

# GLOBAL DYNAMICS OF THE HOŘAVA-LIFSHITZ COSMOLOGY IN THE PRESENCE OF NON-ZERO COSMOLOGICAL CONSTANT IN A FLAT SPACE

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ABSTRACT. Using the qualitative theory of differential equations, the global dynamics of a cosmological model based on Hořava-Lifshitz gravity is studied in the space with zero curvature in the presence of the non-zero cosmological constant. The study shows that there may be three unstable finite equilibrium positions under the Hořava-Lifshitz cosmology model and that the final evolution of the orbits of the cosmological model in the physical region of interest may tend towards some infinite equilibrium position, which may correspond to the late-time state of the universe.

## 1. INTRODUCTION

A decade ago Hořava [1] brought forward a new theory on space-time asymmetric gravitation, called Hořava-Lifshitz gravity, together with the scalar field theory of Lifshitz. This theory's applications to cosmology, dark energy, and black hole have stimulated many studies (See review papers [20], [21] or regular literature [2]-[19]).

Based on whether the value of  $\Lambda$  (cosmological constant) is zero and the flatness of the universe, i.e. whether the space curvature  $k$  is equal to zero, Leon et al. [8, 9, 10, 16] divided Hořava-Lifshitz cosmology into four cases: (1)  $\Lambda = 0, k = 0$ ; (2)  $\Lambda = 0, k \neq 0$ ; (3)  $\Lambda \neq 0, k = 0$ ;

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(4)  $\Lambda \neq 0, k \neq 0$  under the classic FLRW metric. They either studied partially the three-dimensional dynamics of Hořava-Lifshitz cosmology, or analyzed its two-dimensional dynamics under exponential potentials.

For the cosmological constant  $\Lambda$  that scientists have been concerning about, Carlip [22] argued that the vacuum fluctuations under the standard effective field theory produce a huge  $\Lambda$  and produce high  $k$  on the Planck scale, but it is almost invisible at the observable scale. Compared with the prediction in the standard  $\Lambda$  cold dark matter model, Valentino et al. [23] proposed that the cosmological space may be a closed three-dimensional sphere, i.e., the cosmological space's curvature may be positive, based on the enhanced lensing amplitude in the cosmic microwave background power spectra confirmed by Planck Legacy 2018 release [24]. Although this study provides the latest results, the debate about the universe's shape has not yet been settled. An important reason is that the calculation of the critical density of the universe depends on the measurement of the Hubble constant, which is different as estimated from different cosmological data. Then the boundary line of the universe is fuzzy. So it is too early to say that the universe must be closed.

The global dynamics of the Hořava-Lifshitz cosmology under the background of FLRW with  $k = 0, \Lambda = 0$  was studied in [5], and the case of  $k \neq 0, \Lambda = 0$  has also been addressed in [6]. In this paper we will consider the flat universe with  $\Lambda \neq 0$ . More precisely, in this paper we present the global dynamics of the Hořava-Lifshitz cosmology in the three-dimensional space including the infinity, neither partial or local dynamics, nor the two-dimensional case. The motivation for considering the cosmological constant and scalar field here is to fully explain the

universe's dynamic evolution and final state under the Hořava-Lifshitz gravitational model.

## 2. THE COSMOLOGICAL EQUATIONS

In order to describe the cosmological model, we first briefly review the Hořava-Lifshitz theory of gravity proposed in [1]. The field content in this theory can be derived from the space vector  $\mathfrak{N}_i$  and scalar  $\mathfrak{N}$ , see [10, 11]. They are actually common 'lapse' and 'shift' variables in general relativity. From this the complete metric can be expressed as

$$(1) \quad ds^2 = -\mathfrak{N}^2 dt^2 + g_{ij}(dx^i + \mathfrak{N}^i dt)(dx^j + \mathfrak{N}^j dt), \quad \mathfrak{N}_i = g_{ij}\mathfrak{N}^j,$$

where  $g_{ij}$  is a spatial metric, here  $i$  and  $j$  are natural numbers from 1 to 3. The coordinate transformations follow  $t \rightarrow l^3 t$ ,  $x^i \rightarrow lx^i$ . Note that  $g_{ij}$  is invariant, the same as  $\mathfrak{N}$ , but  $\mathfrak{N}^i$  is scaled to  $l^{-2}\mathfrak{N}_i$ .

