ANALYTIC INTEGRABILITY OF QUASI-HOMOGENEOUS SYSTEMS VIA THE YOSHIDA METHOD

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Abstract. The objective of this paper is double. First we do a survey on what we call the Yoshida method for studying the analytic first integrals of the quasi-homogeneous polynomial differential systems. After we apply the Yoshida method for studying the analytic first of all the quasi-homogeneous polynomial differential systems in $\mathbb{R}^3$ of degree 2.

1. Introduction

In 1983 Haruo Yoshida [15, 16] publishes a series of interesting results that establish conditions for the integrability of some classes of differential systems and provide a way for finding first integrals for such systems. Later on several authors [1, 3, 8, 12, 9, 11, 14] have continued to develop his ideas until to have what we call now the *Yoshida method*.

In essence the method is based on the correspondence between certain characteristic values of the first integrals and others inherent to the differential system (the so-called *Kowalevskaya exponents*), being all of them calculable in a finite number of steps.

The main purpose of this work is to analyze the capabilities of the Yoshida method as a tool for the integration of quasi-homogeneous differential systems in the space $\mathbb{R}^3$, a class of differential systems on which these results have been little exploited so far. Additional to this analysis of the Yoshida method, another of our objectives is to carry out a compilation of the main results on this class of quasi-homogeneous differential systems published to date on the subject.

Consider an n-dimensional autonomous polynomial differential system of the form

$$\frac{dx_i}{dt} = \dot{x}_i = P_i\left(\mathbf{x}\right), \qquad \mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n, \qquad i = 1, ..., n, \tag{1}$$

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where P_i belongs to the polynomial ring over $\mathbb{R}$ in the real variables $x_1, ..., x_n$, for $i = 1, ..., n$. As usual the degree of the system is $h = \max\{h_1, ..., h_n\}$, being h_i the degree of P_i.

System (1) is called *quasi-homogeneous* (in the following, simply *QH*) of *weight degree* $d \in \mathbb{Z}^+$ with *weight exponents* $s_1, ..., s_n \in \mathbb{Z}^+$ when for any $\alpha \in \mathbb{R}^+$ the following condition are satisfied for $i = 1, ..., n$:

$$P_i\left(\alpha^{s_1} x_1, ..., \alpha^{s_n} x_n\right) = \alpha^{s_i - 1 + d} P_i\left(x_1, ..., x_n\right). \tag{2}$$

Any vector of positive integers $\mathbf{v} = (s_1, ..., s_n, d)$ for which (2) holds is called *weight vector* of the QH system. Every QH system has an infinite number of weight vectors, because $\mathbf{v} = (s_1, ..., s_n, d)$ is a weight vector if and only if $\mathbf{w} = (ks_1, ..., ks_n, k(d-1)+1)$ is a weight vector for any $k \in \mathbb{Z}^+$. Then the weight vectors of a QH system can be grouped into families (see [6]): the set of weight vectors that verify $s_1 = \lambda_i s_i$ for $i = 2, ..., n$ form the family of ratio $(\lambda_2, ..., \lambda_n) \in \left(\mathbb{Q}^+\right)^{n-1}$. We will use as the representative vector of a family of weight vectors the one that verifies $\gcd(s_1, ..., s_n) = 1$. Besides, according with [6], in the set of weight vectors of a QH system it is possible to define a partial order relation as follows: given two weight vectors, $\mathbf{v} = (a_1, ..., a_{n+1})$ and $\mathbf{w} = (b_1, ..., b_{n+1})$, we say that $\mathbf{v} \leq \mathbf{w}$ when $a_i \leq b_i$ for $i = 1, ..., n+1$. If does exists a weight vector $\mathbf{v_m}$ verifying that $\mathbf{v_m} \leq \mathbf{w}$ for any other weight vector $\mathbf{w}$, we say that $\mathbf{v_m}$ is the *minimum weight vector* of the QH system.

A QH system is called *maximal* if any new monomial added to its structure maintaining the degree of the system prevents it to be QH. Knowing the maximal systems, we can determinate all the QH systems [6].

The QH systems constitute a set within the polynomial differential systems, which includes homogeneous systems as a particular case. For a given degree h the homogeneuous systems coincide with those QH having the weight vector $(1, ..., 1, h)$ among their weight vectors. Starting from the idea of homogeneity and introducing different weights for the variables, the concept of QH is reached in a natural way. Therefore these last years the QH systems have been the subject of research by many authors, especially the planar QH systems and their integrability, see for instance [4] and the references cited therein. However the study of the first integrals of the QH systems in dimensions higher than 2, such as the ones that we are dealing with in this paper, is in general a difficult problem that has been little considered, see for instance [13].

In order to determine the set of normal forms of these systems for a given degree, algorithms have been developed for both the plane [4, 5] and the 3-dimensional space [6].

In some of the preliminary works, Yoshida and others [15, 8] do not strictly deal with QH systems, but with other types of differential equations called *similarity invariants*. This set is characterized by the existence of certain rational numbers g_1, ..., g_n such that

$$P_i\left(\varepsilon^{g_1}x_1,...,\varepsilon^{g_n}x_n\right)=\varepsilon^{g_i+1}P_i\left(x_1,...,x_n\right), \tag{3}$$

it is verified for any nonzero real ε and for $i=1,...,n$. Similarity invariants have two fundamental properties. First, they are invariant under the transformation

$$t\to\varepsilon^{-1}t,\ x_1\to\varepsilon^{g_1}x_1,...,\ x_n\to\varepsilon^{g_n}x_n,$$

for any constant $\varepsilon\in\mathbb{R}\setminus\{0\}$. Moreover if the system of equations

$$P_i\left(c_1,...,c_n\right)=-g_ic_i,\qquad i=1,...,n,$$

has some nonzero solution, then the differential system has a solution of the form

$$\varphi\left(t\right)=\left(c_1t^{-g_1},...,c_nt^{-g_n}\right).$$

We remark that conditions (2) and (3) are equivalent by simply doing $g_i=s_i/\left(d-1\right)$ and taking $\varepsilon=\alpha^{d-1}$. Consequently, a QH system belongs to the set of similarity invariant differential systems as long as it has some weight vector $(s_1,...,s_n,d)$ with $d\neq1$. In these cases $(g_1,...,g_n)$ represents all weight vectors of the family of $(s_1,...,s_n,d)$. From nowon when we talk about QH systems, we will understand that they are within the similarity invariant type.

This work is organized as follows. Section 2 is devoted to the theoretical bases of the Yoshida method, including the most relevant theorems of the publications referring to the topic, several original concepts (Definition 1) and results (Propositions 2, 3, 4, 6 and 8), which add new knowledge and clarity to the method. In section 3 in order to simplify subsequent calculations, the canonical forms of the 20 existing QH systems of degree 2 in dimension 3 are obtained. Finally in section 4, we use the results of what section 2 for studying the analytical integrability of the normal forms obtained in section 3.

2. Yoshida method

2.1. **Yoshida first integrals.** The main goal of the Yoshida's method is the calculation of certain type of polynomial first integrals of a differential system. Given an $n-$dimensional differential system (1), a non-constant real function $H\left(\mathbf{x}\right)$ is a *first integral* of system (1) on an open subset $\Omega\subseteq\mathbb{R}^n$ if H is constant over all solution curves $\varphi\left(t\right)=\left(x_1\left(t\right),...,x_n\left(t\right)\right)$ of system (1) contained in Ω. In case that $H\in C^1\left(\Omega\right)$, then the previous definition is equivalent to

$$\sum_{i=1}^{n}P_i\left(\mathbf{x}\right)\frac{\partial H}{\partial x_i}\left(\mathbf{x}\right)\equiv0 \tag{4}$$

for all $\mathbf{x} \in \Omega$. A first integral is *global* when Ω matches the system's domain, and it is called *polynomial* (resp. *analytic*, resp. *algebraic*) if H is a polynomial (resp. analytic, resp. algebraic) function.

A set of real functions $H_1, ..., H_r$ are *functionally independent* in $\mathbb{R}^n$ if the rank of the $r \times n$ matrix

$$\begin{pmatrix} \dfrac{\partial H_1}{\partial x_1} & \cdots & \dfrac{\partial H_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \dfrac{\partial H_r}{\partial x_1} & \cdots & \dfrac{\partial H_r}{\partial x_n} \end{pmatrix} (\mathbf{x})$$

is r at all points $\mathbf{x} \in \mathbb{R}^n$ where the matrix is defined, with the exception of a zero Lebesgue-measure subset. Otherwise the functions are *functionally dependent*, which geometrically implies that the gradients $\nabla H_i(\mathbf{x})$ are linearly dependent at all almost all the points $\mathbf{x} \in \mathbb{R}^n$.

Several notions of integrability appear in the literature. We will work on this concept through analytical functions. It is said that system (1) is *completely integrable* if there exist $n-1$ functionally independent analytical first integrals, and *partially integrable* if the number of independent analytical first integrals is less than $n-1$. While authors as Yoshida [16] and Goriely [8] focus on the algebraic first integrals. They refer to the differential system (1) as *algebraically integrable in the weak sense* if there exist k ($1 \leq k \leq n-1$) functionally independent algebraic first integrals $H_i(\mathbf{x})$, $i = 1, ..., k$, (which define an $(n-k)$ $-$dimensional algebraic variety $\mathcal{L}$) and other $n-1-k$ independent first integrals given by the integral of closed 1-form defined on $\mathcal{L}$,

$$H_i(\mathbf{x}) = \sum_{j=1}^{n-k} \int^{x_j} \phi_{ij}(\mathbf{x})\, dx_j, \quad i = 1, ..., n-1-k,$$

where ϕ_{ij} are algebraic functions of $\mathbf{x}$. Finally their stronger definition of *algebraically integrable* system is equivalent to the weak one but setting $k = n-1$.

The interest in the search for first integrals lies in the fact that in case we can prove that (1) is completely integrable, with first integrals H_1,...,H_{n-1}, then the orbits of the system are determined by intersecting the invariant set $\mathcal{F}_i = \left\{ H_i^{-1}(x) \mid x \in \mathbb{R} \right\}$, $i = 1, ..., n-1$. Even the case of partial integrability is interesting, because the knowledge of a first integral implies the knowledge of $n-1$-dimension surfaces in which the orbits live, whose behavior can be studied over these surfaces with the advantage of reducing one dimension.

