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# Polynomial Liénard systems with a nilpotent global center

**Isaac A. García**[1] · **Jaume Llibre**[1]

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## Abstract

A center for a differential system $\dot{\mathbf{x}} = f(\mathbf{x})$ in $\mathbb{R}^2$ is a singular point $p$ having a neighborhood $U$ such that $U \setminus \{p\}$ is filled with periodic orbits. A global center is a center $p$ such that $\mathbb{R}^2 \setminus \{p\}$ is filled with periodic orbits. There are three kinds of centers, the centers $p$ such that the Jacobian matrix $Df(p)$ has purely imaginary eigenvalues, the nilpotent centers $p$ such that $Df(p)$ is a nilpotent matrix, and the degenerate centers $p$ such that the matrix $Df(p)$ is the zero matrix. For the first class of centers there are several works studying when such centers are global. As far as we know there are no works for studying the nilpotent global centers. One of the most studied classes of differential systems in $\mathbb{R}^2$ are the polynomial Liénard differential systems. The objective of this paper is to study the nilpotent global centers of the polynomial Liénard differential systems.


**Keywords** Center · Global center · Periodic orbits · Nilpotent singularity

**Mathematics Subject Classification** 34C05

## 1 Introduction and statement of the main results

The study of the global centers of the polynomial differential systems was initiated by Conti and their collaborators, see [4, 5, 7]. There were some previous articles on the global centers before the work of Conti but without this name and for differential equations which were not polynomial. It is known that the polynomial differential systems with a global center have odd degree, see [7, 11]. Also it is known that the unique polynomial differential systems which have a rigid global center are the linear differential centers, see [4]. A center $p$ is *rigid*

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✉ Isaac A. García
isaac.garcia@udl.cat

Jaume Llibre
jllibre@mat.uab.cat

1 Departament de Matemàtica, Universitat de Lleida, Avda. Jaume II, 69, 25001 Lleida, Spain

if in polar coordinates with origin at $p$ has constant angular velocity. For other results on global centers for polynomial differential systems see [8–10, 16].

We recall that Takens proved in [14] that an analytic differential system with a nilpotent singular point at the origin of coordinates can be formally transformed into a Liénard differential system of the form

$$\dot{x} = -y\ ,\ \ \dot{y} = a(x) + y\tilde{b}(x), \tag{1}$$

with $a, \tilde{b} \in \mathbb{R}[[x]]$ formal power series such that $a(x) = a_s x^s(1 + \mathcal{O}(x))$ with $s \geq 2$ and $\tilde{b}(0) = 0$. Later on Strózyna and Zoladek shown in [13] that the formal change can be choosen convergent.

In the monodromic case, i.e. when the orbits in a neighborhood of the origin rotates around it, the exponent $s = 2n - 1$ with $n \geq 2$, and after the change $x \mapsto u$ with

$$u(x) = \left(2n \int_0^x a(z)dz\right)^{1/(2n)} = x(a_{2n-1} + \mathcal{O}(x))^{1/(2n)},$$

and the time rescaling $t \mapsto \tau$ with

$$\frac{dt}{d\tau} = \frac{u^{2n-1}}{a(x)} = a_{2n-1}^{-1/(2n)} + \mathcal{O}(x),$$

the Liénard differential system (1) becomes the analytic differential system

$$\dot{x} = -y\ ,\ \ \dot{y} = x^{2n-1} + yb(x)\ , \tag{2}$$

where $b(x) = \sum_{j\geq\beta} b_j x^j$ and $b_\beta \neq 0$.

Since the momodromy is preserved under orbital analytic conjugation system (2) satisfies one of the following three conditions:

(i) $\beta \geq n$,
(ii) $\beta = n - 1$ and $b_\beta^2 - 4n < 0$, and
(iii) $b(x) \equiv 0$.

These conditions follow easily from the Andreev characterization of those analytic systems having a monodromic nilpotent singular point, see [1] or Theorem 3.5 of [6].

For the differential systems (2) it is not difficult to characterize their centers. The origin of system (2) is a center if and only if $b(x)$ is an odd function, see [3, 12, 15]. Strózyna and Żołądek also prove in [13] that system (2) has a local analytic first integral if and only if $b(x) \equiv 0$.

Notice that the only finite singularity of system (2) is the origin of coordinates, i.e. (0, 0). Since we only analyze polynomial differential systems (2) having a center at the origin of coordinates the polynomial $b(x)$ of degree $m$ must be odd, that is

$$b(x) = \sum_{j=\beta}^{m} b_j x^j, \qquad b(-x) = -b(x), \tag{3}$$

with $m \geq \beta$. In particular, both $m$ and $\beta$ are odd positive integers. When $b(x) \not\equiv 0$, we define the odd integer

$$\alpha = 2n - 2 - m \in \mathbb{Z}\backslash\{0\}. \tag{4}$$

Our first result is the following.

**Theorem 1** *The polynomial differential system (2) satisfying (3) has a center at the origin. Then the following statements hold.*

(i) *The center is not global if* $\alpha < 0$.
(ii) *The center is global if either* $b(x) \equiv 0$*, or* $\beta = n - 1 = m$ *with* $b_\beta^2 - 4n < 0$ *and* $\alpha > 0$.

The next result gives a partial answer on the global center problem for the differential system (2) satisfying (3) in the cases not covered by Theorem 1 for the first possible values of $\alpha$ and the Andreev number $n$. We only analyze the first values of $\alpha$ and $n$ due to the huge computations involved.

**Theorem 2** *The polynomial differential system (2) satisfying (3) has a center at the origin. Assume that* $\alpha > 0$.