According to the detailed-balance condition, the full gravitational action of Hořava-Lifshitz is expressed as

$$(2) \quad S_g = \int dt d^3x \sqrt{g} \mathfrak{N} \left\{ \frac{2}{\kappa^2} (K_{ij}K^{ij} - \lambda K^2) - \frac{\kappa^2}{2w^4} C_{ij}C^{ij} + \frac{\mu\kappa^2}{2w^2} \frac{\epsilon^{ijk}}{\sqrt{g}} R_{il} \nabla_j R_k^l - \frac{\mu^2\kappa^2}{8} R_{ij}R^{ij} - \frac{\mu^2\kappa^2}{8(3\lambda-1)} \left( \frac{1-4\lambda}{4} R^2 + \Lambda R - 3\Lambda^2 \right) \right\},$$

where  $C^{ij} = \epsilon^{ijk} \nabla_k (4R_i^j - R\delta_i^j) / (4\sqrt{g})$  denotes the Cotton tensor,  $K_{ij} = (\dot{g}_{ij} - \nabla_i \mathfrak{N}_j - \nabla_j \mathfrak{N}_i) / (2\mathfrak{N})$  represents the extrinsic curvature,  $\epsilon^{ijk}$  is the totally antisymmetric unit tensor, and  $\epsilon^{ijk} / \sqrt{g}$  is the standard general covariant antisymmetric tensor, the indices can be raised or lowered using the metric  $g_{ij}$ .  $\kappa$ ,  $\lambda$ ,  $w$  and  $\mu$  are all constants, for

more details see [1].

Consider the matter source action due to a scalar field,  $\phi$ , as follows

$$(3) \quad S = \int dt d^3x \sqrt{g} \mathfrak{N} \left( \frac{3\lambda - 1}{4} \frac{\dot{\phi}^2}{\mathfrak{N}^2} - V(\phi) \right),$$

and the metric  $\mathfrak{N}^i = 0$ ,  $g_{ij} = a^2(t) \gamma_{ij}$ ,  $\gamma_{ij} dx^i dx^j = r^2 d\Omega_2^2 + dr^2 / (1 - kr^2)$ . Here  $a(t)$  represents the scale factor of the expanding universe, which is dimensionless, and  $\gamma_{ij}$  refers to the constant curvature metric with maximal symmetry. For the flat space we take  $k = 0$  in this paper.

For the sake of simplicity,  $\kappa^2$  and  $\mathfrak{N}$  are normalized, and then the corresponding cosmological model can be interpreted as

$$(4) \quad \begin{aligned} H^2 &= \frac{\dot{\phi}^2}{24} + \frac{V(\phi)}{6(3\lambda - 1)} - \frac{\mu^2 \Lambda^2}{16(3\lambda - 1)^2}, \\ \dot{H} + \frac{3}{2} H^2 &= -\frac{\dot{\phi}^2}{16} + \frac{V(\phi)}{4(3\lambda - 1)} - \frac{3\mu^2 \Lambda^2}{32(3\lambda - 1)^2}, \\ \ddot{\phi} + 3H\dot{\phi} + \frac{2V'(\phi)}{3\lambda - 1} &= 0, \end{aligned}$$

where  $H$  is the Hubble parameter and has the form  $\dot{a}(t)/a(t)$ .

Considering that  $V(\phi)$  admits various mathematical forms (see [9, 25, 26, 27]), we take  $V(\phi) = (\varrho\phi)^{2n}/2n$  with a natural number  $n$  and a constant  $\varrho > 0$  in this paper. Following [9, 10] we do the dimensionless transformation

$$(5) \quad \begin{aligned} x &= \frac{\dot{\phi}}{2\sqrt{6}H}, & y &= \frac{\sqrt{V(\phi)}}{\sqrt{6}H\sqrt{3\lambda - 1}}, & z &= \frac{\Lambda\mu}{4(3\lambda - 1)H}, \\ s &= -\frac{V'(\phi)}{V(\phi)}, & f(s) &\equiv \frac{V''(\phi)}{V(\phi)} - \frac{V'(\phi)^2}{V(\phi)^2}. \end{aligned}$$

Thus we obtain  $f(s) = -s^2/(2n)$ , which is a power-law potential, so  $ds/dN = \sqrt{6}xs^2/n$ , here  $N$  denotes the new time variable. Furthermore it follows from equations (4) and (5) that

$$(6) \quad \begin{aligned} x^2 + y^2 - z^2 &= 1, \\ \frac{\dot{H}}{H} &= -3x^2. \end{aligned}$$

Therefore the field equations become the dimensionless form

$$(7) \quad \begin{aligned} \frac{dx}{dN} &= \sqrt{6}s(z^2 - x^2 + 1) + 3x(x^2 - 1), \\ \frac{dz}{dN} &= 3zx^2, \\ \frac{ds}{dN} &= \frac{\sqrt{6}}{n}xs^2. \end{aligned}$$

For more details on system (7) see the equations (205)-(207) of [9] or equations (52)-(54) of [16].

