2}y^2 \\ &= -\frac{a^2}{c} + \left(-\frac{ab}{c} + \frac{cd + ab}{c}\right)x + \left(-\frac{b^2}{2c} + \frac{ce + b^2}{2c}\right)x^2 - \frac{c}{2}y^2 - ay - bxy \\ &= -\frac{c}{2}y^2 + \frac{e}{2}x^2 - bxy + dx - ay - \frac{a^2}{c}. \end{aligned}$$

Multiplying this expression by two and removing the constant term, we obtain statement (i) of Theorem 1 when $c \neq 0$.

Subcase (I.2): $\tilde{f} \neq 0$ and $\tilde{e} \neq 0$. Then

$$(14) \quad f \neq -b \text{ and } ce \neq bf.$$

So we do the change of variables

$$(15) \quad X = \frac{\tilde{d}}{\tilde{e}} + x, \quad Y = Y,$$

to system (12) and we obtain

$$(16) \quad \dot{X} = cY, \quad \dot{Y} = \tilde{e}X + \tilde{f}Y,$$

where $\tilde{e} = (ce - bf)/c \neq 0$ and $\tilde{f} = f + b \neq 0$.

System (16) is homogeneous and we can apply to it Corollary 6. Therefore we have to solve

$$ncYH - c\left(Y^2 - \tilde{f}XY/c - \tilde{e}X^2/c\right)H_Y = 0.$$

We consider $T = Y^2 - \tilde{f}XY/c - \tilde{e}X^2/c$, so the previous equation can be written as

$$(17) \quad nYH - TH_Y = 0.$$

Now we have to see that (17) having a polynomial solution requires that T factorizes as $T = (Y + \alpha X)(Y + \beta X)$. We suppose that T does not factorize. From Proposition 3 we know that in order to have a polynomial first integral of (17) it is necessary the existence of $p \in \mathbb{Z}^+$ such that $2p \leq n$ and $H = T^p X^{n-2p}$ and

$$nY - p \left(2Y - \frac{\tilde{f}}{c} X \right) = 0,$$

or equivalently

$$(n - 2p)Y + p \frac{\tilde{f}}{c} X = 0.$$

Then we should impose $n = 2p$ and $\tilde{f} = 0$. But it contradicts our assumptions. So we have to suppose that T factorizes and it means that $T = (Y + \alpha X)(Y + \beta X)$ with

$$(18) \quad \alpha + \beta = -\frac{\tilde{f}}{c} \quad \text{and} \quad \alpha\beta = -\frac{\tilde{e}}{c}.$$

We would like to remind that we are studying the case $\tilde{f} \neq 0$ and $\tilde{e} \neq 0$. Then we have that $\alpha + \beta \neq 0$ and $\alpha\beta \neq 0$.

According to Proposition 3 we obtain the polynomial first integral of (17)

$$H = (-Y - \alpha X)^p (Y + \beta X)^q X^{n-p-q},$$

where $p, q \in \mathbb{Z}^+$, $2 \leq p + q \leq n$, and

$$nY - p(Y + \beta X) - q(Y + \alpha X) = 0.$$

Taking this into account we have that

$$(19) \quad n = p + q \quad \text{and} \quad \alpha q + \beta p = 0.$$

Moreover since $\alpha + \beta \neq 0$, we have that

$$(20) \quad p \neq q.$$

Now if we consider (18), (19) and (20) all together we conclude that

$$(21) \quad \alpha = \frac{p}{c(q-p)} \tilde{f}, \quad \beta = -\frac{q}{c(q-p)} \tilde{f},$$

and from (18) we obtain that

$$(22) \quad c\tilde{e} = \tilde{f}^2 \frac{pq}{(q-p)^2}.$$

In conclusion a polynomial first integral of system (16) is

$$H = (Y + \alpha X)^p (Y + \beta X)^q,$$

because we have (19), (20), (21), (22) and we can reject multiplicative constants as $(-1)^p$.

In order to obtain the first integral of the original system (3) we should undo the changes of variables (11) and (15). So

$$\begin{aligned} Y + \alpha X &= y + \frac{b}{c}x + \frac{a}{c} + \frac{p}{q-p} \frac{\tilde{f}}{c}x + \frac{p}{q-p} \frac{\tilde{f}\tilde{d}}{c\tilde{e}} \\ &= \left(\frac{a}{c} + \frac{p}{q-p} \frac{\tilde{f}\tilde{d}}{\tilde{e}c} \right) + y + \left(\frac{b}{c} + \frac{p}{q-p} \frac{\tilde{f}}{c} \right) x. \end{aligned}$$

But

$$\begin{aligned} \frac{a}{c} + \frac{p}{q-p} \frac{\tilde{d}\tilde{f}}{\tilde{e}c} &= \frac{a}{c} + \frac{p}{q-p} \frac{\tilde{d}\tilde{f}}{\tilde{f}^2 \frac{pq}{(q-p)^2}} = \frac{a}{c} + \frac{\tilde{d}(q-p)}{\tilde{f}q} \\ &= \frac{a}{c} + \frac{(cd-af)(q-p)}{c(f+b)q} \\ &= \frac{1}{c(f+b)q} (a(f+b)q + cdq - cdp - afq + afp) \\ &= \frac{1}{c(f+b)q} (a(pf+qb) + cd(q-p)), \end{aligned}$$

and

$$\frac{b}{c} + \frac{p}{q-p} \frac{\tilde{f}}{c} = \frac{1}{c} \frac{b(q-p) + p(f+b)}{q-p} = \frac{1}{c(q-p)} (bq + fp).$$

Then taking all this into account we get

$$Y + \alpha X = y + \frac{fp + bq}{c(q-p)}x + \frac{a(pf+qb) + cd(q-p)}{cq(f+b)}.$$

Analogously

$$Y + \beta X = y - \frac{bp + fq}{c(q-p)}x + \frac{a(pb+qf) - cd(q-p)}{cp(f+b)}.$$

Finally condition (22) becomes

$$\frac{c(ce - bf)}{c} = (f+b)^2 \frac{pq}{(q-p)^2},$$

so

$$ce = \frac{bf(q-p)^2 + (f+b)^2 pq}{(q-p)^2} = \frac{fq(bq+fp) + bp(fp+bq)}{(q-p)^2} = \frac{(fq+bp)(bq+fp)}{(q-p)^2}.$$

In summary we have proved statement (ii) of Theorem 1.