(i) *If* $\beta \geq n$*, then the following statements hold.*

(i.1) *If* $\alpha = 1$*, then* $n \geq 3$ *and there is no global center if* $n \in \{3, 4\}$.
(i.2) *If* $\alpha = 3$*, then* $n \geq 5$ *and there is no global center if* $n = 5$.

(ii) *If* $\beta = n - 1$ *with* $b_\beta^2 - 4n < 0$*, then* $n - 1 \leq m \leq 2n - 3$*,* $n$ *is even and the following statements hold.*

(ii.1) *If* $n = 2$ *the center is global.*
(ii.2) *If* $n = 4$ *then* $m \in \{3, 5\}$*. When* $m = 5$ *there is no global center.*

The proofs of Theorems 1 and 2 are given in section 2.

From the results of Theorems 1 and 2 we believe that the following conjecture must be true.

**Conjecture**. *The polynomial differential system* (2) *satisfying*(3) *has a center at the origin. Then this center is global if and only if either* $b(x) \equiv 0$*, or* $\beta = n - 1 = m$ *with* $b_\beta^2 - 4n < 0$.

Note that the "only if" part of the conjecture is proved in statement (ii) of Theorem 1.

In other words, if the conjecture holds then the origin of the polynomial system (2) is a global center if and only if the system is $(1, n)$-quasi-homogeneous of degree $n - 1$, that is, invariant under the transformation

$$(x, y, t) \mapsto (\lambda x, \lambda^n y, \lambda^{1-n} t).$$

## 2 The proofs

The first result corresponds with the trivial cases.

**Proposition 3** *The polynomial differential system (2) satisfying (3) has a center at the origin. In the Hamiltonian case, i.e.* $b(x) \equiv 0$*, the center is global. In the non-Hamiltonian case with* $\alpha < 0$ *the center is not global.*

***Proof*** The case $b(x) \equiv 0$ corresponds with a nilpotent center because it is Hamiltonian with the first integral $H(x, y) = \frac{1}{2}y^2 + \frac{1}{2n}x^{2n}$, whose level curves foliates with ovals $\mathbb{R}^2 \backslash \{(0, 0)\}$. Hence the origin is a global center.

If $b(x) \not\equiv 0$ and $\alpha = 2n - 2 - m < 0$, then the degree $d$ of the polynomial differential system (2) is $d = \max\{2n - 1, m + 1\} = m + 1$, an even number. Therefore, from the results of [7] and [11] there are no global centers, because the differential system always has some orbit that goes or comes from the infinity. □

The next result deals with the $(1, n)$-quasihomogeneous case.

**Proposition 4** *The origin of the polynomial differential system (2) satisfying (3) is a global center if* $\beta = n - 1 = m$ *with* $b_\beta^2 - 4n < 0$ *and* $\alpha > 0$.

***Proof*** Under the conditions of the proposition system (2) becomes

$$\dot{x} = p_n(x, y) = -y, \quad \dot{y} = q_{2n-1}(x, y) = x^{2n-1} + b_{n-1}x^{n-1}y,$$

where $p_n$ and $q_{2n-1}$ are $(1, n)$-quasihomogeneous of degrees $n$ and $2n - 1$, respectively. Then system (2) is $(1, n)$-quasihomogeneous of degree $n - 1$. These differential systems are invariant under the similarity transformation

$$(x, y, t) \mapsto (\lambda x, \lambda^n y, \lambda^{1-n} t)$$

for all $\lambda \in \mathbb{R}$. In other words, the foliation generated by $p_n(x, y)dy - q_{2n-1}(x, y)dx = 0$ is invariant under the dilation $(x, y) \mapsto (\lambda x, \lambda^n y)$ and consequently the center needs to be global. □

Any polynomial differential system can be extended analytically to the Poincaré disc with the circle of the infinity invariant by this extended flow. This extension is called the Poincaré compactification, see the details Chapter 5 of [6] for example. Using their notation we see that we need to analyze the infinite singular points of the local chart $U_1$ as well as the origin of the local chart $U_2$. Moreover, if we do the extension of the polynomial differential system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ of degree $d = \max\{\deg P, \deg Q\}$ then we need to study the nature of the following singularities:

(a) In the local chart $U_1$ the singular points on $v = 0$ of the polynomial differential system $\dot{u} = \hat{P}(u, v)$, $\dot{v} = \hat{Q}(u, v)$ obtained after the change to local coordinates in $U_1$ given by $(x, y) \mapsto (u, v)$ with $x = 1/v$ and $y = u/v$.
(b) In the local chart $U_2$ the singularity at the origin of the polynomial differential system $\dot{u} = \bar{P}(u, v)$, $\dot{v} = \bar{Q}(u, v)$ obtained via the change to local coordinates in $U_2$ expressed as $(x, y) \mapsto (u, v)$ with $x = u/v$ and $y = 1/v$.

The next result shows that the analysis of the infinite singular points of the polynomial differential system (2) with $\alpha > 0$ are restricted to the local chart $U_2$.

**Proposition 5** *The polynomial differential system (2) with* $\alpha > 0$ *has no singularities at infinity in the local chart* $U_1$.

***Proof*** In the local chart $U_1$ we perform the change $(x, y) \mapsto (u, v)$ with $x = 1/v$ and $y = u/v$, and we get that system (2) is transformed into

$$\dot{u} = 1 + u^2 v^{2n-2} + u\hat{b}(v), \quad \dot{v} = uv^{2n-1}, \tag{5}$$

where $\hat{b}(v) = v^{2n-2} b(1/v) \in \mathbb{R}[v]$ since $m < 2n - 2$ because $\alpha > 0$. Notice that $\hat{b}$ is odd because $b$ is also odd, in particular $\hat{b}(0) = 0$. This means that system (5) has no singular points on the line $v = 0$, that is, at the infinity of $U_1$. □

We recall that a polynomial differential system with a unique finite singularity (which is a center) and without a line of singular points at infinity, has a global center if and only if all its eventual infinite singular points have local phase portrait formed by two hyperbolic sectors with both separatrices on the invariant infinite circle.