The present paper gives a fully description of the global dynamics of system (7) in the physical area of interest  $G = \{(x, z, s) \in \mathbb{R}^3 : x^2 - z^2 \leq 1\}$ . In sections 3 and 4 we will investigate the phase portraits of system (7) at finite and infinite equilibrium points on invariant planes and surface. In section 5 we will discuss the phase portraits of system (7) inside the Poincaré ball restricted to the region  $G$ . An introduction to the Poincaré ball that can be used to study the dynamics of the system (7) near infinity can be found in the appendix. Based on these sections, considering the symmetry of system (7), we will study the global dynamics of system (7) adding its behavior at infinity. Moreover we will give the final discussion and summary in the last section 6.

3. PHASE PORTRAITS ON TWO INVARIANT PLANES  $z = 0$ ,  $s = 0$   
AND ON THE INVARIANT SURFACE  $x^2 - z^2 = 1$

In order to understand the local phase portraits of equilibrium points (finite and infinite) and the global phase portraits of system (7) in the aforementioned region  $G$  (refer to [9] or [16] again), we begin to describe the phase portraits on the invariant planes  $z = 0$ ,  $s = 0$  and on the invariant surface  $x^2 - z^2 = 1$ .

From a dynamic point of view, if the initial point of universe evolution is on an invariant plane or an invariant surface, then the evolutionary trajectory of the universe will always remain on this plane or surface, which is why we call it “invariant”.

**3.1. The invariant plane  $z = 0$ .** On the plane  $z = 0$  system (7) reduces to

$$(8) \quad \begin{aligned} \frac{dx}{dN} &= (x^2 - 1)(3x - \sqrt{6}s), \\ \frac{ds}{dN} &= \frac{\sqrt{6}}{n}xs^2. \end{aligned}$$

The phase portrait of the above system in the strip  $z = 0$  and  $x^2 - z^2 \leq 1$  has been presented in [5] (see Figure 1). System (8) contains a hyperbolic equilibrium point  $e_0 = (0, 0)$  and two semi-hyperbolic equilibrium points  $e_1 = (1, 0)$ ,  $e_2 = (-1, 0)$ , where  $e_0$  is a saddle point, and the other two are saddle-nodes.

**3.2. The invariant plane  $s = 0$ .** On the plane  $s = 0$  system (7) reduces

$$(9) \quad \frac{dx}{dN} = 3x(x^2 - 1), \quad \frac{dz}{dN} = 3zx^2.$$

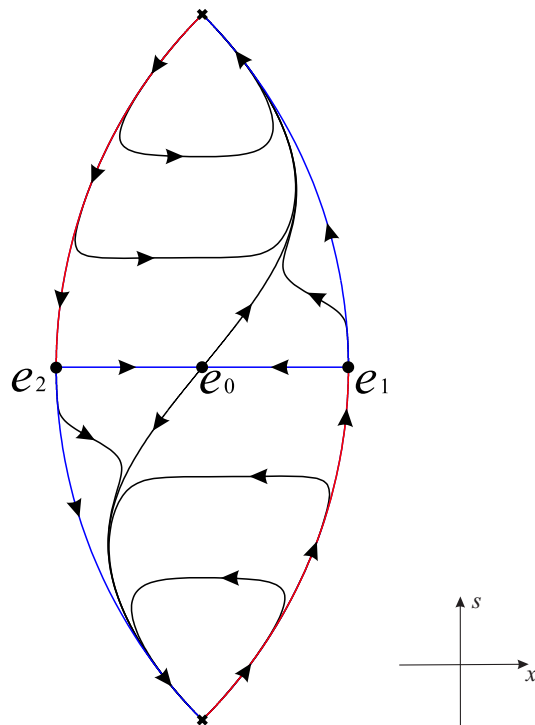


FIGURE 1. The phase portrait of system (7) on the invariant plane  $z = 0$  in the region  $-1 \leq x \leq 1$ .