QH systems should not be mistaken with quasi-homogeneous polynomials. A polynomial $P(x_1, ..., x_n)$ is *quasi-homogeneous* with *weight exponents* $\mathbf{s} =$

$(s_1,...,s_n) \in (\mathbb{Z}^+)^n$ and *weight degree* $k \in \mathbb{Z}^+$ when for any $\alpha \in \mathbb{R}^+$,

$$P\left(\alpha^{s_1}x_1,...,\alpha^{s_n}x_n\right) = \alpha^k P\left(x_1,...,x_n\right). \tag{5}$$

To simplify we will call **s**-*type* all those quasi-homogeneous polynomials with weight exponents **s**, whatever be their weight degree k. It is obvious that a **s**-type polynomial with weight degree k is also of p**s**-type with weight degree pk for every $p \in \mathbb{Z}^+$, so we can assume $\gcd(s_1,...,s_n) = 1$.

The set of **s**-*type* quasi-homogeneous polynomials with weight degree k is constituted by the functions of the form

$$\sum_{(e_1,...,e_n)\in D} A x_1^{e_1}...x_n^{e_n},$$

being $A \in \mathbb{R}$ arbitrary coefficients and D the collection of non-negative solutions of the diophantine equation $s_1e_1 + \cdots + s_ne_n = k$.

Given an arbitrary set of weight exponents $\mathbf{s} = (s_1,...,s_n) \in (\mathbb{Z}^+)^n$, any monomial $x_1^{\beta_1}...x_n^{\beta_n}$ is **s**-type of degree $k = \sum_{i=1}^{n} s_i\beta_i$. As a consequence, given an analytical function $H(x_1,...,x_n)$, it is possible to split H in a unique form $H = \sum_k P^k$, where every P^k is a **s**-type polynomial of weight degree k, that is $P^k(\alpha^{s_1}x_1,...,\alpha^{s_n}x_n) = \alpha^k P^k(x_1,...,x_n)$.

The following result is proved in [12] for polynomial first integrals, although the proof for analytical first integrals is the same.

Proposition 1. *Let* (1) *be a QH differential system with weight exponents* $\mathbf{s} = (s_1,...,s_n) \in (\mathbb{Z}^+)^n$, *and let H be an analytic function whose decomposition into* **s**-*type polynomials is* $H = \sum_k P^k$. *Then H is a first integral of system* (1) *if and only if each polynomial P^k is a first integral of system* (1).

From Proposition 1 if we want to study the analytical first integrals of a QH system, it is enough to know those first integrals that are of **s**-type, being **s** the weight exponent of the system. All other analytical integrals can be built using these.

Definition 1. *Let* (1) *be a QH differential system with weight exponents* $\mathbf{s} = (s_1,...,s_n) \in (\mathbb{Z}^+)^n$. *Any* **s**-*type first integral of system* (1) *is called* Yoshida first integral (YFI) *of system* (1).

With respect to the number of independent YFIs, we have the following result, which although intuitive, has not been proven as far as we know.

Proposition 2. *Let* (1) *be a QH differential system. The number of functionally independent analytical first integrals of system* (1) *matches with the number of functionally independent YFIs.*

Proof. Of course any set of functionally independent polynomial first integrals is in particular a set of analytical independent functions. Then let $\{H_1(\mathbf{x}), ..., H_r(\mathbf{x})\}$ be a set of r functionally independent analytical first integrals. So if we conveniently choose the variables $x_1, ..., x_r$, the value of the determinant

$$D = \begin{bmatrix} \frac{\partial H_1}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial H_1}{\partial x_r}(\mathbf{x}) \\ \vdots & \cdots & \vdots \\ \frac{\partial H_r}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial H_r}{\partial x_r}(\mathbf{x}) \end{bmatrix}$$

is different from zero for all values of $\mathbf{x} \in \mathbb{R}^n$ except at most for a subset of zero Lebesgue measure. Since the first integrals $H_i(\mathbf{x})$, $i = 1, ..., r$, are analytic functions, they can be expressed as sums of YFIs,

$$H_i(\mathbf{x}) = \sum_{k_i} P_i^{k_i}, \ i = 1, ..., r,$$

and taking into account the properties of the determinants, we reach

$$D = \sum_{k_1} \cdots \sum_{k_r} \begin{bmatrix} \frac{\partial P_1^{k_1}}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial P_1^{k_1}}{\partial x_r}(\mathbf{x}) \\ \vdots & \cdots & \vdots \\ \frac{\partial P_r^{k_r}}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial P_r^{k_r}}{\partial x_r}(\mathbf{x}) \end{bmatrix} \neq 0,$$

which means that there must be r first integrals $\left\{P_1^{k_{1_0}}, ..., P_r^{k_{r_0}}\right\}$ that are functionally independent. □

Returning to the objectives of the Yoshida method, it is now understood why it focuses on the search for polynomial first integrals, and in particular on YFIs: there may be many other analytical first integrals, and even others that are also $\mathbf{s}^*$-type for different weight exponents $\mathbf{s}^*$. However, for the purposes of integrability of the system, what interests is to obtain functionally independent sets of first integrals with the greatest possible number of first integrals, and this can be achieved by studying exclusively YFIs.

If the problem is approached from the point of view of algebraic integrability, there are similar results to the previous ones: based on preliminary works of Bruns [2], Yoshida [15] proved that every algebraic first integral is built from rational quasi-homogeneous first integrals. On the other hand, Goriely [8] proved that the highest number of functionally independent algebraic integrals is reached within the subset of the quasi-homogeneous rationals. These last are therefore the equivalent in algebraic integrability to what the YFI mean in the field of analytic integrability.

2.2. **Balances and Kowalevskaya exponents.** When Haruo Yoshida [15, 16] establishes the bases of the integration method we are discussing here, he was recovering the works that Sofia Kowalevskaya [10] had published at the end of the 19th century. The ideas of the Russian mathematician, although not lacking in controversy at the time of its publication, had made a remarkable contribution to the study of integrability in the classical rigid body problem. However and perhaps because eventually the method developed by Kowalevskaya were not appropriate to be applied to other physical phenomena, the fact is that their advances are forgotten since the first World War until in the beginning of the 80's, in its theoretical aspects for the field of integrability.

Now we suppose that system (1) is a QH polynomial differential system of weight degree d with respect to the weight exponents $\mathbf{s} = (s_1, ..., s_n)$. As we advance before, we define $\mathbf{g} = \mathbf{s}/(d-1)$. Then any non-trivial solution $\mathbf{c} = (c_1, ..., c_n) \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ of the polynomial system of equations

$$P_i(c_1, ..., c_n) + g_i c_i = 0, \qquad i = 1, ..., n \tag{6}$$

is called a *balance* of system (1). It is clear that the idea of balance makes no sense if $d = 1$. The balances take different denominations in the literature, and for example they are called *Darboux points* in works of Maciejewski's [14], or *way directions* in Furta [3].

Each balance provides a particular solution of the differential system (1), called *scale-invariant solution* of the form

$$\varphi_{\mathbf{c}}(t) = \left(c_1 t^{-g_1}, ..., c_n t^{-g_n}\right), \tag{7}$$

which is deduced trivially from the fact that $\mathbf{c}$ is a solution of (6) and that, being system (1) QH, because

$$P_i\left(c_1 t^{-g_1}, ..., c_n t^{-g_n}\right) = t^{-g_i - 1} P_i(c_1, ..., c_n), \quad i = 1, ..., n.$$

Proposition 3. *Let* (1) *be a QH differential system withweight degree* $d \neq 1$. *If* H *is a YFI and* $\mathbf{c} = (c_1, ..., c_n)$ *is a balance of the system, then* $H(\mathbf{c}) = 0$.

Proof. Let k be the weight degree of H, and (7) the scale-invariant solution linked to the balance $\mathbf{c}$. If H is a YFI, then by (5) setting $\alpha = t^{\frac{1}{1-d}}$ we get

$$H(\varphi_{\mathbf{c}}(t)) = H\left(c_1 t^{-g_1}, ..., c_n t^{-g_n}\right) = t^{\frac{k}{1-d}} H(c_1, ..., c_n).$$

On the other hand, we know that H is constant along the solution $\varphi_{\mathbf{c}}(t)$, so since $\frac{k}{1-d} \neq 0$, we obtain that $H(c_1, ..., c_n) = 0$. □

From the proof of Proposition 3 it follows that, if there is more than one balance, and therefore more than one scale-invariant solution, all of them live on the same level surface of every YFI. So we have proved the next result.

Corollary 4. *Given a YFI H of a QH differential system with weight degree $d \neq 1$, any scale-invariant solution $\varphi_{\mathbf{c}}(t)$ lives on the level surface determined by $H(x_1, ..., x_n) = 0$.*

For each balance $\mathbf{c}$ the $n \times n$ matrix defined by

$$K(\mathbf{c}) = (K_{ij}(\mathbf{c})) = \left(\frac{\partial P_i}{\partial x_j}(\mathbf{c}) + \delta_{ij} g_j \right), \quad i, j = 1, ..., n,$$

is called the *Kowalevskaya matrix* of the QH differential system associated to the balance $\mathbf{c}$, and the eigenvalues of $K(\mathbf{c})$ are the *Kowalevskaya exponents* of the balance $\mathbf{c}$. These exponents, calculable in a finite number of steps, are the values on which is based all the theory of the Yoshida method for the search of the first integrals of QH systems. It is known that every Kowalevskaya matrix has -1 as eigenvalue. However this Kowalevskaya exponent does not have practical utility in the Yoshida method. This result has been proven in [3].

Proposition 5. *Let* (1) *be a QH differential system with weight degree $d \neq 1$ and let $\mathbf{c} = (c_1, ..., c_n)$ be a balance of this system. Then -1 is a Kowalevskaya exponent of $\mathbf{c}$ and $\mathbf{gc} = (g_1 c_1, ..., g_n c_n)$ is its corresponding eigenvector.*

We will name the useless eigenvector $\lambda_n = -1$ as the *trivial* Kowalevskaya exponent, and the rest of the eigenvalues $(\lambda_1, ..., \lambda_{n-1})$ as *non-trivial* Kowalevskaya exponents.