Subcase (I.3): $\tilde{f} \neq 0$ and $\tilde{e} = 0$. This case agrees with $f \neq -b$ and $ce = bf$. Additionally from (13) we obtain also

$$(23) \quad \tilde{d} \neq 0.$$

We now apply Proposition 4 to solve system (12) taking $P = P_1 = cY$ and $Q = \tilde{d} + \tilde{f}Y$. Thus

$$xQ - yP = \tilde{d}x + \tilde{f}xY - cY^2 = \tilde{d}x + Y(\tilde{f}x - cY),$$

and consequently $T_1 = \tilde{d}x$ and $T_2 = Y(\tilde{f}x - cY)$.

The first equation of (6) can be written in this case as

$$ncYH_n + Y(\tilde{f}x - cY)H_{n,Y} = 0.$$

So dividing by Y we obtain that

$$ncH_n + (\tilde{f}x - cY)H_{n,Y} = 0.$$

From Proposition 3 we know that there exists a polynomial solution H_n if and only if there is $p \in \mathbb{N}$ such that $p \leq n$ and $nc - pc = 0$. In this case we additionally have that

$$H_n = (\tilde{f}x - cY)^p x^{n-p}.$$

But these conditions implies that $(n - p)c = 0$, or equivalently $n = p$ and therefore

$$H_n = (\tilde{f}x - cY)^n.$$

If we use this relation in the second equation of system (6) we obtain

$$(n - 1)cYH_{n-1} + Y(\tilde{f}x - cY)H_{n-1,Y} = -\tilde{d}xn(-c)(\tilde{f}x - cY)^{n-1}.$$

It follows easily that Y should divide the right hand side of this equality. It leads to a contradiction because we have (23), $\tilde{f} \neq 0$ and $c \neq 0$. This shows that in this case does not exist polynomial first integrals.

Case (II): $c = 0$. We do the change of variables $X = y$ and $Y = x$ to system (3) and we have

$$(24) \quad \dot{X} = \bar{a} + \bar{b}X + \bar{c}Y, \quad \dot{Y} = \bar{d} + \bar{e}X + \bar{f}Y,$$

where $\bar{a} = d$, $\bar{d} = a$, $\bar{b} = f$, $\bar{f} = b$, $\bar{c} = e$ and $\bar{e} = c = 0$. Now we distinguish two subcases.

Subcase (II.1): $\bar{c} \neq 0$. But it agrees with case (I) previously discussed. Therefore the polynomial first integral exists.

Subcase (II.1.1): $\bar{f} = -\bar{b}$. It is known that a polynomial first integral is

$$H(X, Y) = -\bar{c}Y^2 - 2\bar{b}XY + 2\bar{d}X - 2\bar{a}Y$$

If we undo the change of variables applied in order to obtain system (24), the condition characterizing this case translates to $b = -f$, and the first integral writes as

$$H(x, y) = -ex^2 + 2bxy + 2ay - 2dx.$$

Therefore statement (i) of Theorem 1 is verified when $c = 0$. So together with the subcase (I.1) it concludes the proof of this subcase.

Subcase (II.1.2): $\bar{f} \neq -\bar{b}$ and $\bar{f}\bar{b} \neq 0$. This subcase coincides with conditions (14) in case (I.2). Therefore in order to have a polynomial first integral \bar{H} it must exist $p, q \in \mathbb{N}$ such that $p \neq q$,

$$\bar{e}\bar{c} = \frac{(p\bar{b} + q\bar{f})(p\bar{f} + q\bar{b})}{(q - p)^2}$$

and $\bar{H}(X, Y) = \bar{F}^p \bar{G}^q$ where

$$\begin{aligned}\bar{F}(X, Y) &= \bar{c}Y + \frac{p\bar{f} + q\bar{b}}{q - p}X + \frac{\bar{a}(p\bar{f} + q\bar{b}) + \bar{d}\bar{c}(q - p)}{q(\bar{f} + \bar{b})}, \text{ and} \\ \bar{G}(X, Y) &= \bar{c}Y - \frac{p\bar{b} + q\bar{f}}{q - p}X + \frac{\bar{a}(p\bar{b} + q\bar{f}) - \bar{d}\bar{c}(q - p)}{p(\bar{f} + \bar{b})}.\end{aligned}$$

We undo now the change of variables. From (24) we obtain that our conditions are translated to $b \neq f$ and $bf \neq 0$, p and q satisfy that

$$(25) \quad (pf + qb)(pb + qf) = 0$$

and the polynomial first integral is $H(x, y) = F^p G^q$ where

$$\begin{aligned}F(x, y) &= ex + \frac{pb + qf}{q - p}y + \frac{d(pb + qf) + ae(q - p)}{q(f + b)}, \text{ and} \\ G(x, y) &= ex - \frac{pf + qb}{q - p}y + \frac{d(pf + qb) - ae(q - p)}{q(f + b)}.\end{aligned}$$

We assume now that (25) is satisfied because $pb + qf = 0$. Hence $f \neq b$, because $n = p + q \in \mathbb{Z}^+$ is the degree of H . Moreover we conclude that

$$p = \frac{-nf}{b - f} \text{ and } q = \frac{nb}{b - f},$$

and therefore

$$q - p = \frac{n(b + f)}{b - f} \text{ and } pf + qb = n(b + f).$$

Taking into account these relations we obtain that

$$\begin{aligned}F(x, y) &= ex + \frac{ae(q - p)}{q(f + b)} = ex + ae \frac{n(b + f)}{b - f} \frac{1}{\frac{nb}{b - f}(f + b)} \\ &= ex + \frac{ae}{b} = e(bx + a), \text{ and} \\ G(x, y) &= ex - \frac{n(f + b)}{\frac{n(f + b)}{b - f}}y + \frac{\frac{dn(f + b) - aen(b + f)}{b - f}}{-\frac{nf}{b - f}(f + b)} \\ &= ex - (b - f)y - d\frac{f - b}{f} - \frac{ae}{f} = ex - (b - f)y - \frac{d(b - f) + ae}{f}.\end{aligned}$$

In conclusion under our assumptions

$$H(x, y) = (a + bx)^p (efx + f(f - b)y + d(f - b) + ae)^q.$$

Following similar computations if (25) is satisfied from the assumption that $pf + qb = 0$, we obtain the same expression for $H(x, y)$. Therefore we have finished the proof of statement (iii) of Theorem 1.