Note that each point on the line  $x = 0$  is an equilibrium point. In order to eliminate the common factor  $x$  of the above equation. Introducing the transformation with respect to time  $d\tau_1 = x dN$  yields

$$(10) \quad \frac{dx}{d\tau_1} = 3(x^2 - 1), \quad \frac{dz}{d\tau_1} = 3zx,$$

which has two hyperbolic equilibrium points  $e_1 = (1, 0)$  and  $e_2 = (-1, 0)$ . Here  $e_1$  is an unstable node and has two eigenvalues 3 and 6, but  $e_2$  is a stable node and has eigenvalues -3 and -6.

According to the Poincaré compactification method (see Chapter 5 of [28] for more details), we perform the Poincaré transformation  $x = 1/v$ ,

$z = u/v$  on the local chart  $U_1$ , then system (9) becomes

$$(11) \quad \frac{du}{dN} = 3uv^2, \quad \frac{dv}{dN} = 3v(v^2 - 1).$$

The transformation can help us to plot the vector field of system (9) on the local chart  $U_1$ , and can also show the trajectory going to or coming from infinity. Since all the points at infinity (i.e. at  $v = 0$ ) of system (11) are equilibrium points, we do the transformation of the time  $d\tau_2 = v dN$ , and the system (11) becomes

$$(12) \quad \frac{du}{d\tau_2} = 3uv, \quad \frac{dv}{d\tau_2} = v^2 - 1.$$

However there is no equilibrium points in system (12).

System (9) on the local chart  $U_2$  (Poincaré transformation  $x = u/v$ ,  $z = 1/v$ ) becomes

$$(13) \quad \frac{du}{dN} = -3uv^2, \quad \frac{dv}{dN} = -3u^2v.$$

Since this system's linear term is always equal to zero, the corresponding topological index is known to be zero by the Poincaré-Hopf Theorem (for more details, see Theorem 6.30 in [28]). To study the local phase portrait of the equilibrium point (0,0) of system (13), we use the vertical blow-up techniques (see Ref. [29]), i.e., let  $w = v/u$  then we have

$$(14) \quad \frac{du}{dN} = -3u^3w^2, \quad \frac{dw}{dN} = 3u^2w(w^2 - 1).$$

Rescaling system (14)'s time variable  $N$  as  $d\tau_3 = 3u^2w dN$ , yields

$$(15) \quad \frac{du}{d\tau_3} = -uw, \quad \frac{dw}{d\tau_3} = w^2 - 1.$$



This system admits two equilibrium points  $(0, -1)$  and  $(0, 1)$  on  $u = 0$ . Both of these two points are hyperbolic unstable saddle points with eigenvalues of  $-2, 1$  and  $2, -1$ , respectively. The local phase portrait around them is shown in Figure 2(a). Note the time rescaling between the above two systems, the local phase portrait of system (14) can be found in Figure 2(b). Additionally, all points on the axes  $u = 0$  and  $w = 0$  are singularities of system (14). Since  $u > 0$  and  $w > 0$  in the first quadrant I of Figure 2(b), then  $v = uw$  will decrease as  $u$  decreases, so the local phase portrait in the quadrant I of the  $u-w$  coordinate system (corresponding to system (14)) can be equivalently converted to the portrait in the first quadrant of the  $u-v$  coordinate system (corresponding to system (13)). Similarly the local phase portraits in the quadrants II, III and IV of the  $u-w$  coordinate system can also be equivalently converted to these portraits in the third, second and fourth quadrants of the  $u-v$  coordinate system, respectively. Therefore the local phase portrait of system (13) is displayed in Figure 2(c), and the corresponding local phase portrait at the origins of  $U_2$  and the symmetrical  $V_2$  in the invariant plane  $s = 0$  can be found in Figure 2(d).

In summary, considering that the straight lines  $x = 0$  and  $z = 0$  are invariant under the flow of system (9), we can obtain that in the Poincaré disk with  $s = 0$ , the global phase portrait is restricted to the strip  $-1 \leq x \leq 1$  in Figure 3.