The proof of the forthcoming Propositions 7 and 12 is based in the following desingularization algorithm described first in [2]. The method is different from the classical blow-up technique of blowing-up in two different directions, and gluing the two blow-ups. It consists in adding a linear change previous to the blow-up to avoid making the blow-up in an already characteristic direction. In this way all the blow-ups can be done always in the same direction creating a more methodical way of work.

First we take the coordinates $(u, v)$ of the local chart $U_2$ defined via the change $(x, y) \mapsto (u, v)$ with $x = u/v$ and $y = 1/v$. We know that the only singularity on the line $v = 0$ of the resulting system is the origin, which is linearly zero. Next, since $u = 0$ is not a characteristic direction at the origin, we perform a vertical blow up $(u, v) \mapsto (u_1, v_1) = (u, v/u)$ together with a time-rescaling dividing the vector field by $u_1^{2n-3}$ to obtain the polynomial differential system $\dot u_1 = P_1(u_1, v_1)$, $\dot v_1 = Q_1(u_1, v_1)$ with coprime components. We analyze the singularities of this differential system on the line $u_1 = 0$ as follows:

(i) If there is one linearly zero singularity then we translate it to the origin. Next we check that both $u_1 = 0$ and $v_1 = 0$ are characteristic directions at the origin and then, before doing a vertical blow-up, we translate the direction $u_1 = 0$ to the direction $u_1 = v_1$ doing the change of variables $(u_1, v_1) \mapsto (u_2, v_2) = (u_1 + v_1, v_1)$. After checking that the negative bisectrix is not also a characteristic direction (otherwise one must do a different twist as it is described in [2]) we continue the blow-up process. We do the vertical blow up $(u_2, v_2) \mapsto (u_3, v_3) = (u_2, v_2/u_2)$, and after we rescale the time dividing the vector field by a convenient power $u_3^\gamma$ with $\gamma \in \mathbb{N}$ in such a way that the obtained polynomial differential system $\dot u_3 = P_3(u_3, v_3)$, $\dot v_3 = Q_3(u_3, v_3)$ has coprime components.

(ii) We repeat the former steps transforming the system $\dot u_i = P_i(u_i, v_i)$, $\dot v_i = Q_i(u_i, v_i)$ into $\dot u_{i+j} = P_{i+j}(u_{i+j}, v_{i+j})$, $\dot v_{i+j} = Q_{i+j}(u_{i+j}, v_{i+j})$ for some $j \in \mathbb{N}$ until we reach that all the singularities on $u_{i+j} = 0$ are hyperbolic or semi-hyperbolic. In case that one parabolic sector appears in this process then the origin of system (2) cannot be a global center. The local phase portrait at the hyperbolic and semihyperbolic singular points are classified, see for instance Theorems 2.15 and 2.19 of [6], respectively.

***Remark 6*** There is another possibility for the desingularization which consists that along the different blow-up, it could appear a singularity with an infinite number of characteristic directions. This possibility is also nicely described in [2] and called star-like singularities. In the cases analyzed in our study this possibility never occurs.

**Proposition 7** *The polynomial differential system (2) satisfying (3) has a center at the origin. Assume that $b(x) \not\equiv 0$, $\beta \geq n$ and $\alpha > 0$.*

(i) *When $\alpha = 1$ then $n \geq 3$ and there is no global center if $n \in \{3, 4\}$.*
(ii) *When $\alpha = 3$ then $n \geq 5$ and there is no global center if $n = 5$.*

***Proof*** The degree of system (2) is $d = 2n-1$. Moreover all the restrictions can be summarized as

$$n > 2, \quad n \leq \beta \leq m < 2n - 2. \tag{6}$$

In the local chart $U_2$ we do the change $(x, y) \mapsto (u, v)$ with $x = u/v$ and $y = 1/v$ so system (2) becomes

$$\dot u = -u^{2n} - v^{2n-2} - ub^\dagger(u, v), \quad \dot v = -u^{2n-1}v - vb^\dagger(u, v), \tag{7}$$

being $b^\dagger(u,v) = v^{2n-2} b(u/v)$. We have that

$$b(u/v) = \sum_{j=\beta}^{m} b_j (u/v)^j = u^\beta v^{-m} P_{m-\beta}(u,v),$$

where $P_{m-\beta}$ is a homogeneous polynomial of even degree $m-\beta$ such that $P_{m-\beta}(u,0) = b_m u^{m-\beta}$ and $P_{m-\beta}(0,v) = b_\beta v^{m-\beta}$. Then taking into account that $m < 2n-2$, from (6) it follows that $b^\dagger(u,v)$ is a homogeneous polynomial of degree $2n-2$ with $b^\dagger(u,0) = 0$. This implies that the only singularity of (7) on the line $v = 0$ is the origin.