Subcase (II.2): $\bar{e} = 0$ and $\bar{c} = 0$. This is the last subcase that we must consider to finish the proof. But if we undo the change of variables taking into account (24), system (3) writes as

$$\dot{x} = a + bx, \quad \dot{y} = d + fy,$$

and the general hypothesis is reduced to $b^2 + f^2 \neq 0$.

We see that in fact we are just interested in the case $bf \neq 0$. We assume the contrary to our claim, that $b = 0$. Hence $af \neq 0$ in order to avoid common factors in the differential system. As the variables can be separated in the system we obtain

$$\frac{1}{a}dx - \frac{1}{d+fy}ay = 0.$$

Integrating this equality we get

$$\frac{x}{a} - \frac{1}{f} \ln(d+fy) = k$$

where k is a constant. Straightforward computations provide the relation

$$d+fy = k \cdot \exp\left(\frac{f}{a}x\right).$$

Therefore a polynomial first integral does not exist.

Analogously it can be shown that the polynomial first integral cannot be found also for the case $f = 0$. Therefore in order that a polynomial first integral can exist we must have $bf \neq 0$. However if $b = -f$ it corresponds to statement (i) of Theorem 1, whereas if $b \neq -f$ it corresponds to statement (iii) of Theorem 1 for $c = e = 0$. \square

3. PROOF OF THEOREM 2

Theorem 2 focuses on giving bounds on the number of limit cycles of discontinuous piecewise linear differential systems separated by a straight line. Therefore, although it would not be necessary, in order to reduce the computations solving these bounds, we shall apply Theorem 1 to the canonical forms introduced in [10].

Hence, from now on, we consider the discontinuous piecewise linear differential systems with real coefficients

$$(26) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2\ell & -1 \\ \ell^2 - \alpha^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ g \end{pmatrix},$$

defined in $x \leq 0$, and

$$(27) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2r & -1 \\ r^2 - \beta^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} j \\ k \end{pmatrix},$$

defined in $x \geq 0$, where $\alpha, \beta \in \{i, 0, 1\}$. Of course $i^2 = -1$. Note that both systems together depend on five parameters. We remark that if $\alpha = i$ then the equilibrium point of system (26) has eigenvalues $\ell \pm i$, so it is a focus if $\ell \neq 0$, and a center if $\ell = 0$. If $\alpha = 0$ then system (26) is a node with eigenvalue $\ell \neq 0$ of multiplicity 2 whose linear part does not diagonalize. If $\alpha = 1$ then system (26) is a saddle with eigenvalues $\ell - 1$ and $\ell + 1$ when $|\ell| < 1$, and a node with eigenvalues $\ell - 1$ and $\ell + 1$ whose linear part diagonalize when $|\ell| > 1$.

The homeomorphism $\Gamma : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a *topological equivalence* between the discontinuous piecewise linear differential system (1)+(2) and the discontinuous piecewise linear differential system (26)+(27) if Γ applies orbits of system (1)+(2) into orbits of system (26)+(27) and $\Gamma(\{x = 0\}) = \{x = 0\}$. From Propositions 1 and 2 of [10] it follows that there exists a topological equivalence between the phase portraits of

the discontinuous piecewise linear differential systems (1)+(2) and the phase portraits of the discontinuous piecewise linear differential systems (26)+(27) restricted to the orbits that do not have points in common with the sliding set of these systems. Therefore since we are interested in studying the algebraic limit cycles of the planar discontinuous piecewise linear differential systems (1)+(2) which do not intersect its sliding set, it will be sufficient to study the algebraic limit cycles of the planar discontinuous piecewise linear differential systems (26)+(27), for a definition of the sliding set see Filippov [9].

We shall now study using Theorem 1 the polynomial first integrals of the canonical differential systems (26) and (27).

Proposition 7. *We consider the linear differential system*

$$(28) \quad \begin{aligned} \dot{x} &= 2lx - y + s, \\ \dot{y} &= (l^2 - \alpha^2)x + t, \end{aligned}$$

where $l, s, t \in \mathbb{R}$ and $\alpha \in \{i, 0, 1\}$. Then (28) has a polynomial first integral, $H(x, y)$, just in the following cases:

(i) $l = 0$ and in this case

$$(29) \quad H(x, y) = y^2 - \alpha^2 x^2 + 2tx - 2sy;$$

(ii) $l = \frac{q-p}{q+p} \neq 0$ with $p, q \in \mathbb{Z}^+$ and $\alpha = 1$. In this case

$$(30) \quad H(x, y) = \left(y - \frac{2q}{q+p}x + \frac{t(q+p)}{2q} - s \right)^p \left(y + \frac{2p}{q+p}x - \frac{t(q+p)}{2p} - s \right)^q.$$

Proof. Theorem 1 implies that we should consider two cases, $l = 0$ and $l \neq 0$.

If $l = 0$ then system (28) satisfies the condition $f = -b = 0$ of statement (i) of Theorem 1, and (29) is obtained straightforward from that statement.

We consider now that $l \neq 0$. If we identify systems (28) and (3) of Theorem 1, we obtain that $a = s$, $f = 0 \neq -b = -2l$, $c = -1 \neq 0$ and $e = l^2 - \alpha^2$. Therefore the existence of a polynomial first integral is satisfied just under the conditions of statement (ii) of Theorem 1. Hence it should be studied the condition $ce \neq bf$, where

$$(31) \quad ce = \alpha^2 - l^2 \text{ and } bf = 0.$$

So it follows easily that $\alpha \neq \pm l$.

We also must study the case $p, q \in \mathbb{Z}^+$ such that $p \neq q$ and $ce = \frac{(pb + qf)(pf + qb)}{(q - p)^2}$.

This right hand side satisfies

$$\frac{(pb + qf)(pf + qb)}{(q - p)^2} = \frac{(2lp)(2lq)}{(q - p)^2} = \frac{4pq}{(q - p)^2} l^2.$$

So from (31) we get

$$\alpha^2 - l^2 = \frac{4qp}{(q - p)^2} l^2.$$

But it means that

$$\alpha^2 = \left(\frac{4pq}{(q - p)^2} + 1 \right) l^2 = \frac{(p + q)^2}{(q - p)^2} l^2.$$

Therefore in this case there exists a polynomial first integral if and only if $l^2 = \frac{(q-p)^2}{(q+p)^2} \alpha^2$ or, equivalently, $l = \pm \frac{q-p}{q+p} \alpha$.