2.2.1. *Some notes on the number of balances and Kowalevskaya exponents.* In a QH system each family of weight vectors determines a single vector $\mathbf{g}$, so the number of balances depends, first on the number of families of the system. Note that the only QH systems with more than one family of weight vectors are special types of homogeneous QH [13]. Having only one family of weight vectors in the system, the polynomial character of (6) and Bezout's Theorem guarantee that the amount of balances cannot exceed $\prod_{i=1}^{n} h_i$, being h_i the degree of P_i (always provided that $d \neq 1$). Obviously it may also be the case that the system lacks them.

We note that the balances are grouped into equivalence classes whose cardinal depends on the weight degree d of the corresponding weight vector. The relevant fact about this is that all balances of a certain equivalence class provide the same essential information, that is, the same Kowalevskaya exponents:

Proposition 6. *Let* (1) *be a QH differential system with weight vector $\mathbf{v} = (s_1, ..., s_n, d)$, $d \neq 1$, and let $\mathbf{c} = (c_1, ..., c_n)$ be a balance. Then all the* *elements of*

$$B_{\mathbf{c}}^{\mathbf{v}}=\left\{\left(r_p^{s_1}c_1,...,r_p^{s_n}c_n\right)\mid p=1,...,d-1\right\}$$

are balances of system (1)*, where* r_p *runs in* $G_d=\left\{r\in\mathbb{C}\mid r^{d-1}=1\right\}$.

Proof. Let $r\in\mathbb{C}$ be such that $r^{d-1}=1$. Since $\mathbf{v}$ is a weight vector, then

$$P_i\left(r^{s_1}c_1,...,r^{s_n}c_n\right)=r^{s_i-1+d}P_i\left(c_1,...,c_n\right),\qquad i=1,...,n. \tag{8}$$

Since $\mathbf{c}$ is a balance by (6) and (8) we have

$$P_i\left(r^{s_1}c_1,...,r^{s_n}c_n\right)=r^{s_i-1+d}\left(-g_ic_i\right)=-g_i\left(r^{s_i}c_i\right),\qquad i=1,...,n,$$

and as a consequence $(r^{s_1}c_1,...,r^{s_n}c_n)$ is a balance. □

Remark 7. Note that if $\mathbf{v}$ and $\mathbf{w}$ are two weight vectors of the same family, the sets $B_{\mathbf{c}}^{\mathbf{v}}$ and $B_{\mathbf{c}}^{\mathbf{w}}$ coincide: if $\mathbf{v}_m=(s_1,...,s_n,d)$ is the representative vector of the family ($\gcd(s_1,...,s_n)=1$), any other vector of it has the form $(ks_1,...,ks_n,k(d-1)+1)$, being $k\in\mathbb{Z}^+$. Therefore the set of complex values $\left\{r^{ks_i}\in\mathbb{C}\mid r^{k(d-1)}=1\right\}$ trivially matches the set $\left\{r^{s_i}\in\mathbb{C}\mid r^{d-1}=1\right\}$.

Proposition 8. *Under the assumptions of Proposition* 6 *all balances of the set* $B_{\mathbf{c}}^{\mathbf{v}}$ *give rise to the same Kowalevskaya exponents.*

Proof. Let $K(\mathbf{c})$ be the Kowalevskaya matrix of balance $\mathbf{c}$, i.e.

$$K(\mathbf{c})=\left(K_{ij}(\mathbf{c})\right)=\left(\frac{\partial P_i}{\partial x_j}(\mathbf{c})+\delta_{ij}g_j\right),\quad i,j=1,...,n,$$

and, fixed $r\in G_d$, let $K(r^{\mathbf{s}}\mathbf{c})$ be the Kowalevskaya matrix of balance $r^{\mathbf{s}}\mathbf{c}=(r^{s_1}c_1,...,r^{s_n}c_n)$, that is

$$K_{ij}\left(r^{\mathbf{s}}\mathbf{c}\right)=\frac{\partial P_i}{\partial x_j}\left(r^{\mathbf{s}}\mathbf{c}\right)+\delta_{ij}g_j,\quad i,j=1,...,n.$$

We define the functions $f_i(x_1,...,x_n)=P_i(r^{s_1}x_1,...,r^{s_n}x_n)$ for $i=1,...,n$, with which we have that $f_i=P_i\circ g$ for $i=1,...,n$, being $g(x_1,...,x_n)=(r^{s_1}x_1,...,r^{s_n}x_n)$. Since $(s_1,...,s_n,d)$ is a weight vector and $r^{d-1}=1$, it follows that $f_i(x_1,...,x_n)=r^{s_i}P_i(x_1,...,x_n)$, so

$$\nabla f_i\left(x_1,...,x_n\right)=r^{s_i}\nabla P_i\left(x_1,...,x_n\right),\quad i=1,...,n. \tag{9}$$

On the other hand, the chain rule sets that $\nabla f_i(x_1,...,x_n)=\nabla P_i(g(x_1,...,x_n))\cdot J_g(x_1,...,x_n)$ for $i=1,...,n$, and since $J_g(x_1,...,x_n)=diag(r^{s_1},...,r^{s_n})$ is a regular matrix with inverse $J_g^{-1}(x_1,...,x_n)=\operatorname{diag}(r^{-s_1},...,r^{-s_n})$, we can write

$$\nabla P_i\left(g\left(x_1,...,x_n\right)\right)=\nabla f_i\left(x_1,...,x_n\right)\cdot J_g^{-1}\left(x_1,...,x_n\right),\quad i=1,...,n. \tag{10}$$

This implies, due to (9) and (10), that

$$\frac{\partial P_i}{\partial x_j}\left(r^{s_1}x_1,...,r^{s_n}x_n\right)=r^{s_i-s_j}\frac{\partial P_i}{\partial x_j}\left(x_1,...,x_n\right),\quad i,j=1,...,n,$$

and consequently, setting $(x_1, ..., x_n) = \mathbf{c}$ and taking into account that on the main diagonal the powers of r are equal to 1, we obtain

$$K_{ij}\left(r^{\mathbf{s}}\mathbf{c}\right) = r^{s_i - s_j} K_{ij}\left(\mathbf{c}\right), \quad i, j = 1, ..., n.$$

Therefore we only have to take the matrix $J_g\left(x_1, ..., x_n\right)$ as the change matrix to see that the matrices $K\left(\mathbf{c}\right)$ and $K\left(r^{\mathbf{s}}\mathbf{c}\right)$ are similar. Therefore they will have the same spectrum. □

2.3. **Main results of the Yoshida method.** Since the early 80's, when the first outcomes of what could be called the Yoshida method are published, until today, contributions to the subject have been copious. A large group of researchers have expanded, not without obstacles neither difficulties, the collection of useful theorems. In this section we will present in chronological order of publication those that in our opinion are the most relevant results, together with some comments on how the weaknesses found in the theory have been partially solved.

Some of these theorems were published focusing on the search for algebraic first integrals, but as YFIs are algebraic, they are also valid results for the study of analytic first integrals.

The following theorem, due to Yoshida [15], constitutes the origin of the whole theory and relates the Kowalevskaya exponents of a system with the weight degrees of its potential first integrals. In its original publication, intended for algebraic first integrals, it included the additional condition that $\nabla H\left(\mathbf{c}\right)$ be finite, which is unnecessary for polynomial first integrals. Additionally, Yoshida exactly equates weight degrees with Kowalevskaya exponents, because he takes as the weight exponent of H the vector $\mathbf{g} = \mathbf{s}/\left(d-1\right)$ instead of $\mathbf{s}$.

Theorem 9. *Let* (1) *be a QH differential system with weight degree* $d \neq 1$ *and let* $\mathbf{c}$ *be a balance whose non-trivial Kowalevskaya exponents are* $\left\{\lambda_1, ..., \lambda_{n-1}\right\}$. *If* H *is a YFI verifying* $\nabla H\left(\mathbf{c}\right) \neq \mathbf{0}$, *then its weight degree is* $\lambda_j\left(d-1\right)$ *for some* $j \in \left\{1, ..., n-1\right\}$.

The practical use of this result is powerful. Thus knowing the Kowalevskaya exponents of a balance, we limit the search for YFIs to those whose weight degrees adjust to certain values. But this idea has an important weakness, only partially solved in subsequent publications: any YFI that does not verify $\nabla H\left(\mathbf{c}\right) \neq \mathbf{0}$ will be "hidden", outside our search radius. For example, any power H^p of a YFI H will remain hidden, because from Proposition 3, $\nabla H^p\left(\mathbf{c}\right) = pH^{p-1}\left(\mathbf{c}\right) \cdot \nabla H\left(\mathbf{c}\right) = \mathbf{0}$. This is not relevant for the purposes of studying the integrability of the system, because H^p is functionally dependent on H, but unfortunately it does exist YFIs that are functionally independent of the rest and whose gradient is null on all balances:

Example 1. The differential system

$$\dot{x} = x^2 + 3z^2, \quad \dot{y} = 2xu, \quad \dot{z} = -2xz - y^2 - u^2, \quad \dot{u} = -2xy$$

is QH with weight vector $(s_1, s_2, s_3, s_4, d) = (1, 1, 1, 1, 2)$ and has the following balances:

$$\mathbf{c}_1 = (-1, 0, 0, 0)\,, \quad \mathbf{c}_2 = (1/2, 0, i/2, 0)\,, \quad \mathbf{c}_3 = (1/2, 0, -i/2, 0)\,.$$

The non-trivial Kowalevskaya exponents corresponding to $\mathbf{c}_1$ are $\{3, 1+2i, 1-2i\}$ while those corresponding to $\mathbf{c}_2$ and $\mathbf{c}_3$ are $\{3, 1+i, 1-i\}$. The YFI $H_1(x, y, z, u) = x^2z + xy^2 + xu^2 + z^3$ of weight degree 3, concerns the Kowalevskaya exponent $\lambda = 3(d-1) = 3$, which appears in the three balances. However there is also the YFI $H_2(x, y, z, u) = y^2 + u^2$ of weight degree 2. We note that the Kowalevskaya exponent $\lambda = 2(d-1) = 2$ does not appear in any of the three balances, as would be expected, and this is because the aforementioned condition is not met:

$$\nabla H_2(\mathbf{c}_i) = \mathbf{0}, \quad i = 1, 2, 3.$$

Furthermore it is easy to verify that H_1 and H_2 are functionally independent, so if we do not detect H_2 we are missing relevant information regarding integrability.

