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Dynamics of the Secant map near infinity

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ABSTRACT

We investigate the root finding algorithm given by the Secant method applied to a real polynomial p of degree k as a discrete dynamical system defined on $\mathbb{R}^2$. We extend the Secant map to the real projective plane $\mathbb{RP}^2$. The line at infinity ℓ_∞ is invariant, and there is one (if k is odd) or two (if k is even) fixed points at ℓ_∞. We show that these are of saddle type, and this allows us to better understand the dynamics of the Secant map near infinity.

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1. Introduction and preliminaries

Root finding algorithms are not only an efficient way to find numerical solutions of non-linear equations which cannot be solved explicitly but a fruitful source of interesting dynamical systems. The common idea behind any such algorithm is to construct sequences converging to the solutions of the equation, and these sequences correspond to orbits of discrete dynamical systems.

Let X be a topological space. Roughly speaking a discrete dynamical system over X (known as *phase space*) is a map $f : X \to X$ and the orbits induced by this map starting at $x_0 \in X$, $\{x_n := f^n(x_0)\}_{n\in\mathbb{N}}$. The main goal is to describe the *phase portrait*, that is the description of the asymptotic behaviour of those orbits when x_0 runs over all X. We say that ζ in X is a fixed point if $f(\zeta) = \zeta$. We say that ζ is attracting or repelling depending if all nearby seeds correspond to orbits converging or diverging to ζ. It is well known that fixed (as well as periodic) points play a key role to understand the global dynamics, specially when f models a root finding algorithm. In the case that ζ in X is an attracting fixed point we define its *basin of attraction* by

$$A(\zeta) = \{x \in X \mid f^n(x) \to \zeta,\ \text{as } n \to \infty\},$$

or, in other words, $A(\zeta)$ is the maximal set where orbits converge to ζ under iteration. The connected component of $A(\zeta)$ which contains x_0 is called the *immediate basin of attraction of* ζ and it is denoted by $A^\star(\zeta)$.

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