System (28) has real coefficients, so $l \in \mathbb{R}$, and since $l \neq 0$, it follows that $\alpha \neq 0$ and $\alpha \neq i$. Thus $\alpha = 1$. Therefore from statement (ii) of Theorem 1 if a polynomial first integral exists it must be

$$H(x, y) = \left(-y + \frac{2lq}{q-p}x + s - \frac{t(q-p)}{2lq} \right)^p \left(-y - \frac{2lp}{q-p}x + s + \frac{t(q-p)}{2lp} \right)^q.$$

If $l = \frac{q-p}{q+p}$ then the first integral H writes

$$(32) \quad H(x, y) = \left(-y + \frac{2q}{q+p}x + s - \frac{t(q+p)}{2q} \right)^p \left(-y - \frac{2p}{q+p}x + s + \frac{t(q+p)}{2p} \right)^q.$$

Analogously if $l = \frac{p-q}{q+p}$ then the first integral H becomes

$$(33) \quad H(x, y) = \left(-y - \frac{2q}{q+p}x + s + \frac{t(q+p)}{2q} \right)^p \left(-y + \frac{2p}{q+p}x + s - \frac{t(q+p)}{2p} \right)^q.$$

Observe that (32) and (33) are the same polynomial first integral just switching p and q . Actually, multiplying $H(x, y)$ by $(-1)^{p+q}$, it follows (30). This completes the proof of the proposition. \square

Corollary 8. (i) *System (26) has a polynomial first integral if and only if*

(i.1) $l = 0$, with $\hat{H}_1(x, y) = y^2 - \alpha^2 x^2 + 2gx$; or

(i.2) $l = \frac{q-p}{q+p} \neq 0$ where $p, q \in \mathbb{Z}^+$ and $\alpha = 1$, with

$$\hat{H}_2(x, y) = \left(y - \frac{2q}{q+p}x + \frac{g(q+p)}{2q} \right)^p \left(y + \frac{2p}{q+p}x - \frac{g(q+p)}{2p} \right)^q.$$

(ii) *System (27) has a polynomial first integral if and only if*

(ii.1) $r = 0$, with $\hat{H}_3(x, y) = y^2 - \beta^2 x^2 + 2kx - 2jy$; or

(ii.2) $r = \frac{q-p}{q+p} \neq 0$ where $p, q \in \mathbb{Z}^+$ and $\beta = 1$, with

$$\hat{H}_4(x, y) = \left(y - \frac{2q}{q+p}x + \frac{k(q+p)}{2q} - j \right)^p \left(y + \frac{2p}{q+p}x - \frac{k(q+p)}{2p} - j \right)^q.$$

Proof. The proof is straightforward from Proposition 7. \square

Remark 9. *Note that cases (i.1) and (ii.1) are Hamiltonian cases. Meanwhile, any*

system of case (i.2) has a saddle point located at $\left(\frac{h(p+q)}{2pq}, \frac{h(q-p)}{pq} \right)$ where $h = \frac{g(p+q)}{2}$, and its separatrices cut $x = 0$ at $\left(0, -\frac{h}{q} \right)$ and $\left(0, \frac{h}{p} \right)$. Finally systems

of case (ii.2) has a saddle point too, now located at $\left(\frac{\bar{h}(p+q)}{2pq}, \frac{\bar{h}(q-p)}{pq} + j \right)$ with $\bar{h} = \frac{k(p+q)}{2}$, and its separatrices cut $x = 0$ at $\left(0, -\frac{\bar{h}}{q} + j \right)$ and $\left(0, \frac{\bar{h}}{p} + j \right)$.

Focusing again on the location of piecewise limit cycles, we start with some geometrical ideas. Under the assumptions of Theorem 1 let $H^-(x, y)$ (respectively $H^+(x, y)$) a polynomial first integral of the linear differential system in $x \leq 0$

(respectively $x \geq 0$). Any limit cycle must intersect the straight line $x = 0$ in two distinct points $(0, y)$ and $(0, Y)$ satisfying the system of equations

$$\begin{aligned} H^-(0, y) - H^-(0, Y) &= 0, \\ H^+(0, y) - H^+(0, Y) &= 0. \end{aligned}$$

Hence if we can count the pairs of solutions (y, Y) we can give an upper bound of the number of limit cycles of a piecewise differential system under the hypotheses of Theorem 2. But this will be just an upper bound because the connection from branches of the first integrals could not provide a closed curve, or a closed curve which is not a periodic orbit because the two pieces are not travelled in the same sense. Some examples of these phenomena can be appreciated in Figure 1.

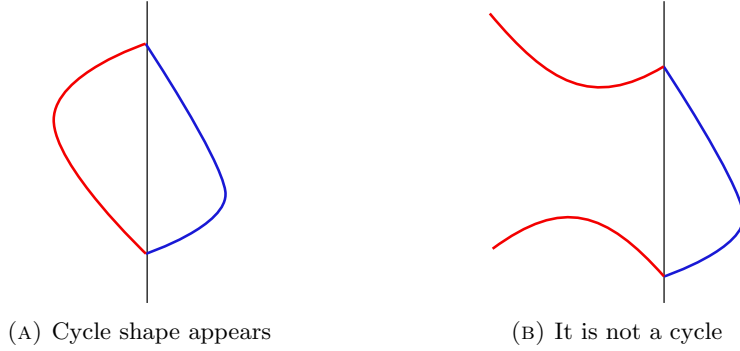


FIGURE 1. Some possible connections between both sides branches.

In order to prove Theorem 2 we shall use the next results.