Since (7) adopts the form $\dot{u} = -v^{2n-2} + \cdots$, $\dot{v} = \cdots$ where the dots mean terms of higher degree than $2n-2$, the origin is a linearly zero singular point and so we need to do a blow up. Since $u = 0$ is not a characteristic direction of the origin of system (7), because the unique characteristic direction is $v = 0$, we perform a vertical blow up: $(u,v) \mapsto (u_1, v_1) = (u, v/u)$. In these new coordinates system (7) becomes

$$\dot{u}_1 = -u_1^{2n} - u_1^{2n-2} v_1^{2n-2} - u_1^{2n-1} b^*(v_1), \quad \dot{v}_1 = u_1^{2n-3} v_1^{2n-1}, \tag{8}$$

where

$$b^*(v_1) = v_1^{2n-2} b(1/v_1) = v_1^\alpha Q(v_1) \in \mathbb{R}[v_1] \tag{9}$$

and $Q$ is an even polynomial with $Q(0) = b_m$ so that $b^*(0) = 0$. We reescale the time dividing the differential system by $u_1^{2n-3}$ and we obtain the system

$$\dot{u}_1 = P_1(u_1,v_1) = -u_1^3 - u_1 v_1^{2n-2} - u_1^2 b^*(v_1), \quad \dot{v}_1 = Q_1(u_1,v_1) = v_1^{2n-1}, \tag{10}$$

where the origin is the unique singularity of (10) on the line $u_1 = 0$, and the origin is still a linearly zero point. So one more blow-up is necessary.

Both $u_1 = 0$ and $v_1 = 0$ are characteristic directions at the origin of system (10). Therefore before doing a vertical blow-up, we translate the direction $u_1 = 0$ to the direction $u_1 = v_1$ doing the change of variables $(u_1, v_1) \mapsto (u_2, v_2) = (u_1 + v_1, v_1)$. System (10) is transformed into

$$\dot{u}_2 = v_2^{2n-2}(2v_2 - u_2) - (u_2 - v_2)^2 (u_2 - v_2 + b^*(v_2)), \quad \dot{v}_2 = v_2^{2n-1}. \tag{11}$$

Again the origin is the unique singular point of system (11) on the straight line $u_2 = 0$, and it is linearly zero. Since $u_2 = 0$ is not a characteristic direction at the origin we do the vertical blow up $(u_2, v_2) \mapsto (u_3, v_3) = (u_2, v_2/u_2)$ and rescale the time dividing the vector field by $u_3^2$. System (11) becomes

$$\begin{aligned}
\dot{u}_3 &= P_3(u_3,v_3) = u_3(-1+v_3)^3 + u_3^{2n-3} v_3^{2n-2}(-1+2v_3) \\
&\quad -(-1+v_3)^2 b^*(u_3 v_3), \\
\dot{v}_3 &= Q_3(u_3,v_3) = -(-1+v_3)^3 v_3 + u_3^{2n-4} v_3^{2n-1}(1-v_3) \\
&\quad [1 + u_3^{2n-3}(1-v_3)^{2n-3}] + (1-v_3)^2 v_3 b^*(u_3 v_3)/u_3.
\end{aligned} \tag{12}$$

Notice that system (12) is a polynomial differential system and $\dot{u}_3|_{\{u_3=0\}} = 0$, because $b^*(0) = 0$. We recall that $b^*(u_3 v_3)/u_3 = u_3^{\alpha-1} v_3^\alpha (b_m + \cdots)$ with $\alpha \geq 1$. Then the number of singular points of system (12) on the straight line $u_3 = 0$ depends on the values of the odd number $\alpha$.

The case $\alpha = 1$: The polynomial $b^*(u_3 v_3)/u_3$ restricted to $u_3 = 0$ is $b_m v_3$, because $\alpha = 1$. Hence $\dot{v}_3|_{\{u_3=0\}} = v_3(-1+v_3)^2(1+(b_m-1)v_3)$ and the singularities of system (12) on the straight line $u_3 = 0$ are the points $(0,0)$, $(0,1)$ and $(0, 1/(1-b_m))$ when $b_m \neq 1$.

The eigenvalues of $(0,0)$ are $\pm 1$ and therefore $(0,0)$ is a hyperbolic saddle point.

The eigenvalues of the singular point $(0, 1/(1-b_m))$ are 0 and $-b_m^2/(b_m-1)^2 < 0$ and consequently it is semi-hyperbolic. We analyze its local flow using Theorem 2.19 of [6]. First we translate the singularity to the origin with the change $(u_3, v_3) \mapsto= (u_3, v_3 - 1/(1-b_m))$ and next we put the linear part in the Jordan canonical form $\dot{u}_3 = \cdots$, $\dot{v}_3 = v_3 + \cdots$ just by the time-rescaling dividing the vector field by $-b_m^2/(b_m-1)^2$. The outcome is that system (12) is transformed into

$$\dot{u}_3 = \bar{A}(u_3, v_3), \quad \dot{v}_3 = v_3 + \bar{B}(u_3, v_3), \tag{13}$$

for some polynomials $\bar{A}$ and $\bar{B}$ with only nonlinear terms. Let $v_3 = \bar{f}(u_3)$ be the solution of $v_3 + \bar{B}(u_3, v_3) = 0$ near the origin, and $\bar{g}(u_3) = \bar{A}(u_3, f(u_3))$. Some computations reveal that $\bar{f}$ has order at least 2 at the origin and $\bar{g}(u_3) = \beta_k u_3^{2k+1} + \cdots$ for some integer $k \geq 1$ and coefficient $\beta_k < 0$. By Theorem 2.19 in [6], the former implies that the origin of system (13) is a topological saddle.