**3.3. The invariant surface  $x^2 - z^2 = 1$ .** Under the flow of system (7), we first verify that  $x^2 - z^2 = 1$  is the invariant surface. If  $\mathfrak{l} = \mathfrak{l}(x, z, s) = x^2 - z^2 - 1$ , then the surface  $x^2 - z^2 = 1$  is invariant, only

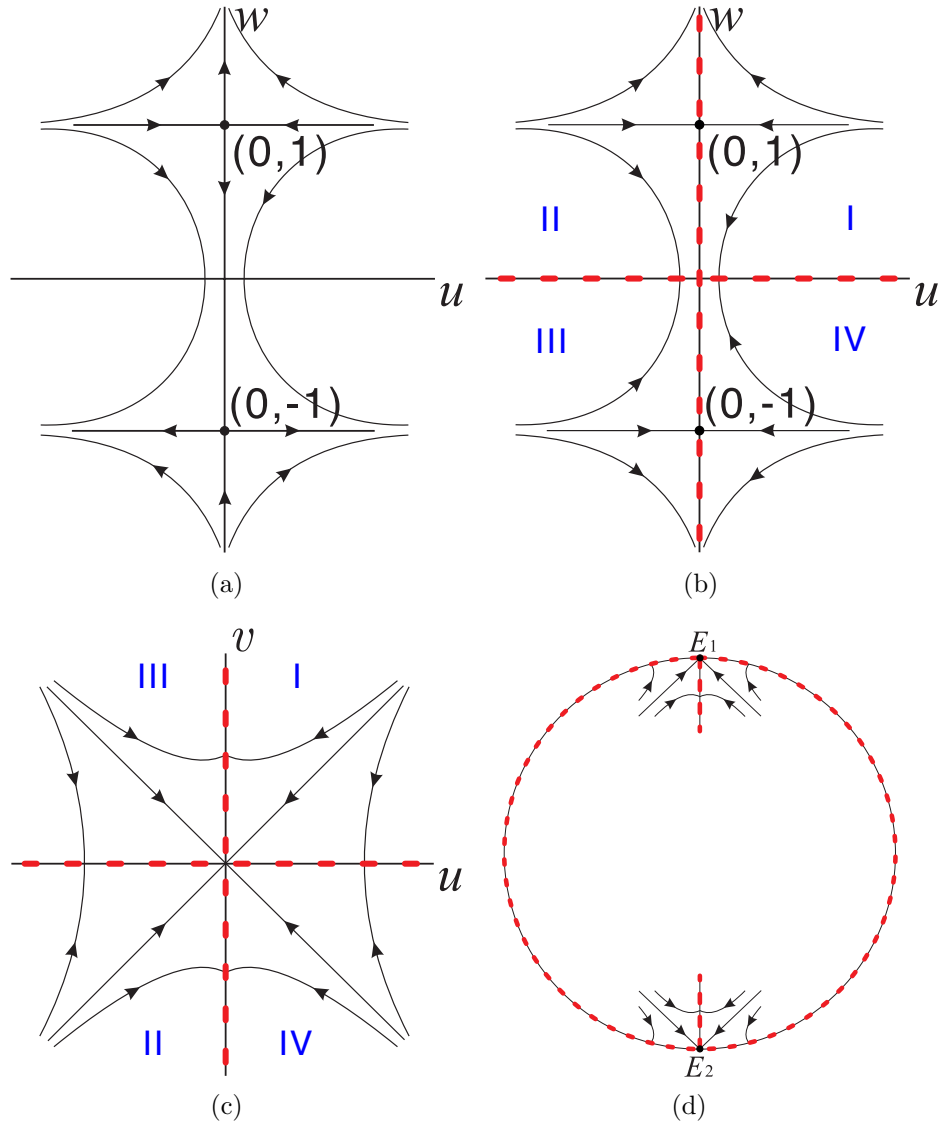


FIGURE 2. In (a), (b), and (c), there are the local phase portraits of the equilibrium points of systems (15), (14), and (13), respectively. In (d), there are the local phase portraits at the origins of  $U_2$  and  $V_2$  for  $s = 0$ .

if there exists a polynomial  $\mathcal{P}$  satisfying the equation

$$\frac{\partial \mathfrak{I}}{\partial x} \frac{dx}{dN} + \frac{\partial \mathfrak{I}}{\partial z} \frac{dz}{dN} + \frac{\partial \mathfrak{I}}{\partial s} \frac{ds}{dN} = \mathcal{P} \mathfrak{I},$$