Proposition 10. *Let γ be a limit cycle of a discontinuous piecewise linear differential system (26)+(27) having polynomial first integrals at each side of $\Sigma = \{x = 0\}$. If one of the first integrals corresponds to a linear differential system with a saddle, therefore γ intersects Σ in two points located between the two points of Σ which belong to the separatrices of the saddle.*

Proof. As a linear differential system without common factors having a polynomial first integral is topologically equivalent to a linear Hamiltonian system (see Proposition C in [12]), the ω and α -limits of the orbits in the saddle case will be restricted to the limits of the separatrices. Hence any orbit out of its separatrices and far from the equilibrium point has a similar behaviour to them, who are two straight lines according to Corollary 8. It implies that any orbit will cross Σ twice if and only if both separatrices cross Σ and the orbit is located at the hyperbolic region between the branches of the separatrices crossing Σ . See Figure 2. \square

Lemma 11. *We consider the function $f_{p,\alpha}(x) = \left(\frac{x-\alpha}{x+\alpha}\right)^p$ for all $x \in \mathbb{R} \setminus \{-\alpha\}$ with $0 < \alpha$ and $p \in \mathbb{Z}^+$. Then it is satisfied that $f_{p,\alpha}(\alpha) = 0$, $f_{p,\alpha}(0) = (-1)^p$, $\lim_{x \rightarrow \pm\infty} f_{p,\alpha}(x) = 1$ and*

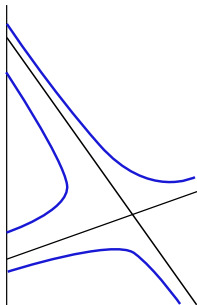


FIGURE 2. Graphical representation of a polynomial saddle phase portrait.

- (i) if p is even, $f_{p,\alpha}(x) > 0$ for all $x \in \mathbb{R} \setminus \{-\alpha, \alpha\}$, $f_{p,\alpha}$ is decreasing for $x \in (-\alpha, \alpha)$ and it is increasing outside, having a local minimum at $x = \alpha$ and an inflexion point at $x = p\alpha$; and,
- (ii) if p is odd, $f_{p,\alpha}(x) > 0$ if and only if $x \in \mathbb{R} \setminus (-\alpha, \alpha)$, $f_{p,\alpha}$ is increasing for all $x \in \mathbb{R} \setminus \{-\alpha\}$, having an inflexion point at $x = \alpha$ and, if $p > 1$, another one at $x = p\alpha$.

Proof. First, straightforward computations show that $f_{p,\alpha}(\alpha) = 0$ and $f_{p,\alpha}(0) = (-1)^p$.

Since

$$f'_{p,\alpha}(x) = p \left(\frac{x-\alpha}{x+\alpha} \right)^{p-1} \frac{x+\alpha-x+\alpha}{(x+\alpha)^2} = \frac{2p\alpha}{(x+\alpha)^2} \left(\frac{x-\alpha}{x+\alpha} \right)^{p-1},$$

if we multiply both sides by $(x-\alpha)(x+\alpha)$ we obtain

$$(34) \quad (x-\alpha)(x+\alpha)f'_{p,\alpha}(x) = 2p\alpha f_{p,\alpha}(x).$$

We compute the second derivative

$$(x+\alpha+x-\alpha)f'_{p,\alpha}(x) + (x-\alpha)(x+\alpha)f''_{p,\alpha}(x) = 2p\alpha f'_{p,\alpha}(x),$$

then

$$(35) \quad 2(x-p\alpha)f'_{p,\alpha}(x) + (x-\alpha)(x+\alpha)f''_{p,\alpha}(x) = 0.$$

From (34) we conclude that if $f'_{p,\alpha}(x) = 0$ then also $f_{p,\alpha}(x) = 0$. So $x = \alpha$ is the unique possible relative extreme. From (35) we conclude that $f''_{p,\alpha}(x) = 0$ if, and only if, or $f'_{p,\alpha}(x) = 0$ or $2(x-p\alpha) = 0$. Hence there are two possible inflexion points $x = \alpha$ and $x = p\alpha$.

- (i) We suppose that p is even. It is obvious that $f_{p,\alpha}(x) > 0$ for all $x \in \mathbb{R} \setminus \{-\alpha, \alpha\}$ and $f_{p,\alpha}(\alpha) = 0$. From (34) we see that

$$(x-\alpha)(x+\alpha)f'_{p,\alpha}(x) > 0,$$

and it implies that either $(x-\alpha)(x+\alpha) > 0$ and $f'_{p,\alpha}(x) > 0$, or $(x-\alpha)(x+\alpha) < 0$ and $f'_{p,\alpha}(x) < 0$. Then we can conclude that

$$\begin{aligned} f'_{p,\alpha}(x) &> 0 \text{ if } x \in (-\infty, -\alpha) \cup (\alpha, \infty), \text{ and} \\ f'_{p,\alpha}(x) &< 0 \text{ if } x \in (-\alpha, \alpha). \end{aligned}$$

Moreover the sign of $f'_{p,\alpha}(x)$ changes at $x = \alpha$, it is negative before α and positive after it, so at this point $f_{p,\alpha}$ has a local minimum.

Since $(x - \alpha)(x + \alpha)f'_{p,\alpha}(x) > 0$ is equivalent to $\frac{f'_{p,\alpha}(x)}{(x-\alpha)(x+\alpha)} > 0$, then (35) can be rewritten as

$$f''_{p,\alpha}(x) = \frac{-f'_{p,\alpha}(x)}{(x-\alpha)(x+\alpha)}2(x-p\alpha),$$

hence $f''_{p,\alpha}(x) > 0$ if, and only if, $2(x-p\alpha) < 0$ or, equivalently, if $x < p\alpha$. And this means that $f_{p,\alpha}$ has an inflexion point at $x = p\alpha$.

- (ii) We assume now that p is odd. In this case $f_{p,\alpha}(x)$ will be positive if, and only if, $(x - \alpha)(x + \alpha) > 0$, i.e. $x \in (-\infty, -\alpha) \cup (\alpha, \infty)$. It implies that $f_{p,\alpha}(x) < 0$ if, and only if, $x \in (-\alpha, \alpha)$.

In order to study the monotonicity we shall use (34),

$$f'_{p,\alpha}(x) = 2p\alpha \frac{f_{p,\alpha}(x)}{(x-\alpha)(x+\alpha)}.$$

The previous paragraph implies that $\frac{f_{p,\alpha}(x)}{(x-\alpha)(x+\alpha)} > 0$ for all $x \in \mathbb{R} \setminus \{-\alpha, \alpha\}$. Since $\alpha > 0$ we conclude that $f'_{p,\alpha}(x) > 0$ for all $x \in \mathbb{R} \setminus \{-\alpha, \alpha\}$. So $f_{p,\alpha}$ is an increasing function in the whole domain.