The singular point $(0, 1)$ of system (12) is linearly zero, so more blow-ups are needed. We translate the singularity $(0, 1)$ to the origin with the change $(u_3, v_3) \mapsto (u_4, v_4) = (u_3, v_3 - 1)$, and next we check that both $u_4 = 0$ and $v_4 = 0$ are characteristic directions at the origin of the transformed system. Hence we do the linear change of variables $(u_4, v_4) \mapsto (u_5, v_5) = (u_4 + v_4, v_4)$ so that $u_5 = 0$ is not characteristic direction at the origin and the blow-up $(u_5, v_5) \mapsto (u_6, v_6) = (u_5, v_5/u_5)$, that transforms the differential system (12) into

$$\begin{aligned}
\dot{u}_6 &= u_6^{2n-3}(1-v_6)^{2n-4}(1+u_6v_6)^{2n-2}(1-3v_6+2u_6v_6-4u_6v_6^2) \\
&\quad -u_6v_6^2(1-u_6+2u_6v_6)[u_6^2v_6 + b^*(\xi(u_6,v_6))/(v_6-1)], \\
\dot{v}_6 &= v_6\big[-u_6^{2n-4}(1-v_6)^{2n-3}(1+u_6v_6)^{2n-2}(3+4u_6v_6) + v_6(1+2u_6v_6) \\
&\quad \times(-u_6^2(1-v_6)v_6 + b^*(\xi(u_6,v_6)))\big],
\end{aligned} \tag{14}$$

where $\xi(u_6, v_6) = u_6(1-v_6)(1+u_6v_6) = u_6 + \cdots$. From (9) we get that $b^*(\xi) \in \mathbb{R}[\xi]$ with the lower order term $b^*(\xi) = \xi(b_m + \cdots)$. Thus we have $b^*(\xi(u_6, v_6))/(v_6-1) \in \mathbb{R}[u_6, v_6]$ so (14) is a polynomial differential system. Indeed

$$b^*(\xi(u_6, v_6)) = (\xi(u_6, v_6))Q(\xi(u_6, v_6)) = (\xi(u_6, v_6))(b_m + O(\xi)),$$

hence system (14) becomes $\dot{u}_6 = u_6^{m_1}\hat{P}(u_6, v_6)$, $\dot{v}_6 = u_6^{m_2}\hat{Q}(u_6, v_6)$, where $m_1, m_2 \in \mathbb{N}$, $\hat{P}$ and $\hat{Q}$ are polynomials non divisible by $u_6$ and

$$m_1 = \min\{2n-3, 3, 2n-1-m\}, \quad m_2 = \min\{2n-4, 2, \alpha\}.$$

Doing a time-rescaling we divide the polynomial differential system (14) by $u_6^\gamma$ with $\gamma = \min\{m_1, m_2\} = \alpha = 1$ and system (14) becomes

$$\begin{aligned}
\dot{u}_6 &= P_6(u_6, v_6) = u_6^{2n-4}(1-v_6)^{2n-4}(1+u_6v_6)^{2n-2}(1-3v_6+2u_6v_6-4u_6v_6^2) \\
&\quad -v_6^2(1-u_6+2u_6v_6)[u_6^2v_6 + b^*(\xi(u_6,v_6))/(v_6-1)], \\
\dot{v}_6 &= Q_6(u_6, v_6) = v_6\big[-u_6^{2n-5}(1-v_6)^{2n-3}(1+u_6v_6)^{2n-2}(3+4u_6v_6) \\
&\quad +v_6(1+2u_6v_6) \times (-(1-v_6)v_6u_6 + b^*(\xi(u_6,v_6)/u_6))\big],
\end{aligned} \tag{15}$$

with $b^*(\xi(u_6, v_6)) = \xi(u_6, v_6)Q(\xi(u_6, v_6)) = u_6(1-v_6)(1+u_6v_6)(b_m + \cdots)$. Then, for system (15) we have $\dot{u}_6|_{\{u_6=0\}} = 0$, and the equation $\dot{v}_6|_{\{u_6=0\}} = 0$ becomes $v_6^2(1-v_6) = 0$. The former means that system (15) has the singular points $(0,0)$ and $(0,1)$ on the line $u_6 = 0$. Moreover, the eigenvalues associated to the point $(0,1)$ are $\pm b_m$ so it is a hyperbolic saddle. On the contrary the point $(0,0)$ still remains linearly zero and more blow-ups are needed.

We analyze the singularity $(0, 0)$. Both $u_6 = 0$ and $v_6 = 0$ are characteristic directions and so we perform the linear change $(u_6, v_6) \mapsto (u_7, v_7) = (u_6 + v_6, v_6)$ which transforms the system into another of the form $\dot{u}_7 = b_m v_7^2 + \cdots$, $\dot{v}_7 = b_m v_7^2 + \cdots$ if $n \geq 4$, or $\dot{u}_7 = u_7^2 - 5u_7v_7 + (4+b_m)v_7^2 + \cdots$, $\dot{v}_7 = -v_7(3u_7 - (3+b_m)v_7) + \cdots$ when $n = 3$. Hence this new system has not $u_7 = 0$ as characteristic direction at the origin except in the particular case $n = 3$ and $b_m = -4$ which will be analyzed later in Remark 8. So we continue assuming $(n, b_m) \neq (3, -4)$ and we perform the blow-up $(u_7, v_7) \mapsto (u_8, v_8) = (u_7, v_7/u_7)$. These changes transform system (15) into a differential system with a common factor $u_8$ which can be removed by a time-rescaling and we obtain a system

$$\dot{u}_8 = P_8(u_8, v_8), \quad \dot{v}_8 = Q_8(u_8, v_8), \tag{16}$$

such that $P_8(0, v_8) = 0$ and $Q_8(0, v_8) = (1 - v_8)v_8(-4 + 4v_8 + b_m v_8)$ if $n = 3$, or $b_m(1 - v_8)v_8^2$ when $n \geq 4$. Therefore the singularities of (16) on the straight line $u_8 = 0$ are the points $(0, 0)$ and $(0, 1)$ with the additional singular point $(0, 4/(4 + b_m)$ when $n = 3$ and $b_m \neq -4$. This last point is a node with eigenvalues $\{\lambda_1 = b_m/(4 + b_m), 4\lambda_1\}$ and this proves statement (i) of the proposition in the case $n = 3$ and $b_m \neq -4$.