Despite this problem Theorem 9 is very useful because it serves as an orientation regarding possible weight degrees of existing YFIs. A similar result but focused on the search for sets of independent YFIs with the same weight degree, is as follows:

Theorem 10. *Let* (1) *be a QH differential system with weight degree* $d \neq 1$ *and let* $\mathbf{c}$ *be a balance whose non-trivial Kowalevskaya exponents are* $\{\lambda_1, ..., \lambda_{n-1}\}$. *Let* $H_1, ..., H_r$ *be functionally independent YFI with the same weight degree* k. *If the vectors* $\nabla H_1(\mathbf{c}), ..., \nabla H_r(\mathbf{c})$ *are linearly independent, then the common weight degree is* $k = \lambda_j(d-1)$ *for some* $j \in \{1, ..., n-1\}$, *being* λ_j *a Kowalevskaya exponent of multiplicity at least* r.

The following theorem provides a necessary condition for the complete integrability, it has been experienced many modifications. It was originally published by Yoshida [16] based on the weak conception of algebraic integrability, although Gascón [7] warned that the proof was actually only valid for systems in the plane. Later on Bessis [1] shows providing counterexamples that this theorem is false in dimensions higher than 2. Finally Goriely [8] proves that the result is valid as long as the idea of integrability used is the strong algebraic (or equivalently, our complete integrability with analytical functions).

Theorem 11. *Let* (1) *be a completely integrable QH differential system with weight degree* $d \neq 1$ *and let* $\mathbf{c}$ *be a balance whose non-trivial Kowalevskaya exponents are* $\{\lambda_1, ..., \lambda_{n-1}\}$. *Then* $\lambda_j \in \mathbb{Q}$ *for all* $j \in \{1, ..., n-1\}$.

Let X be either $\mathbb{N}$ or $\mathbb{Z}$. A set of complex values $\lambda_1, ..., \lambda_n$ is $X-$*independent* when

$$\sum_{j=1}^{n} \alpha_j \lambda_j = 0, \qquad \alpha_j \in X \quad \forall j \in \{1, ..., n\},$$

implies that $\alpha_j = 0 \quad \forall j \in \{1, ..., n\}$. Otherwise the set is called $X-$*dependent*.

The following theorem, due to Furta [3], is useful to determine the partial integrability when we already have a YFI.

Theorem 12. *Let* (1) *be a QH differential system with weight degree* $d \neq 1$ *and let* $\mathbf{c}$ *be a balance whose non-trivial Kowalevskaya exponents are* $\{\lambda_1, ..., \lambda_{n-1}\}$. *Let* H *be a YFI verifying* $\nabla H(\mathbf{c}) \neq \mathbf{0}$ *whose weight degree is* $\lambda_j (d-1)$ *for some* $j \in \{1, ..., n-1\}$. *If the set* $\{-1, \lambda_2, ..., \lambda_{j-1}, \lambda_{j+1}, ..., \lambda_n\}$ *is* $\mathbb{Z}-$*independent, then any other first integral* H' *of system* (1) *is a function of* H*, i.e.* $H' = \mathcal{F}(H)$*, where* $\mathcal{F}$ *is a smooth function.*

The following result and its corollary once again relates the Kowalevskaya exponents with the weight degrees of YFIs, but with an interesting novelty: $\nabla H(\mathbf{c}) \neq \mathbf{0}$ is no longer required from the first integrals. Therefore, this theorem captures all YFIs, leaving no "hidden" ones. His credit goes to Goriely [8], although after it has been improved by Llibre and Zhang [12], and finally by Liu, Wu and Yang [11].

Theorem 13. *Let* (1) *be a QH differential system with weight degree* $d \neq 1$ *and let* $\mathbf{c}$ *be a balance whose non-trivial Kowalevskaya exponents are* $\{\lambda_1, ..., \lambda_{n-1}\}$. *Let* H *be a YFI whose weight degree is* k. *Then there exist natural numbers* $\alpha_1, ..., \alpha_{n-1}$ *verifying* $0 < \alpha_1 + \cdots + \alpha_{n-1} \leq k$ *such that*

$$\sum_{j=1}^{n-1} \alpha_j \lambda_j = \frac{k}{d-1}.$$

Corollary 14. *Let* (1) *be a QH differential system with weight degree* $d \neq 1$ *and let* $\mathbf{c}$ *be a balance. If the Kowalevskaya exponents of* $\mathbf{c}$ *are* $\mathbb{N}-$*independent, then there is no analytical first integral of system* (1).

Finally the contribution by Maciejewski [14], which links the weight vector of the system with the Kowalevskaya exponents of each existing balances, it is useful when obtaining the Kowalevskaya matrix presents computational difficulties.

Theorem 15. *Let* (1) *be a QH differential system with weight exponent* $\mathbf{s} = (s_1, ..., s_n) \in (\mathbb{Z}^+)^n$ *and weight degree* $d \neq 1$*, and let* $\{\mathbf{c}_{\mathrm{i}}\}_{i \in I}$ *be the set of balances of the system, whose non-trivial Kowalevskaya exponents are, respectively,* $\left\{\lambda_{i1}, ..., \lambda_{i(n-1)}\right\}$. *Then*

$$\sum_{i \in I} \frac{\left(\lambda_{i1} + \cdots + \lambda_{i(n-1)}\right)^j}{\lambda_{i1} \cdot \ldots \cdot \lambda_{i(n-1)}} = \frac{(g_1 + \cdots + g_n + 1)^j}{g_1 \cdot \ldots \cdot g_n}, \qquad j = 0, ...n-1,$$

being $g_i = s_i/(d-1)$, $i = 1, ..., n$.

3. Canonical forms

In [6] an algorithm was published to determine, fixed a given degree, all QH systems in dimension 3. Such algorithm provides all normal forms of maximal systems. Then all other quasi-homogeneous systems can be trivially deduced from the maximal set, because they can be considered as particular cases of the maximal normal forms in which some monomials are zero. Some interesting properties of the 3-dimensional maximal systems are proved in [6], like the fact that a maximal QH system has a unique weight vector family, or that a maximal QH system always has a minimum weight vector. In fact, the minimum vector has a simple characterization in the unique-family case: a weight vector (s_1, s_2, s_3, d) is the minimum weight vector if and only if $\gcd(s_1, s_2, s_3, d) = 1$. The existence of the minimum vector $\mathbf{v_m}$, added to the fact that two different maximal QH systems of the same degree n have no common weight vectors, confers to $\mathbf{v_m}$ the character of unique identifier within the set of maximal systems of degree n.

The simplest type of maximal 3-dimensional QH system is that of degree 2. The algorithm published in [6] provides 20 normal forms of this type, to which the trivial case of the homogeneous system should be added. They are the following, accompanied by their respective minimum vectors:

$$\mathsf{N}_1: \begin{array}{l} \dot{x} = ay^2 + byz + cz^2 + dx \\ \dot{y} = ey + fz \\ \dot{z} = gy + hz \\ \mathbf{w}_m = (2,1,1,1) \end{array}$$

$$\mathsf{N}_6: \begin{array}{l} \dot{x} = ayz + bx \\ \dot{y} = cz^2 + dy \\ \dot{z} = ez \\ \mathbf{w}_m = (3,2,1,1) \end{array}$$

$$\mathsf{N}_2: \begin{array}{l} \dot{x} = az^2 + bx + cy \\ \dot{y} = dz^2 + ex + fy \\ \dot{z} = gz \\ \mathbf{w}_m = (2,2,1,1) \end{array}$$

$$\mathsf{N}_7: \begin{array}{l} \dot{x} = ay^2 + bx \\ \dot{y} = cz^2 + dy \\ \dot{z} = ez \\ \mathbf{w}_m = (4,2,1,1) \end{array}$$

$$\mathsf{N}_3: \begin{array}{l} \dot{x} = ay^2 + byz + cz^2 \\ \dot{y} = dx \\ \dot{z} = ex \\ \mathbf{w}_m = (3,2,2,2) \end{array}$$

$$\mathsf{N}_8: \begin{array}{l} \dot{x} = ayz \\ \dot{y} = bz^2 + cx \\ \dot{z} = dy \\ \mathbf{w}_m = (4,3,2,2) \end{array}$$

$$\mathsf{N}_4: \begin{array}{l} \dot{x} = az^2 \\ \dot{y} = bz^2 \\ \dot{z} = cx + dy \\ \mathbf{w}_m = (3,3,2,2) \end{array}$$

$$\mathsf{N}_9: \begin{array}{l} \dot{x} = axy + bxz \\ \dot{y} = cy^2 + dyz + ez^2 + fx \\ \dot{z} = gy^2 + hyz + iz^2 + jx \\ \mathbf{w}_m = (2,1,1,2) \end{array}$$

$$\mathsf{N}_5: \begin{array}{l} \dot{x} = az^2 \\ \dot{y} = bx \\ \dot{z} = cy \\ \mathbf{w}_m = (5,4,3,2) \end{array}$$

$$\mathsf{N}_{10}: \begin{array}{l} \dot{x} = ay^2 \\ \dot{y} = bz^2 + cx \\ \dot{z} = 0 \\ \mathbf{w}_m = (6,4,3,3) \end{array}$$

$$\mathsf{N}_{11}: \begin{array}{l} \dot{x} = ay^2 \\ \dot{y} = bz^2 \\ \dot{z} = cy \\ \mathbf{w}_m = (5,3,2,2) \end{array} \qquad \mathsf{N}_{16}: \begin{array}{l} \dot{x} = axz + by^2 \\ \dot{y} = cyz \\ \dot{z} = dz^2 + ex \\ \mathbf{w}_m = (4,3,2,3) \end{array}$$