Finally we use (35) to study the convexity,

$$f''_{p,\alpha}(x) = \frac{-f'_{p,\alpha}(x)}{(x-\alpha)(x+\alpha)}2(x-p\alpha).$$

Since $f'_{p,\alpha}(x) > 0$ for all x , $f''_{p,\alpha}(x) > 0$ for $(x-\alpha)(x+\alpha) > 0$ and $2(x-p\alpha) > 0$, or $(x-\alpha)(x+\alpha) < 0$ and $2(x-p\alpha) < 0$. First case if $x \in (-\infty, -\alpha) \cup (\alpha, \infty)$ and $x < p\alpha$, then $f''_{p,\alpha}(x) > 0$ if $x \in (-\infty, -\alpha) \cup (\alpha, p\alpha)$. Second case if $x \in (-\alpha, \alpha)$ and $x > p\alpha$ but $p\alpha \geq \alpha$, and since $p \in \mathbb{Z}^+$ and $\alpha > 0$, then there are no other solution for x where $f''_{p,\alpha}(x) > 0$. In conclusion $f''_{p,\alpha}(x) > 0$ if, and only if, $x \in (-\infty, -\alpha) \cup (\alpha, p\alpha)$. In a similar way we conclude that $f''_{p,\alpha}(x) < 0$ if, and only if, $x \in (-\alpha, \alpha) \cup (p\alpha, \infty)$, and the proof is complete.

We include in Figure 3 some graphical representations summarizing the results obtained in this proof. \square

Now we are ready to proceed with the proof of Theorem 2.

Proof of Theorem 2. Taking into account Corollary 8, it is enough to check Theorem 2 for the planar discontinuous piecewise linear differential systems (26)+(27). We divide the proof in three cases according with Corollary 8, it controls when the linear differential systems (26) and (27) have polynomial first integrals.

Case 1: Systems (26) and (27) are both Hamiltonian. Therefore from Proposition 7 in system (27) we have that $r = 0$ and similarly in system (26) $\ell = 0$. In this case if the discontinuous piecewise linear differential systems (26)+(27) have a limit cycle this must intersect the straight line $x = 0$ in two distinct points $(0, y)$ and $(0, Y)$

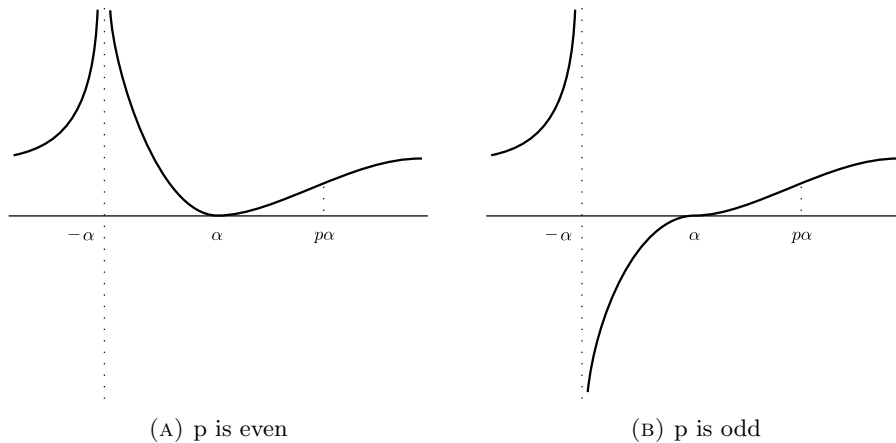


FIGURE 3. Graphical representations of $f_{p,\alpha}(x)$ from Lemma 11.

satisfying the system

$$\begin{aligned} e_1 &= \hat{H}_1(0, y) - \hat{H}_1(0, Y) = (y - Y)(y + Y) = 0, \\ e_2 &= \hat{H}_3(0, y) - \hat{H}_3(0, Y) = (y - Y)(y + Y - 2j) = 0, \end{aligned}$$

where we are using the notation of Corollary 8.

We only are interested in the solutions such that $y \neq Y$. So system $e_1 = 0, e_2 = 0$ either has no solutions, or has infinitely many solutions when $j = 0$. Consequently has no isolated solutions, and therefore in this case the discontinuous piecewise linear differential systems (26)+(27) has no limit cycles.

Case 2: Only system (26) is Hamiltonian. Then from Proposition 7 system (27) has $\beta = 1$ and $r = \frac{q-p}{q+p} \neq 0$ with $p, q \in \mathbb{Z}^+$, and system (26) has $\ell = 0$. If the discontinuous piecewise linear differential systems (26)+(27) have a limit cycle this must intersect the straight line $x = 0$ in two distinct points $(0, y)$ and $(0, Y)$ satisfying the system

$$\begin{aligned} e_1 &= \hat{H}_1(0, y) - \hat{H}_1(0, Y) = (y - Y)(y + Y) = 0, \\ e_2 &= \hat{H}_4(0, y) - \hat{H}_4(0, Y) \\ &= \left(y + \frac{k(q+p)}{2q} - j\right)^p \left(y - \frac{k(q+p)}{2p} - j\right)^q - \\ &\quad \left(Y + \frac{k(q+p)}{2q} - j\right)^p \left(Y - \frac{k(q+p)}{2p} - j\right)^q = 0. \end{aligned}$$

As in the previous case we only are interested in the solutions such that $y \neq Y$. Then $e_1 = 0$ implies that $y = -Y$, so $e_2 = 0$ implies that

$$\begin{aligned} &\left(-Y + \frac{k(q+p)}{2q} - j\right)^p \left(-Y - \frac{k(q+p)}{2p} - j\right)^q \\ &= \left(Y + \frac{k(q+p)}{2q} - j\right)^p \left(Y - \frac{k(q+p)}{2p} - j\right)^q, \end{aligned}$$

or, equivalently

$$\left(\frac{-Y + \frac{k(q+p)}{2q} - j}{Y + \frac{k(q+p)}{2q} - j} \right)^p = \left(\frac{Y - \frac{k(q+p)}{2p} - j}{-Y - \frac{k(q+p)}{2p} - j} \right)^q.$$

From Lemma 11 it means that we will have to study equation

$$f_{p,\alpha}(Y) = (-1)^{p+q} f_{q,\beta}(Y),$$

where $\alpha = \frac{k(p+q)}{2q} - j$ and $\beta = j + \frac{k(p+q)}{2p}$. We can assume k is positive, otherwise to switch $\bar{k} = -k > 0$, $\bar{p} = q$ and $\bar{q} = p$ will expand same equation, $e_2 = 0$, under this assumption. In this case $-\alpha < j < \beta$ and according to Proposition 10, we should look for our solutions $Y \in (-\alpha, \beta)$. Furthermore, as $y = -Y$ would be a solution in $(-\alpha, \beta)$, it implies that $(-|Y|, |Y|) \subset (-\alpha, \beta)$, so α and β are both positive.