***Remark 8*** The complementary case $(n, b_m) = (3, -4)$ yields $m = 3$ by the condition $\alpha = 1$ and system (2) becomes $\dot{x} = -y$, $\dot{y} = x^5 - 4x^3 y$. In this particular case we do not perform the former linear change $(u_6, v_6) \mapsto (u_7, v_7) = (u_6 + v_6, v_6)$. Instead we do $(u_6, v_6) \mapsto (u_7, v_7) = (2u_6 + v_6, v_6)$ that transforms the system into $\dot{u}_7 = (u_7^2 - 5u_7v_7 - 4v_7^2)/2 + \cdots$, $\dot{v}_7 = -v_7(3u_7 + 5v_7)/2 + \cdots$. Since $u_7 = 0$ is not a characteristic direction at the origin, now we do the blow-up $(u_7, v_7) \mapsto (u_8, v_8) = (u_7, v_7/u_7)$ and remove the common factor $u_8$ by a time-rescaling. The outcome is a system (16) with $P_8(0, v_8) = 0$ and $Q_8(0, v_8) = 2(v_8 - 1)v_8(1 + v_8)$, hence the singularities of (16) on the straight line $u_8 = 0$ are the points $(0, 0)$ and $(0, \pm 1)$. The points $(0, 0)$ and $(0, 1)$ are hyperbolic saddles but $(0, -1)$ is a node which proves statement (i) of the proposition in the case $n = 3$ and $b_m = -4$.

When $n \geq 4$ the point $(0, 1)$ is a hyperbolic saddle of system (16) with eigenvalues $\pm b_m$, and the point $(0, 0)$ is still linearly zero and more blow-ups are needed. Before doing the blow-up at $(0, 0)$ we check, from the lower order terms of system (16), that both $u_8 = 0$ and $v_8 = 0$ are characteristic directions at the origin and therefore we do the linear change $(u_8, v_8) \mapsto (u_9, v_9) = (u_8 + v_8, v_8)$. The resulting differential system has the form $\dot{u}_9 = b_m v_9^2 + \cdots$, $\dot{v}_9 = b_m v_9^2 + \cdots$ and $u_9 = 0$ is not a characteristic direction at the origin. Then we do the blow-up $(u_9, v_9) \mapsto (u_{10}, v_{10}) = (u_9, v_9/u_9)$ and system (16) becomes, after performing the time-rescaling dividing the system by $u_{10}$, the polynomial differential system

$$\dot{u}_{10} = P_{10}(u_{10}, v_{10}), \quad \dot{v}_{10} = Q_{10}(u_{10}, v_{10}), \tag{17}$$

with $P_{10}(0, v_{10}) = 0$ and $Q_{10}(0, v_{10})$ having the expression $b_m(1 - v_{10})v_{10}^2$. Therefore the singularities of system (17) on the straight line $u_{10} = 0$ are the point $(0, 1)$ which is a hyperbolic saddle with eigenvalues $\pm b_m$, and the point $(0, 0)$ which is still linearly zero and more blow-ups are needed. Since both $u_{10} = 0$ and $v_{10} = 0$ are characteristic directions at the origin of system (17), we do the linear change $(u_{10}, v_{10}) \mapsto (u_{11}, v_{11}) = (u_{10} + v_{10}, v_{10})$. The transformed system has the form $\dot{u}_{11} = b_m v_{11}^2 + \cdots$, $\dot{v}_{11} = b_m v_{11}^2 + \cdots$ if $n \geq 5$, or $\dot{u}_{11} = u_{11}^2 - 7u_{11}v_{11} + (6 + b_m)v_{11}^2 + \cdots$, $\dot{v}_{11} = v_{11}(-5u_{11} + (5 + b_m)v_{11}) + \cdots$ when $n = 4$. Hence this system has not the characteristic direction $u_{11} = 0$ at the origin except when $n = 4$ and $b_m = -6$ which will be analyzed later in Remark 9. So we continue assuming $(n, b_m) \neq (4, -6)$ and we perform the blow-up $(u_{11}, v_{11}) \mapsto (u_{12}, v_{12}) = (u_{11}, v_{11}/u_{11})$.

In these new coordinates and after the time-rescaling dividing this system by $u_{12}$, system (17) is written as the polynomial differential system

$$\dot{u}_{12} = P_{12}(u_{12}, v_{12}), \quad \dot{v}_{12} = Q_{12}(u_{12}, v_{12}), \tag{18}$$

such that $P_{12}(0, v_{12}) = 0$ and $Q_{12}(0, v_{12}) = (1 - v_{12})v_{12}(-6 + 6v_{12} + b_m v_{12})$ if $n = 4$, or $b_m(1 - v_{12})v_{12}^2$ when $n \geq 5$. Therefore the singularities of system (18) on the straight line $u_{12} = 0$ are the points $(0, 0)$ and $(0, 1)$ with the additional singular point $(0, 6/(6+b_m))$ when $n = 4$ and $b_m \neq -6$. This last point is a node with eigenvalues $\{\lambda_1 = b_m/(6 + b_m), 6\lambda_1\}$ and this proves statement (i) of the proposition in the case $n = 4$ and $b_m \neq -6$.