$$\mathsf{N}_{12}: \begin{array}{l} \dot{x} = ayz \\ \dot{y} = bz^2 \\ \dot{z} = cx \\ \mathbf{w}_m = (5,4,3,3) \end{array} \qquad \mathsf{N}_{17}: \begin{array}{l} \dot{x} = ay^2 \\ \dot{y} = bxz \\ \dot{z} = cy \\ \mathbf{w}_m = (4,3,1,3) \end{array}$$

$$\mathsf{N}_{13}: \begin{array}{l} \dot{x} = ay^2 \\ \dot{y} = bz^2 \\ \dot{z} = cx \\ \mathbf{w}_m = (7,5,4,4) \end{array} \qquad \mathsf{N}_{18}: \begin{array}{l} \dot{x} = ay^2 \\ \dot{y} = bxz \\ \dot{z} = cx \\ \mathbf{w}_m = (5,4,2,4) \end{array}$$

$$\mathsf{N}_{14}: \begin{array}{l} \dot{x} = axz + byz \\ \dot{y} = cxz + dyz \\ \dot{z} = ez^2 + fx + gy \\ \mathbf{w}_m = (2,2,1,2) \end{array} \qquad \mathsf{N}_{19}: \begin{array}{l} \dot{x} = axy \\ \dot{y} = by^2 + cxz \\ \dot{z} = dyz + ex \\ \mathbf{w}_m = (3,2,1,3) \end{array}$$

$$\mathsf{N}_{15}: \begin{array}{l} \dot{x} = axz + by^2 \\ \dot{y} = cyz + dx \\ \dot{z} = ez^2 + fy \\ \mathbf{w}_m = (3,2,1,2) \end{array} \qquad \mathsf{N}_{20}: \begin{array}{l} \dot{x} = 0 \\ \dot{y} = axz \\ \dot{z} = by^2 + cx \\ \mathbf{w}_m = (4,2,1,4) \end{array}$$

Since these are maximal systems, their coefficients a, b, ... i, j can take any real value other than zero.

Our intention is to use these families of systems to test the capabilities of the Yoshida method on three-dimensional QH systems. In order to simplify the calculations as much as possible, we will carry out a reduction on the number of parameters. Doing a linear change of variables we will preserve their topological equivalence and they remain in the class of the QH systems.

Theorem 16. *Any maximal quadratic QH system in $\mathbb{R}^3$ can be written, after a rescaling of the variables and time, as one of the following systems:*

$$\mathsf{S}_{1}: \begin{array}{l} \dot{x} = y^2 + yz + pz^2 + x \\ \dot{y} = qy + rz \\ \dot{z} = sy + uz \\ \mathbf{w}_m = (2,1,1,1) \end{array} \qquad \mathsf{S}_{4}: \begin{array}{l} \dot{x} = pz^2 \\ \dot{y} = z^2 \\ \dot{z} = x + y \\ \mathbf{w}_m = (3,3,2,2) \end{array}$$

$$\mathsf{S}_{2}: \begin{array}{l} \dot{x} = z^2 + x + y \\ \dot{y} = pz^2 + qx + ry \\ \dot{z} = sz \\ \mathbf{w}_m = (2,2,1,1) \end{array} \qquad \mathsf{S}_{5}: \begin{array}{l} \dot{x} = z^2 \\ \dot{y} = x \\ \dot{z} = y \\ \mathbf{w}_m = (5,4,3,2) \end{array}$$

$$\mathsf{S}_{3}: \begin{array}{l} \dot{x} = py^2 + qyz + z^2 \\ \dot{y} = x \\ \dot{z} = x \\ \mathbf{w}_m = (3,2,2,2) \end{array} \qquad \mathsf{S}_{6}: \begin{array}{l} \dot{x} = yz + px \\ \dot{y} = z^2 + qy \\ \dot{z} = z \\ \mathbf{w}_m = (3,2,1,1) \end{array}$$

$$\mathsf{S}_7: \begin{array}{l} \dot{x} = y^2 + x \\ \dot{y} = \pm z^2 + py \\ \dot{z} = qz \\ \mathbf{w}_m = (4,2,1,1) \end{array}$$

$$\mathsf{S}_8: \begin{array}{l} \dot{x} = pyz \\ \dot{y} = z^2 + x \\ \dot{z} = y \\ \mathbf{w}_m = (4,3,2,2) \end{array}$$

$$\mathsf{S}_9: \begin{array}{l} \dot{x} = xy + xz \\ \dot{y} = py^2 + qyz + rz^2 + x \\ \dot{z} = sy^2 + uyz + vz^2 + wx \\ \mathbf{w}_m = (2,1,1,2) \end{array}$$

$$\mathsf{S}_{10}: \begin{array}{l} \dot{x} = y^2 \\ \dot{y} = \pm z^2 + x \\ \dot{z} = 0 \\ \mathbf{w}_m = (6,4,3,3) \end{array}$$

$$\mathsf{S}_{11}: \begin{array}{l} \dot{x} = y^2 \\ \dot{y} = z^2 \\ \dot{z} = y \\ \mathbf{w}_m = (5,3,2,2) \end{array}$$

$$\mathsf{S}_{12}: \begin{array}{l} \dot{x} = yz \\ \dot{y} = z^2 \\ \dot{z} = x \\ \mathbf{w}_m = (5,4,3,3) \end{array}$$

$$\mathsf{S}_{13}: \begin{array}{l} \dot{x} = y^2 \\ \dot{y} = z^2 \\ \dot{z} = x \\ \mathbf{w}_m = (7,5,4,4) \end{array}$$

$$\mathsf{S}_{14}: \begin{array}{l} \dot{x} = pxz + qyz \\ \dot{y} = rxz + syz \\ \dot{z} = z^2 + x + y \\ \mathbf{w}_m = (2,2,1,2) \end{array}$$

$$\mathsf{S}_{15}: \begin{array}{l} \dot{x} = xz + y^2 \\ \dot{y} = pyz + x \\ \dot{z} = qz^2 + ry \\ \mathbf{w}_m = (3,2,1,2) \end{array}$$

$$\mathsf{S}_{16}: \begin{array}{l} \dot{x} = xz \pm y^2 \\ \dot{y} = pyz \\ \dot{z} = qz^2 + x \\ \mathbf{w}_m = (4,3,2,3) \end{array}$$

$$\mathsf{S}_{17}: \begin{array}{l} \dot{x} = \pm y^2 \\ \dot{y} = \pm xz \\ \dot{z} = \pm y \\ \mathbf{w}_m = (4,3,1,3) \end{array}$$

$$\mathsf{S}_{18}: \begin{array}{l} \dot{x} = y^2 \\ \dot{y} = xz \\ \dot{z} = x \\ \mathbf{w}_m = (5,4,2,4) \end{array}$$

$$\mathsf{S}_{19}: \begin{array}{l} \dot{x} = pxy \\ \dot{y} = qy^2 \pm xz \\ \dot{z} = yz \pm x \\ \mathbf{w}_m = (3,2,1,3) \end{array}$$

$$\mathsf{S}_{20}: \begin{array}{l} \dot{x} = 0 \\ \dot{y} = xz \\ \dot{z} = y^2 + x \\ \mathbf{w}_m = (4,2,1,4) \end{array}$$

where the coefficients p, q, r, s, u, v, w are nonzero real numbers in all systems.

Proof. As stated any maximal QH system in $\mathbb{R}^3$ can be expressed as one of the normal forms N_m. We denote its components by

$$P_i\left(x,y,z\right) = \sum_{k=1}^{n_i} h_i^k x^{A_i^k} y^{B_i^k} z^{C_i^k}, \qquad i = 1,2,3, \tag{11}$$

where n_i is the number of monomials of the polynomial P_i, $h_i^k \in \{a,b,c,...,j\} \subseteq \mathbb{R} \setminus \{0\}$ is the coefficient of the $k-$th monomial of P_i, and A_i^k, B_i^k, C_i^k are respectively the exponents of the variables x, y, z in the $k-$th monomial of P_i, for k between 1 and n_i.

To reduce the greatest possible number of parameters, but obtaining systems that are topologically equivalent to the starting ones, we make the following linear change of variables:

$$x = \alpha X, \quad y = \beta Y, \quad z = \gamma Z, \quad t = \delta T, \tag{12}$$

being α, β, γ, δ nonzero real numbers to be determined. With these new variables, system (1) with $n = 3$ is transformed into

$$\frac{dX}{dT} = \bar{P}_1(X,Y,Z), \quad \frac{dY}{dT} = \bar{P}_2(X,Y,Z), \quad \frac{dZ}{dT} = \bar{P}_3(X,Y,Z), \tag{13}$$

where

$$\bar{P}_i(X,Y,Z) = \frac{\delta}{g(i)} P_j(\alpha X, \beta Y, \gamma Z), \qquad i = 1,2,3, \tag{14}$$

being $g(1) = \alpha$, $g(2) = \beta$, $g(3) = \gamma$. Taking into account (11) and (14) we get

$$\bar{P}_i(x,y,z) = \sum_{k=1}^{n_i} \frac{\delta}{g(i)} h_i^k \alpha^{A_i^k} \beta^{B_i^k} \gamma^{C_i^k} X^{A_i^k} Y^{B_i^k} Z^{C_i^k}, \qquad i = 1,2,3.$$

Now we look for the values of α, β, γ, δ based on the $h_i^k \in \{a, b, c, ..., j\}$ coefficients, so that the change (12) provides a system (13) as simplified as possible, that is, with the largest number of unitary coefficients. We therefore propose the system of $n_1 + n_2 + n_3$ equations in the variables α, β, γ, δ:

$$\left\{ \frac{\delta}{g(i)} h_i^k \alpha^{A_i^k} \beta^{B_i^k} \gamma^{C_i^k} = 1 \text{ such that } 1 \leq k \leq n_i\, ;\ i = 1,2,3 \right\}.$$

In case that this system is compatible only for certain values of the h_i^k, we will successively eliminate equations from it until it has a solution. Each equation eliminated means a parameter that we cannot convert to ± 1. Finally, we get the solution $(\alpha_0, \beta_0, \gamma_0, \delta_0)$, valid for any h_i^k, which provides the optimal change (12): $x = \alpha_0 X,\ y = \beta_0 Y,\ z = \gamma_0 Z,\ t = \delta_0 T$. Note that this solution always exists if we reduce equations conveniently, because h_i^k is nonzero and the equation

$$\frac{\delta}{g(i)} h_i^k \alpha^{A_i^k} \beta^{B_i^k} \gamma^{C_i^k} = 1$$

has the solution $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\left(g(i)/h_i^k\right)^{1/A_i^k}, 1, 1, 1 \right)$.