The proof is completed by showing that the graphs of $f_{p,\alpha}(x)$ and $(-1)^{p+q} f_{q,\beta}(x)$ cut each other in at most one non-vanishing value of $x \in (-\alpha, \beta)$. In order to check it, we must take into account the parity of p and q and the relative positions between α and β . Lemma 11 is the key of this analysis.

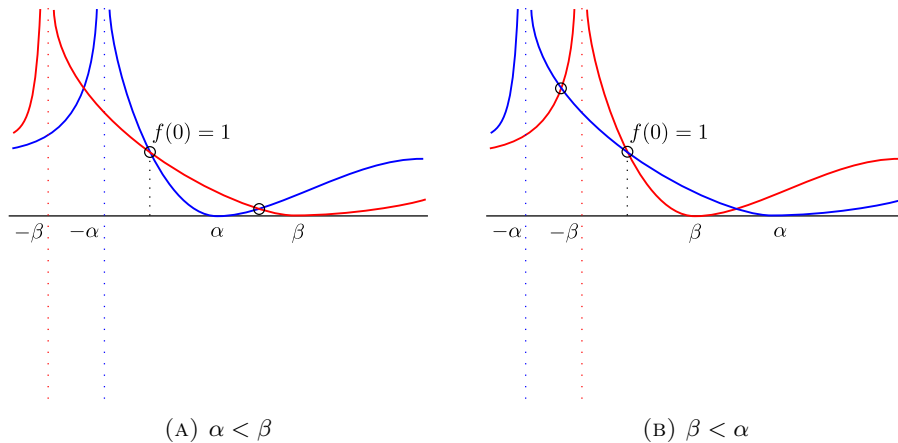
Nevertheless, we remark that we only need to consider one case: p and q even. Note that in statements (ii) and (iii) of Theorem 1, without loss of generality, we can assume that p and q are both even. On the contrary, we would have $p' = 2p$ and $q' = 2q$ being both even, satisfying the same hypotheses of the theorem for the same differential system and giving a polynomial first integral, H'_j , such that $H'_j = (H_j)^2$ for any $j = 2, 3$.

Let p and q be even integers. In this case $(-1)^{p+q} = 1$, so we will compare $f_{p,\alpha}$ and $f_{q,\beta}$. We first consider $\alpha < \beta$. In this case we should divide $(-\alpha, \beta)$ studying separately intervals $(-\alpha, \alpha)$ and (α, β) . In both intervals $f_{p,\alpha}$ and $f_{q,\beta}$ are positive functions, but at $(-\alpha, \alpha)$ both functions are decreasing while at (α, β) $f_{p,\alpha}$ is increasing and $f_{q,\beta}$ is decreasing. Since 0 belongs to $(-\alpha, \alpha)$ and both functions has the same value at this point, 1, meanwhile there are no inflexion points in this interval. So there are no other point in common between both graphs in this interval. In the interval (α, β) the monotonicity is enough to assure the existence of a point in common because $f_{p,\alpha}(\alpha) = 0$, $f_{p,\alpha}(\beta) > 0$, $f_{q,\beta}(\alpha) > 0$ and $f_{q,\beta}(\beta) = 0$. Figure 4 summaries our proof.

If $\beta < \alpha$ similar arguments can be developed but analyzing separately the functions at the intervals $(-\alpha, -\beta)$ and $(-\beta, \beta)$.

This completes the desired conclusion, having at most one limit cycle.

Case 3: Only system (27) is Hamiltonian. Again from Proposition 7 system (27) has $r = 0$ and system (26) has $\alpha = 1$ and $\ell = \frac{q-p}{q+p} \neq 0$ with $p, q \in \mathbb{Z}^+$. If the discontinuous piecewise linear differential systems (26)+(27) have a limit cycle this must intersect the straight line $x = 0$ in two distinct points $(0, y)$ and $(0, Y)$

FIGURE 4. p and q even.

satisfying the system

$$\begin{aligned}
 e_1 &= \hat{H}_2(0, y) - \hat{H}_2(0, Y) \\
 &= \left(y + \frac{g(q+p)}{2q} \right)^p \left(y - \frac{g(q+p)}{2p} \right)^q - \left(Y + \frac{g(q+p)}{2q} \right)^p \left(Y - \frac{g(q+p)}{2p} \right)^q = 0. \\
 e_2 &= \hat{H}_3(0, y) - \hat{H}_3(0, Y) = (y - Y)(y + Y) = 0,
 \end{aligned}$$

Similar arguments as the ones used in Case 2 show that the piecewise differential system has at most one limit cycle.

The proof of Theorem 2 is completed. \square

4. EXAMPLES

Now we provide a discontinuous piecewise linear differential system separated by a straight line having both linear differential systems a polynomial first integral and having exactly one limit cycle. We consider $A^- = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$, $A^+ = \begin{pmatrix} 1/2 & 1 \\ 1 & 1/2 \end{pmatrix}$, $b^- = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $b^+ = \begin{pmatrix} a \\ d \end{pmatrix}$ with $a, d \in \mathbb{R}$. So we can consider piecewise linear system given by

$$(36) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A^- \begin{pmatrix} x \\ y \end{pmatrix} + b^-,$$

if $x \leq 0$, and

$$(37) \quad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A^+ \begin{pmatrix} x \\ y \end{pmatrix} + b^+,$$

if $x \geq 0$.

System (36) satisfies the hypotheses of Theorem 1 statement (i). It means that System (36) is a Hamiltonian system and has $H^- = 3x^2 + y^2 - 4xy + 2x - 2y$ as a first integral. Moreover this system has a saddle located at $(-1, -1)$ and its

separatrices are $y = x$ (unstable) and $y = 3x + 2$ (stable). We remark that these separatrices cut the vertical axis at $(0, 0)$ and $(0, 2)$.