***Remark 9*** The complementary case $(n, b_m) = (4, -6)$ gives $m = 5$ by the condition $\alpha = 1$, system (2) becomes $\dot{x} = -y$, $\dot{y} = x^7 - 6x^5 y$. For this system we do not perform the former linear change $(u_{10}, v_{10}) \mapsto (u_{11}, v_{11}) = (u_{10} + v_{10}, v_{10})$. Instead we do $(u_{10}, v_{10}) \mapsto (u_{11}, v_{11}) = (2u_{10}+v_{10}, v_{10})$ transforming the system into $\dot{u}_{11} = (u_{11}^2 - 7u_{11}v_{11} - 6v_{11}^2)/2 + \cdots$, $\dot{v}_{11} = -v_{11}(5u_{11} + 7v_{11})/2 + \cdots$. Since $u_{11} = 0$ is not a characteristic direction at the origin, now we do the blow-up $(u_{11}, v_{11}) \mapsto (u_{12}, v_{12}) = (u_{11}, v_{11}/u_{11})$ and remove the common factor $u_{12}$ rescaling the time. In this way we obtain a differential system (18) with $P_{12}(0, v_{12}) = 0$ and $Q_{12}(0, v_{12}) = 3(v_{12} - 1)v_{12}(1 + v_{12})$, so having the singularities on the straight line $u_{12} = 0$ given by the points $(0, 0)$ and $(0, \pm 1)$. The points $(0, 0)$ and $(0, 1)$ are hyperbolic saddles but $(0, -1)$ is a node which proves statement (i) of the proposition in the case $(n, b_m) = (4, -6)$.

The computations with $\alpha = 1$ and arbitrary Andreev number $n \geq 5$ becomes heavy so we do not do them.

THE CASE $\alpha = 3$: Clearly in this case $n \geq 5$.

*The case* $n = 5$: The singularities of system (12) on the straight line $u_3 = 0$ are a hyperbolic saddle point at $(0, 0)$ and a linearly zero point at $(0, 1)$. Translating the point $(0, 1)$ to the origin with the change $(u_3, v_3) \mapsto (u_4, v_4) = (u_3, v_3 - 1)$, doing the linear change of variables $(u_4, v_4) \mapsto (u_5, v_5) = (u_4 + v_4, v_4)$ because both $u_4 = 0$ and $v_4 = 0$ are characteristic directions at the origin. Now $u_5 = 0$ is not a characteristic direction at the origin and we perform the blow-up $(u_5, v_5) \mapsto (u_6, v_6) = (u_5, v_5/u_5)$, and finally rescaling the time dividing the system by $u_6^2$, system (12) is transformed into

$$\dot{u}_6 = P_6(u_6, v_6), \quad \dot{v}_6 = Q_6(u_6, v_6), \tag{19}$$

with $P_6(0, v_6) = 0$ and $Q_6(0, v_6) = (-1 + v_6)v_6^3$. Therefore the singularities of system (19) on the straight line $u_6 = 0$ are the point $(0, 1)$ which is a hyperbolic saddle and the linearly zero point $(0, 0)$.

We check that both $u_6 = 0$ and $v_6 = 0$ are characteristic directions at the origin of system (19). Then we do the linear change of variables $(u_6, v_6) \mapsto (u_7, v_7) = (u_6 + v_6, v_6)$ transforming the system into $\dot{u}_7 = v_7^2(b_m u_7 - (1 + b_m)v_7) + \cdots$, $\dot{v}_7 = v_7^2(b_m u_7 - (1 + b_m)v_7) + \cdots$. We see that if $b_m \neq -1$ then $u_7 = 0$ is not a characteristic direction at the origin. The special case $b_m = -1$ will be analyzed later in Remark 10. Continuing with the generic case $b_m \neq -1$, we perform the blow-up $(u_7, v_7) \mapsto (u_8, v_8) = (u_7, v_7/u_7)$, and the rescaling of the time dividing the system by $u_8^2$. The outcome is that system (19) is transformed into

$$\dot{u}_8 = P_8(u_8, v_8), \quad \dot{v}_8 = Q_8(u_8, v_8), \tag{20}$$

with $P_8(0, v_8) \equiv 0$ and $Q_8(0, v_8) = (-1 + v_8)v_8^2(-b_m + v_8 + b_m v_8)$. Therefore the singularities of system (20) on the line $u_8 = 0$ are a hyperbolic saddle at $(0, 1)$, a semi-hyperbolic point at $(0, b_m/(1 + b_m))$ and a linearly zero point at $(0, 0)$. This last point will be desingularized following the standard process as follows: since $u_8 = 0$ and $v_8 = 0$ are characteristic directions at the origin of system (20) we do first the linear change of variables $(u_8, v_8) \mapsto (u_9, v_9) = (u_8 + v_8, v_8)$, since $u_9 = 0$ is not a characteristic direction at the origin we perform the blow-up $(u_9, v_9) \mapsto (u_{10}, v_{10}) = (u_9, v_9/u_9)$, and the rescaling of the time dividing the system by $u_{10}$. This sequence of changes brings system (20) to system

$$\dot{u}_{10} = P_{10}(u_{10}, v_{10}), \quad \dot{v}_{10} = Q_{10}(u_{10}, v_{10}), \tag{21}$$

with $P_{10}(0, v_{10}) = 0$ and $Q_{10}(0, v_{10}) = -b_m(-1 + v_{10})v_{10}^2$. Then the singularities on $u_{10} = 0$ are a hyperbolic saddle at $(0, 1)$, and a linearly zero point at $(0, 0)$. Repeating again the desingularization process of the point $(0, 0)$, namely we do the linear change of variables $(u_{10}, v_{10}) \mapsto (u_{11}, v_{11}) = (u_{10} + v_{10}, v_{10})$ because $u_{10} = 0$ and $v_{10} = 0$ are characteristic directions at the origin of system (21), and the system is transformed into $\dot{u}_{11} = u_{11}^2 - 7u_{11}v_{11} + (6 + b_m)v_{11}^2 + \cdots$, $\dot{v}_{11} = -v_{11}(5u_{11} - 5v_{11} - b_m v_{11}) + \cdots$. We see that if $b_m \neq -6$ then $u_{11} = 0$ is not a characteristic direction at the origin. The special case $b_m = -6$ will be analyzed later in Remark 11. In the generic case $b_m \neq -6$, we do the blow-up $(u_{11}, v_{11}) \mapsto (u_{12}, v_{12}) = (u_{11}, v_{11}/u_{11})$, and the rescaling of the time dividing the system by $u_{12}$. After these changes system (21) is transformed into

$$\dot{u}_{12} = P_{12}(u_{12}, v_{12}), \quad \dot{v}_{12} = Q_{12}(u_{12}, v_{12}), \tag{22}$$

with a singularity of type node on $u_{12} = 0$. In short the center cannot be global.