Following this procedure with the 20 normal forms $\mathsf{N_m}$ it is possible to reduce a maximum of three parameters in each of them. Next, for each normal form, both the optimal change found and the values taken by the parameters of the canonical forms $\mathsf{S_m}$ that have not been possible to reduce to unit values are provided:

S1: $(\alpha_0, \beta_0, \gamma_0, \delta_0) = \left(\frac{a}{d}, 1, \frac{a}{b}, \frac{1}{d} \right)$ $\quad p = \frac{ac}{b^2},\ q = \frac{e}{d},\ r = \frac{af}{bd},\ s = \frac{bg}{ad},\ u = \frac{h}{d}$

S2: $(\alpha_0,\beta_0,\gamma_0,\delta_0)=\left(\frac{a}{b},\frac{a}{c},1,\frac{1}{b}\right)$ $\quad p=\frac{cd}{ab},\ q=\frac{ce}{b^2},\ r=\frac{f}{b},\ s=\frac{g}{b}$

S3: $(\alpha_0,\beta_0,\gamma_0,\delta_0)=\left(\frac{1}{ce^2},\frac{d}{ce^2},\frac{1}{ce},1\right)$ $\quad p=\frac{ad^2}{ce^2},\ q=\frac{bd}{ce}$

S4: $(\alpha_0,\beta_0,\gamma_0,\delta_0)=\left(\frac{1}{bcd},\frac{1}{bd^2},\frac{1}{bd},1\right)$ $\quad p=\frac{ac}{bd}$

S5: $(\alpha_0,\beta_0,\gamma_0,\delta_0)=\left(\frac{1}{ab^2c^2},\frac{1}{abc^2},\frac{1}{abc},1\right)$

S6: $(\alpha_0,\beta_0,\gamma_0,\delta_0)=\left(\frac{ac}{e^2},\frac{c}{e},1,\frac{1}{e}\right)$ $\quad p=\frac{b}{e},\ q=\frac{d}{e}$

S7: $(\alpha_0,\beta_0,\gamma_0,\delta_0)=\left(\frac{a}{b},1,\sqrt{|\frac{b}{c}|},\frac{1}{b}\right)$ $\quad p=\frac{d}{b},\ q=\frac{e}{b}$

S8: $(\alpha_0,\beta_0,\gamma_0,\delta_0)=\left(\frac{1}{bcd^2},\frac{1}{bd^2},\frac{1}{bd},1\right)$ $\quad p=\frac{ac}{bd}$

S9: $(\alpha_0,\beta_0,\gamma_0,\delta_0)=\left(\frac{1}{af},\frac{1}{a},\frac{1}{b},1\right)$ $\quad p=\frac{c}{a},\ q=\frac{d}{b},\ r=\frac{ae}{b^2},\ s=\frac{bg}{a^2},\ u=\frac{h}{a},\ v=\frac{i}{b},\ w=\frac{bj}{af}$

S10: $(\alpha_0,\beta_0,\gamma_0,\delta_0)=\left(\frac{1}{ac^2},\frac{1}{ac},\frac{1}{\sqrt{|abc|}},1\right)$

S11: $(\alpha_0,\beta_0,\gamma_0,\delta_0)=\left(\frac{a}{b^{1/3}c^{2/3}},1,\frac{c^{1/3}}{b^{1/3}},\frac{1}{b^{1/3}c^{2/3}}\right)$

S12: $(\alpha_0,\beta_0,\gamma_0,\delta_0)=\left(\frac{a^{1/3}b^{1/3}}{c^{2/3}},\frac{b^{2/3}}{a^{1/3}c^{1/3}},1,\frac{1}{a^{1/3}b^{1/3}c^{1/3}}\right)$

S13: $(\alpha_0,\beta_0,\gamma_0,\delta_0)=\left(\frac{1}{a^{1/3}b^{2/3}c^{4/3}},\frac{1}{a^{2/3}b^{1/3}c^{2/3}},\frac{1}{a^{1/3}b^{2/3}c^{1/3}},1\right)$

S14: $(\alpha_0,\beta_0,\gamma_0,\delta_0)=\left(\frac{1}{ef},\frac{1}{eg},\frac{1}{e},1\right)$ $\quad p=\frac{a}{e},\ q=\frac{bf}{eg},\ r=\frac{cg}{ef},\ s=\frac{d}{e}$

S15: $(\alpha_0,\beta_0,\gamma_0,\delta_0)=\left(\frac{1}{bd^2},\frac{1}{bd},\frac{1}{a},1\right)$ $\quad p=\frac{c}{a},\ q=\frac{e}{a},\ r=\frac{af}{bd}$

S16: $(\alpha_0,\beta_0,\gamma_0,\delta_0)=\left(\frac{1}{ae},\frac{1}{\sqrt{|abe|}},\frac{1}{a},1\right)$ $\quad p=\frac{c}{a},\ q=\frac{d}{a}$

S17: $(\alpha_0,\beta_0,\gamma_0,\delta_0)=\left(\frac{1}{bc},\frac{1}{\sqrt{|abc|}},\sqrt{|\frac{c}{ab}|},1\right)$

S18: $(\alpha_0,\beta_0,\gamma_0,\delta_0)=\left(\frac{1}{a^{1/3}b^{2/3}c^{2/3}},\frac{1}{a^{2/3}b^{1/3}c^{1/3}},\frac{c^{1/3}}{a^{1/3}b^{2/3}},1\right)$

S19: $(\alpha_0,\beta_0,\gamma_0,\delta_0)=\left(\frac{1}{\sqrt{|cde|}},\frac{1}{d},\sqrt{|\frac{e}{cd}|},1\right)$ $\quad p=\frac{a}{d},\ q=\frac{b}{d}$

S20: $(\alpha_0,\beta_0,\gamma_0,\delta_0)=\left(\frac{1}{a^{2/3}b^{1/3}c^{1/3}},\frac{c^{1/3}}{a^{1/3}b^{2/3}},\frac{c^{2/3}}{a^{2/3}b^{1/3}},1\right)$

On the other hand, since the systems of each canonical form $\mathsf{S_m}$ are a subset of the normal form $\mathsf{N_m}$, their corresponding minimum vector does not change. □

Example 2. As a sample of the procedure followed in the proof of the previous theorem, detailed calculations for the reduction of parameters in the normal form $\mathsf{N_3}$ are given below. The maximal QH differential system $\mathsf{N_3}$ is

$$\dot{x} = ay^2 + byz + cz^2, \quad \dot{y} = dx, \quad \dot{z} = ex.$$

Applying the change of variables (12) the system is transformed into

$$\dot{X} = \frac{\beta^2\delta}{\alpha} aY^2 + \frac{\beta\gamma\delta}{\alpha} bYZ + \frac{\gamma^2\delta}{\alpha} cZ^2, \quad \dot{Y} = \frac{\alpha\delta}{\beta} dX, \quad \dot{Z} = \frac{\alpha\delta}{\gamma} eX. \tag{15}$$

We are now looking for values for α, β, γ, δ based on a, b, c, d, e so that the change (12) provides a system (15) as simplified as possible; that is, with the highest number of unitary coefficients. We therefore consider the system of equations

$$\frac{\beta^2\delta}{\alpha} a = 1, \quad \frac{\beta\gamma\delta}{\alpha} b = 1, \quad \frac{\gamma^2\delta}{\alpha} c = 1, \quad \frac{\alpha\delta}{\beta} d = 1, \quad \frac{\alpha\delta}{\gamma} e = 1.$$

Not all values of a, b, c, d, e give rise to a compatible system in the variables α, β, γ, δ. If we eliminate one of the five equations, this situation remains. But eliminating two equations we have compatible systems. We should get rid of equations trying that the solutions obtained afterwards have no roots, which would force us to consider as positive some of the parameters a, b, c, d, e. We remember that our only condition for these values is that they are not null. Thereby if we eliminate the first two equations then we obtain a compatible system whose δ-based solutions are

$$\alpha = \frac{1}{ce^2\delta^3}, \quad \beta = \frac{d}{ce^2\delta^2}, \quad \gamma = \frac{1}{ce\delta^2}.$$

With this result, for whatever value $\delta \neq 0$ we take, and in particular for $\delta = 1$, the normal form $\mathsf{N_3}$ is transformed into (we have returned, in order to simplicity, to the variables x, y, z, t):

$$\mathsf{S_3}: \begin{array}{l} \dot{x} = py^2 + qyz + z^2 \\ \dot{y} = x \\ \dot{z} = x \end{array}$$

where $p = \dfrac{ad^2}{ce^2} \in \mathbb{R} \setminus \{0\}$, $q = \dfrac{bd}{ce} \in \mathbb{R} \setminus \{0\}$. In short we have eliminated three of the five parameters and found a canonical form.

4. Application of the method to 3-dimensional QH systems of degree 2

The study of the integrability, that is of the existence of first integrals in a differential system depending on parameters, is generally a difficult problem.

Except for some simple cases, this task is very hard and there are no completely satisfactory methods to solve it. One of the procedures available for this, provided we treat with QH systems, is the Yoshida method discussed in this article. With the intention of evaluating the capabilities of this method, without the support of any other integration tool, we will apply it to every 3-dimensional QH of degree 2, a wide set of systems whose canonical forms were obtained in the previous section. Note that in this paper and in dimension 3 a system is considered completely integrable when there are two functionally independent analytic first integrals, and from Proposition 2 this is equivalent to the existence of two functionally independent YFIs.