System (37) satisfies the hypotheses of Theorem 1 statement (ii) with $p = 1$ and $q = 3$. It means that system (37) has the polynomial first integral

$$H^+ = \left(y + x + \frac{2}{3}(a + d) \right) (y - x + 2(a - d))^3.$$

So this system has a saddle located at $\left(\frac{2a-4d}{3}, \frac{2d-4a}{3} \right)$ and its separatrices are $y = -x - \frac{2}{3}(a + d)$ (stable) and $y = x - 2(a - d)$ (unstable). We remark that these separatrices cut vertical axis at $(0, -\frac{2}{3}(a + d))$ and $(0, -2(a - d))$.

We denote $\gamma = \frac{2}{3}(a + d)$ and $\theta = 2(a - d)$. In this way, as we have mention before, we can characterize the limit cycles solving system

$$(38) \quad \begin{aligned} H^+(0, y_1) - H^+(0, y_0) &= 0, \\ H^-(0, y_1) - H^-(0, y_0) &= 0, \end{aligned}$$

where y_1 and y_0 are unknown and identify points at $\Sigma_0 = \{(x, y) \in \mathbb{R}^2 : x = 0\}$ that characterize both level curves shaping the cycle, in case it exists. If we compute the resultant between both left-hand side expressions with respect to y_1 , we conclude that in order to have a limit cycle a necessary condition is that y_0 satisfies the equation

$$R(y_0) = -(4 + \gamma + 3\theta)y_0^2 + 2(4 + \gamma + 3\theta)y_0 - \theta^3 - 3\gamma\theta^2 - 6\theta^2 - 6\gamma\theta - 12\theta - 4\gamma - 8 = 0.$$

The discriminant of this quadratic equation is

$$D(\gamma, \theta) = -4(1 + \theta)^2(4 + 3\gamma + \theta)(4 + \gamma + 3\theta).$$

Figure 5 shows the set of points where $D(\gamma, \theta)$ vanishes. So these straight lines bound the regions where discriminant has a sign and $R(y_0) = 0$ has or has not real solutions.

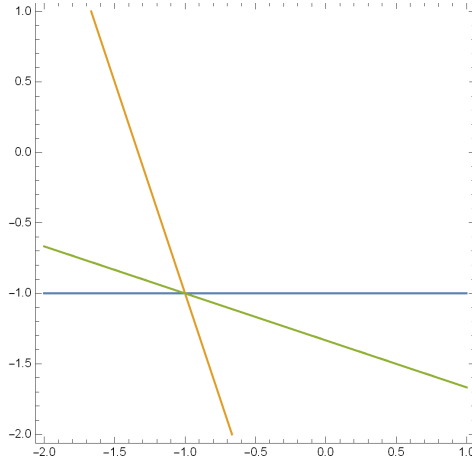


FIGURE 5. Regions delimited by $D(\gamma, \theta) = 0$.

As far as $(0, y_0)$ is an intersection point of the limit cycle with Σ_0 and any limit cycle requires two of these points, we look for the region where $R(y_0) = 0$ has

two real different solutions. It means that we are interested in $D(\gamma, \theta) > 0$, or equivalently,

$$(4 + 3\gamma + \theta)(4 + \gamma + 3\theta) < 0.$$

In order to assure the existence of a limit cycle, it is required that the solutions of $R(y_0) = 0$ should be located at $(-2, 0)$ and between $-\gamma$ and $-\theta$, the intersection points of the separatrices and Σ_0 . The equations

$$\begin{aligned} R(-\gamma) &= -(2 + \gamma + \theta)^3 = 0, \\ R(-\theta) &= -4(1 + \theta)^2(2 + \gamma + \theta) = 0, \\ R(0) &= -8 - 4\gamma - 12\theta - 6\gamma\theta - 6\theta^2 - 3\gamma\theta^2 - \theta^3 = 0, \\ R(-2) &= -40 - 12\gamma - 36\theta - 6\gamma\theta - 6\theta^2 - 3\gamma\theta^2 - \theta^3 = 0, \end{aligned}$$

characterize the regions what must be studied. It is a simple matter to check that we will found algebraic limit cycles if we take γ and θ satisfying

$$\begin{aligned} 2 + \gamma + \theta &> 0, \\ 4 + 3\gamma + \theta &< 0, \\ -8 - 4\gamma - 12\theta - 6\gamma\theta - 6\theta^2 - 3\gamma\theta^2 - \theta^3 &< 0. \end{aligned}$$

Figure 6 shows the region described above. It also shows the limit cycle found for $\gamma = -2$ and $\theta = 1$. In this case we see that the limit cycle pass through the points $(0, 1 - \frac{2\sqrt{5}}{5}) \approx (0, 0.105573)$ and $(0, 1 + \frac{2\sqrt{5}}{5}) \approx (0, 1.89443)$.

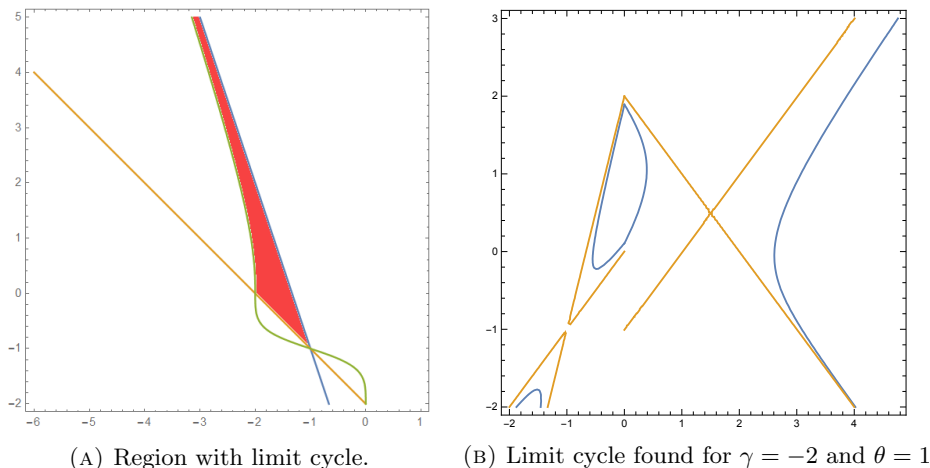


FIGURE 6

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