***Remark 10*** The complementary case $(n, b_m) = (5, -1)$ gives $m = 5$ by the condition $\alpha = 3$ and system (2) becomes $\dot{x} = -y$, $\dot{y} = x^9 - x^5 y$. For this system we do not perform the previous linear change $(u_6, v_6) \mapsto (u_7, v_7) = (u_6 + v_6, v_6)$ and we do $(u_6, v_6) \mapsto (u_7, v_7) = (2u_6 + v_6, v_6)$ so that $u_7 = 0$ is not a characteristic direction at the origin. Now we do the blow-up $(u_7, v_7) \mapsto (u_8, v_8) = (u_7, v_7/u_7)$ and remove the common factor $u_8^2$ rescaling the time. In this way we obtain a differential system (20) with $P_8(0, v_8) = 0$ and $Q_8(0, v_8) = (-1 + v_8)v_8^2(1 + v_8)/2$, so having the singularities on the straight line $u_8 = 0$ given by the points $(0, 0)$ and $(0, \pm 1)$. The point $(0, 1)$ is a hyperbolic saddle, the point $(0, -1)$ is semi-hyperbolic and the point $(0, 0)$ is still linearly zero. After the linear transformation $(u_8, v_8) \mapsto (u_9, v_9) = (u_8 + v_8, v_8)$, $u_9 = 0$ is not a characteristic direction at the origin and we do the blow-up $(u_9, v_9) \mapsto (u_{10}, v_{10}) = (u_9, v_9/u_9)$ and the rescaling of the time dividing the system by $u_{10}$. The resulting system has, on the line $u_{10} = 0$, the singular points $(0, 1)$ which is a hyperbolic saddle and the linearly zero point $(0, 0)$. Doing the linear transformation $(u_{10}, v_{10}) \mapsto (u_{11}, v_{11}) = (u_{10} + v_{10}, v_{10})$, $u_{11} = 0$ is no longer a characteristic direction at the origin and we do the blow-up $(u_{11}, v_{11}) \mapsto (u_{12}, v_{12}) = (u_{11}, v_{11}/u_{11})$ and the rescaling of the time dividing the system by $u_{12}$. The resulting system has, on the line $u_{12} = 0$ the singular points $(0, -3)$ which is a hyperbolic node. This proves statement (ii) when $b_m = -1$.

***Remark 11*** The specific case $(n, b_m) = (5, -6)$ gives $m = 5$ by the condition $\alpha = 3$ and system (2) becomes $\dot{x} = -y$, $\dot{y} = x^9 - 6x^5 y$. For this system, instead of the change $(u_{10}, v_{10}) \mapsto (u_{11}, v_{11}) = (u_{10} + v_{10}, v_{10})$ we do $(u_{10}, v_{10}) \mapsto (u_{11}, v_{11}) = (2u_{10} + v_{10}, v_{10})$ transforming the system into a system where $u_{11} = 0$ is not a characteristic direction at the origin. Now we do the blow-up $(u_{11}, v_{11}) \mapsto (u_{12}, v_{12}) = (u_{11}, v_{11}/u_{11})$ and remove the common factor $u_{12}$ rescaling the time. The resulting system has a node on the straight line $u_{12} = 0$ and this proves statement (ii) of the proposition in the case $b_m = -6$.

□

**Proposition 12** *The polynomial differential system (2) satisfying (3) has a center at the origin. Assume that* $b(x) \not\equiv 0$, $\beta = n-1$, $b_\beta^2 - 4n < 0$, $\alpha > 0$, *and* $m > n-1$. *If* $n = 4$ *then* $m \in \{3, 5\}$, *and when* $m = 5$ *the origin is not a global center.*

***Proof*** Under the hypothesis $b(x) \not\equiv 0$, $\beta = n-1$, $b_\beta^2 - 4n < 0$ and $\alpha > 0$ we have that $n-1 \le m \le 2n-3$ and $n$ is even. In particular $b(x) = b_{n-1}x^{n-1} + \cdots + b_m x^m$ with $b_{n-1}^2 + b_m^2 \neq 0$ and

$$b_{n-1}^2 - 4n < 0. \tag{23}$$

The degree of system (2) is $d = 2n-1$ and

$$\beta = n-1 \le m \le 2n-3. \tag{24}$$

Most of the computations already done for system (2) in the proof of Proposition 7 related with the local chart $U_2$ are repeated verbatim here.

We do not consider the case $n = 2$ because then $m = 1$ and the center is global by Proposition 4.

If $n = 4$ then $m \in \{3, 5\}$ and we only consider $m = 5$. Then the system has no global center because the analogous to system (18), that is, $\dot{u}_{12} = P_{12}(u_{12}, v_{12})$, $\dot{v}_{12} = Q_{12}(u_{12}, v_{12})$ has a node on the line $u_{12} = 0$. Therefore the origin is not a global center of system (2). □

### 2.1 Proof of theorem 1

The proof of statement (i) follows by Proposition 3 whereas statement (ii) is a consequence of Propositions 3 and 4.

### 2.2 Proof of theorem 2

The proof follows from Propositions 7 and 12.

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