Theorem 17. *The differential systems corresponding to the canonical form* S_i *are completely integrable, and respectively have the functionally independent YFIs* H_i *and* G_i *indicated below, for* $i = 3, 4, 8, 10, 18$*.*

(a) $H_3(x, y, z) = y - z$
$G_3(x, y, z) = 3x^2 - (1 + 2p)\, y^3 + 3y^2 z - 3(1 + q)\, yz^2 + (q - 1)\, z^3$
(b) $H_4(x, y, z) = x - py$
$G_4(x, y, z) = 3x^2 + 3py^2 - 2pz^3$
(c) $H_8(x, y, z) = 2x - pz^2$
$G_8(x, y, z) = 6xz - 3y^2 + (2 - 2p)\, z^3$
(d) $H_{10}(x, y, z) = z$
$G_{10}(x, y, z) = 3x^2 + 6pxz^2 - 2y^3$
(e) $H_{18}(x, y, z) = 2y - z^2$
$G_{18}(x, y, z) = 15x^2 - 30y^2 z + 20yz^3 - 4z^5$

Proof. (a) The only balance of the S_3 normal form is

$$\mathbf{c} = (-12, 6, 6) \,/\, (1 + p + q)\,,$$

and its corresponding Kowalevskaya matrix is

$$K(\mathbf{c}) = \begin{pmatrix} 3 & \dfrac{12p + 6q}{p + q + 1} & \dfrac{12 + 6q}{p + q + 1} \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix},$$

which provides the non-trivial Kowalevskaya exponents $\lambda_1 = 2$, $\lambda_2 = 6$. Note that when $p + q = -1$ there is no balance, and therefore no possibility of applying the method.

Following Theorem 9, and taking into account that $d = 2$, these exponents could provide YFIs of weight degrees, respectively, $k_1 = 2$ and $k_2 = 6$. We will now work with the first of them. We denote by H_3 a possible YFI of S_3 with weight degree $k_1 = 2$. In this case both H_3 and its monomials must verify (5) for $\mathbf{s} = (3, 2, 2)$. That is, if $Ax^a y^b z^c$ is a monomial of H_3, with $A \in \mathbb{R}$ and $a, b, c \in \mathbb{N}$, it must be verified that

$$A\alpha^{3a+2b+2c} x^a y^b z^c = \alpha^2 A x^a y^b z^c,$$

for any $\alpha \in \mathbb{R}^+$. Then the solutions in the domain of the natural numbers of the diophantine equation $3a + 2b + 2c = 2$ will provide the exponents of the potential monomials of H_3. The only solutions are $(0,1,0)$ and $(0,0,1)$, so to be quasi-homogeneous with weight vector $(3,2,2,2)$, the polynomial must be of the form $H_3(x,y,z) = Ay + Bz$, being A and B real coefficients. Furthermore H_3 must meet (4) to be a first integral, this implies that $x(A+B) \equiv 0$ for all $x \in \mathbb{R}$. Taking $A = 1$, $B = -1$, we obtain the YFI

$$H_3(x,y,z) = y - z.$$

Now let G_3 be a possible YFI of S_3 with weight degree $k_2 = 6$. If $Ax^a y^b z^c$ is a monomial of G_3, it must verify $A\alpha^{3a+2b+2c}x^a y^b z^c = \alpha^6 A x^a y^b z^c$ for any $\alpha \in \mathbb{R}^+$, so we need the natural solutions of $3a + 2b + 2c = 6$. These are $(2,0,0)$, $(0,3,0)$, $(0,2,1)$, $(0,1,2)$ and $(0,0,3)$, therefore the polynomial must be of the form $G_3(x,y,z) = A^2 + By^3 + Cy^2 z + Dyz^2 + Ez^3$, being A, B, C, D and E real coefficients. It is easy to prove that for G_3 to also meet (4), its coefficients must verify the following system of equations:

$$2A + D + 3E = 0, \quad 2qA + 2C + 2D = 0, \quad 2pA + 3B + C = 0.$$

The solution $(A,B,C,D,E) = (3, -1-2p, 3, -3-3q, q-1)$ provides the YFI

$$G_3(x,y,z) = 3x^2 - (1+2p)\,y^3 + 3y^2 z - 3(1+q)\,yz^2 + (q-1)\,z^3.$$

Regarding the case $p + q = -1$, where as we have pointed out, the method cannot be applied, we observe that the two YFIs obtained are also valid in this case.

Finally the YFIs H_3 and G_3 are functionally independent, because the matrix

$$\begin{pmatrix} \nabla H_3(x,y,z) \\ \nabla G_3(x,y,z) \end{pmatrix} = \begin{pmatrix} 0 & 6x \\ 1 & -3(1+2p)\,y^2 + 6yz - 3(1+q)\,z^2 \\ -1 & 3y^2 - 6(1+q)\,yz + 3(q-1)\,z^2 \end{pmatrix}^t$$

has rank 2 at all points except at most for the null measure set $x = 0$.

To obtain the YFIs of the rest of the normal forms, the same procedure must be followed. The main data in the calculations of each case are the following:

(b) The balance is $\mathbf{c} = \left(-12p/(1+p)^2, -12/(1+p)^2, 6/(1+p)\right)$, which provides the non-trivial Kowalevskaya exponents $\lambda_1 = 3$ and $\lambda_2 = 6$ and, being $d = 2$, the same possible weight degrees for the YFIs. With the weight degree 3 we have H_4 and with 6 we obtain G_4, both being functionally independent. When $p = -1$ the system given has no balances, but the two YFIs achieved for $p \neq -1$ are also valid.

(c) The balance is $\mathbf{c} = \left(72p/(2+p)^2, -24/(2+p), 12/(2+p)\right)$, which provides the non-trivial Kowalevskaya exponents $\lambda_1 = 4$ and $\lambda_2 = 6$ and, being $d = 2$, the same possible weight degrees for the YFIs. With the weight degree 4 we have H_8 and with 6 we obtain G_8, both being functionally independent. When $p = 2$ the system has no balances, but the two YFIs achieved for $p \neq -2$ are also valid.

(d) Since the weight degree of this normal form is $d = 3$, two equivalent balances could be expected for each equivalence class (Proposition 6). However, because the third component is canceled, both coincide at $\mathbf{c} = (-12, 6, 0)$. The Kowalevskaya exponents are $\lambda_1 = 3/2$ and $\lambda_2 = 6$, and therefore the potential weight degrees for the YFIs are 3 and 12. From the first we obtain H_{10} and from the second G_{10}, both functionally independent.

(e) Now $d = 4$ and three equivalent complex balances appear:

$$\mathbf{c_1} = \left(\frac{-4\gamma}{\delta^5}, \frac{2\gamma^2}{\delta^4}, \frac{2\gamma}{\delta^2} \right), \ \mathbf{c_2} = \left(\frac{4\alpha}{\delta^5}, \frac{2\alpha^2}{\delta^4}, \frac{-2\alpha}{\delta^2} \right),$$

$$\mathbf{c_3} = \left(\frac{-4\beta^2\gamma}{\delta^5}, \frac{-2\beta\gamma^2}{\delta^4}, \frac{2\beta^2\gamma}{\delta^2} \right),$$

being $\alpha = \sqrt[3]{-5}$, $\beta = \sqrt[3]{-1}$, $\gamma = \sqrt[3]{5}$ and $\delta = \sqrt[3]{3}$. The common Kowalevskaya exponents are $\lambda_1 = 4/3$ and $\lambda_2 = 10/3$, so the possible weight degrees for the YFIs are 4 and 10. From the first we obtain H_{18} and from the second G_{18}, both functionally independent. □

In all cases of Theorem 17 the Yoshida method provides the maximum level of information. All Kowalevskaya exponents give rise to a YFI, and therefore there can be no "hidden" analytic first integrals that are functionally independent of the rest. Based on Theorem 9 it is evident that for this to happen, the gradients of these YFIs cannot be annulled on the balances. This is a matter that, as a mere confirmation we can verify now

$$\nabla H_{18}(\mathbf{c_1}) = \left(0, 2, -\frac{4\sqrt[3]{5}}{\left(\sqrt[3]{3}\right)^2} \right) \neq \mathbf{0}.$$

Theorem 18. *The differential systems corresponding to the canonical form* S_i *are not completely integrable for* $i = 5, 12, 13, 17$.

Proof. The only balance of the S_5 normal form is $\mathbf{c} = (-720, 180, -60)$, and its corresponding Kowalevskaya matrix is

$$K(\mathbf{c}) = \begin{pmatrix} 5 & 0 & -120 \\ 1 & 4 & 0 \\ 0 & 1 & 3 \end{pmatrix},$$

which provides the non-trivial Kowalevskaya exponents $\lambda_{1,2} = \dfrac{13 \pm \sqrt{71}i}{2} \notin \mathbb{Q}$. The proof for this normal form concludes by applying Theorem 11.

The proof is identical for the normal forms S_{12}, S_{13} and S_{17}, which have non-rational Kowalevskaya exponents, respectively, $\dfrac{7 \pm \sqrt{11}i}{2}$, $\dfrac{19 \pm \sqrt{199}i}{6}$ and $\dfrac{5 \pm \sqrt{13}}{2}$. □

We are mainly interested in determining if there is complete integrability or not, because only in this case all the trajectories of the system can be controlled. However we can ask ourselves if any of these four normal forms is partially integrable. That is, if they have one and only one functionally independent YFI. Obviously, if it exist it must be "hidden" for the fact of not fulfilling the gradient condition. We are going to study, as an example, the normal form S_5: first, we exclude using Corollary 14 to prove the nonexistence of analytic first integrals, because their conditions are not met $(\lambda_1 + \lambda_2 - 13 = 0)$. Therefore without demanding the gradient condition, we are left with Theorem 13: if there is a YFI of weight degree $k \in \mathbb{Z}^+$, there must be α_1, $\alpha_2 \in \mathbb{N}$ such that

$$\alpha_1 \left(\frac{13 + \sqrt{71}i}{2} \right) + \alpha_2 \left(\frac{13 - \sqrt{71}i}{2} \right) = \frac{k}{2-1}, \qquad 0 < \alpha_1 + \alpha_2 \leq k.$$

Then $\sqrt{71}i\,(\alpha_1 - \alpha_2) = 2k - 13\,(\alpha_1 + \alpha_2) \in \mathbb{Z}$, and as a consequence $\alpha_1 = \alpha_2 = \alpha \in \mathbb{Z}^+$. We conclude that $k = 13\alpha$, and this means that the only possible weight degrees k would be the multiples of 13.

Following a procedure of construction of first integrals identical to that carried out in the proof of Theorem 17, and with the help of software of computational algebra, we have verified that there are no YFI of weight degree less than or equal to 104 for this normal form.

We carried out the same study with the forms S_{12}, S_{13} and S_{17} and it is proved that, if they are partially integrable, the YFI's weight degrees would be multiples of 14, 19 and 10, respectively. And based on this, also as in the previous case we have exclude the existence of first integrals up to 112, 114 and 100 weight degrees, respectively.

In the same way as in the normal forms of Theorem 17, we can affirm that the Yoshida method provides a powerful answer in the cases covered by the previous Theorem: there is no complete integrability. However the rest of the normal forms present difficulties of various kinds for the method, which we classify below.

4.0.1. *Problematic cases.*

i) Not similarity invariant systems (d=1). When all the weight degrees of a QH system verify $d = 1$, it is not a similarity invariant system. In this case the vector $\mathbf{g}$ cannot be defined, and therefore none of the results of the

Yoshida method can be applied. We are in this situation with the normal forms S_1, S_2, S_6 and S_7.

ii) There are no balances. When the system (6) only has the trivial solution, there are no balances, and therefore there is no possibility of applying the method either. Within our family of normal forms, this situation occurs in S_{20}, whose system (6) is as follows and only has the solution $\mathbf{0}$:

$$\frac{4x}{3} = 0, \qquad \frac{2y}{3} + xz = 0, \qquad x + y^2 + \frac{z}{3} = 0.$$

It is noteworthy that this normal form has a very evident YFI of weight degree 4, $H_{20}(x, y, z) = x$. Furthermore, the YFI $G_{20}(x, y, z) = 6xy - 3xz^2 + 2y^3$, of weight degree 6, can be easily found. As both first integrals are functionally independent, this normal form is completely integrable. This shows that the Yoshida method can even "fail" in very trivial cases.

iii) Not all Kovalevskaya exponents provide a YFI. The normal form S_{11} has a unique balance $\mathbf{c} = (-144/5, -12, 6)$, which in turn provides the non-trivial Kowalevskaya exponents $\lambda_1 = 5$ and $\lambda_2 = 6$. From the second of them we obtain, using Theorem 9, the YFI $H_{11}(x, y, z) = 3y^2 - 2z^3$. However the Kowalevskaya exponent $\lambda_1 = 5$ does not provide any first integral, which does not mean that there are no more. In fact, using alternative integration methods to that of Yoshida, we know that it exists, because the divergence of S_{11} is null and we can apply Corollary 6 of [13]. It is evident that, if analytical, this hidden first integral G_{11} would verify $\nabla G_{11}(\mathbf{c}) = \mathbf{0}$. However G_{11} has been searched using software running all its possible weight degrees up to 20, and the only YFI found are the powers of H_{11}. We conjecture that G_{11} is not analytical.

iv) Kowalevskaya exponents depend on parameters. The Yoshida method does not seem efficient to study the integrability of these kind of normal forms, which moreover coincide in our study group with those in which there are degree 2 monomials in the three components $\dot{x}$, $\dot{y}$, $\dot{z}$. These are cases S_9, S_{14}, S_{15}, S_{16} and S_{19}. The first problem with these normal forms arises with balances. In all five cases, very complicated parameter-dependent balances appear, obviously obtained through the use of software. As they are made up of too many lines each, we cannot reproduce them here. However in all forms except S_9 there are also simple balances that allow to work with them. In summary:

- Form S_9 has five unrepeatable balances.
- Form S_{14} has three balances. Two of them are unrepeatable and a third is $\mathbf{c}_{14} = (0, 0, -1)$.
- Form S_{15} has three balances. Two of them are unrepeatable and a third is $\mathbf{c}_{15} = (0, 0, -1/q)$.
- Form S_{16} has six balances, although they are really reduced to three, because the weight degree is $d = 3$ (see Propositions 6 and 8).

Of these three, one is unrepeatable and the other two are $\mathbf{c}_{16}^1 = (0, 0, -1/r)$ and $\mathbf{c}_{16}^2 = (2 - 4r, 0, -2)$.

- Form S_{19} has four balances, although they are really reduced to two, because the weight degree of the form is $d = 3$. Of these one is unrepeatable and the other is $\mathbf{c}_{19} = (0, -1/q, 0)$.

In any case a treatable Kowalevskaya exponent is obtained from one of those unrepeatable balances (except trivial -1, which always appears), reaching up to 50 pages in some cases. From the simple balances, existing in the forms S_{14}, S_{15}, S_{16} and S_{19}, simple exponents do emerge, although all of them with the common characteristic of being dependent on the parameters of the normal form. It is noteworthy that when there are parameters of type ± 1 (S_{16} and S_{19} forms), the choice made does not affect the Kowalevskaya exponents that are obtained. Summarizing:

- Form S_{14} has the non-trivial Kowalevskaya exponents, obtained from $\mathbf{c}_{14}$,
$$\lambda_{1,2} = \left(4 - p - s \pm \sqrt{p + 4qr - 2ps + s^2}\right)/2.$$
- Form S_{15} has the non-trivial Kowalevskaya exponents, obtained from $\mathbf{c}_{15}$,
$$\lambda_1 = 3 - {}^1\!/_q, \quad \lambda_2 = 2 - {}^p\!/_q.$$
- Form S_{16} has two balances. From $\mathbf{c}_{16}^1$ are obtained
$$\lambda_1 = 2 - {}^1\!/_r, \quad \lambda_2 = {}^3\!/_2 - {}^q\!/_r$$
and from $\mathbf{c}_{16}^2$ we have
$$\lambda_1 = {}^3\!/_2 - 2q, \quad \lambda_2 = 2 - 4r.$$
- Form S_{19} has the non-trivial Kowalevskaya exponents, obtained from $\mathbf{c}_{19}$,
$$\lambda_1 = {}^1\!/_2 - {}^1\!/_q, \quad \lambda_2 = {}^3\!/_2 - {}^p\!/_q.$$

In general the information that we can extract from these exponents is very scarce. Based on Theorem 11 we can establish the most relevant result, which is applied to three of the normal forms:

Proposition 19. *The differential systems corresponding to the canonical forms* S_{15}*,* S_{16} *and* S_{19} *are not completely integrable when any of their parameters is irrational.*

But the rest of the theory provides little more: with Theorems 9 and 10 we can establish conditions for potential weight degrees, but the "hidden by the gradient" YFIs will always be left out. For example it is trivial that a necessary condition for the normal form S_{15} to have two not hidden YFIs is that its parameters p, q verify
$$(p, q) \in \{({}^m\!/_n, {}^{-1}\!/_n) \mid m, n \in \mathbb{Z} \setminus \{0\}\,;\ -1 \le m;\ -2 \le n\}\,.$$

Similar conditions can be obtained for S_{16} and S_{19}. Another requirement, which only adds information to the above if there are non-rational parameters, can be extracted from Theorem 13:

Proposition 20. *A necessary condition so that the differential systems corresponding to the canonical form* S_{15} *(respectively* S_{16}*, respectively* S_{19}*) can have some analytic first integral is that* q *be an affine function of* p*,* $q = Ap + B$*, being* A *and* B *rationals (respectively* r *of* q*, respectively* q *of* p*).*

Proof. To be an analytic first integral of S_{15} there must be a YFI of a certain weight degree $k \in \mathbb{Z}^+$. In such a case there must exist $\alpha_1, \alpha_2 \in \mathbb{N}$ verifying $0 < \alpha_1 + \alpha_2 \leq k$ and $\alpha_1 (3 - 1/q) + \alpha_2 (2 - p/q) = k$. This is equivalent to have $q = Ap + B$ with $A = \dfrac{\alpha_2}{3\alpha_1 + 2\alpha_2 - k}$, $B = \dfrac{\alpha_1}{3\alpha_1 + 2\alpha_2 - k}$. The cases S_{16} and S_{19} are identical. □

We can know little about the normal forms S_9 and S_{14}. If we disregard the method and test with possible low weight degrees, we observe for example that S_9 has a YFI of weight degree 1 if and only if the components $\dot{y}$ and $\dot{z}$ are proportional, i.e., if $s/p = u/q = v/r = w$. This first integral is $wy - z$. Some other partial results can be obtained.

In summary, the problem with these normal forms in order to apply the method is that their Kowalevskaya exponents depend on the parameters of the form, in addition to being, in many cases, intractable due to their complexity, even using *Mathematica* software. In those cases the only knowledge that the Yoshida method detects are some situations of non-integrability, along with particular results that have little value and we do not reproduce.

4.0.2. *Conclusions.* Yoshida method has traditionally been undervalued as mere theory with no practical application. This is because, in principle, it does not provide information about the shape of the searched first integrals, but only about their weight degrees. Furthermore, the main result (Theorem 9) presents the mentioned problem of “hidden” YFIs. Thus, until recently, almost all articles devoted to the topic were purely theoretical works, which did not carry out the effective calculation of first integrals. But closer in time, works like [13] do contribute with practical applications. A technique is developed that, based on the knowledge of the weight exponents that all YFI must have, is capable of constructing the first integral through the resolution of diophantine equations. However, when this technique is used, we need some clarification of the theory which gives it a formal structure and which proves some results that are being used as true. We hope to have achieved that clarification in subsection 2.1, mainly contributing with the concept of Yoshida First Integal (YFI) and Proposition 2, which equates the number of analytic integrals of interest with that of YFIs.

From these considerations it is necessary to evaluate the degree of utility of the method as a tool for the integration of quasi-homogeneous systems. In our study we have obtained conclusive results for 9 of the 20 analyzed forms. For this reason, we can conclude that the Yoshida method is useful in the practical study of analytical integrability, but that, like any other tool dedicated to this complex task, it has notable limitations that make it necessary to complement it with other techniques.

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