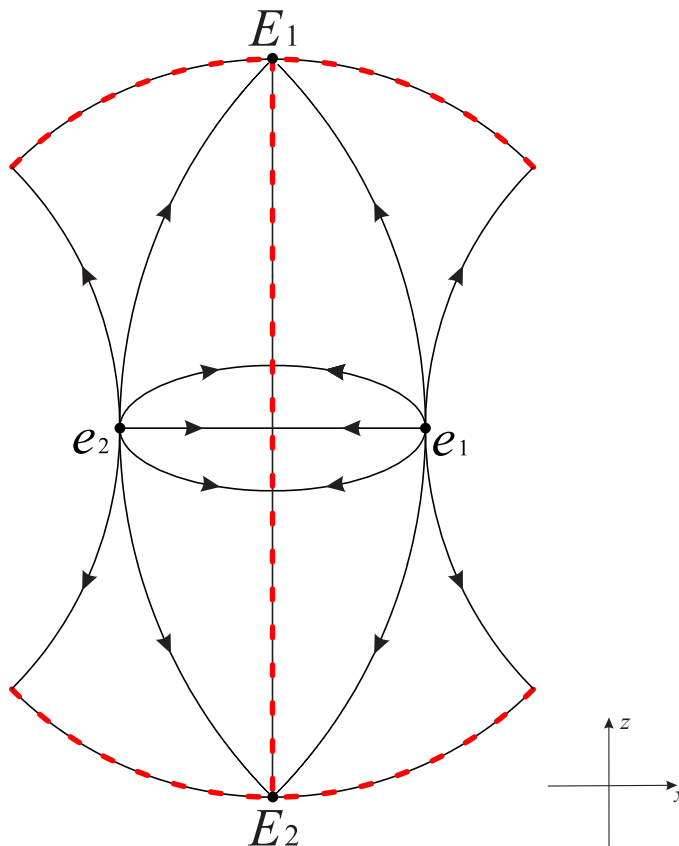


FIGURE 3. The phase portrait restricted to the region  $x^2 - z^2 \leq 1$  on the invariant plane  $s = 0$  inside the Poincaré disc.

which is exactly the case with  $\mathcal{P} = 2x(3x - \sqrt{6}s)$ . The polynomial here is derived with the help of the software Mathematica, and more information on invariance can be found in [28].

On the surface  $x^2 - z^2 = 1$  system (7) writes

$$(16) \quad \frac{dx}{dN} = 3x(x^2 - 1), \quad \frac{ds}{dN} = \frac{\sqrt{6}}{n} xs^2.$$

Then except for  $s$ -axis that is filled with equilibrium points, the above system also has two finite semi-hyperbolic equilibrium points  $e_1 = (1, 0)$

and  $e_2 = (-1, 0)$ . It follows from Theorem 2.19 of [28] that both  $e_1$  and  $e_2$  are saddle-nodes.

System (16) on the local chart  $U_1$  can be written as

$$(17) \quad \frac{du}{dN} = u \left[ \frac{\sqrt{6}}{n}u + 3(v^2 - 1) \right], \quad \frac{dv}{dN} = 3v(v^2 - 1).$$

This system admits two infinite hyperbolic equilibrium points  $e_5 = (0, 0)$  and  $e_6 = (\sqrt{6}n/2, 0)$ , where  $e_5$  is a stable node and has eigenvalues  $-3$  of multiplicity two, and the other point  $e_6$  is an unstable saddle and has two eigenvalues  $\pm 3$ .

System (16) on the local chart  $U_2$  can be written as

$$(18) \quad \frac{du}{dN} = u \left[ -\frac{\sqrt{6}}{n}u + 3(u^2 - v^2) \right], \quad \frac{dv}{dN} = -\frac{\sqrt{6}}{n}uv.$$

Rescaling the time  $d\tau_4 = udN$  we have

$$(19) \quad \frac{du}{d\tau_4} = -\frac{\sqrt{6}}{n}u + 3(u^2 - v^2), \quad \frac{dv}{d\tau_4} = -\frac{\sqrt{6}}{n}v.$$

The origin  $(0, 0)$  of the above system is a hyperbolic stable node with eigenvalues  $-\sqrt{6}/n$  and a multiplicity of 2. In this way, the origin  $e_7 = (0, 0)$  of system (18) has a local phase portrait as shown in Figure 4.

In summary the global phase portraits of system (16) is integrated in Figure 5.

**3.4. The finite equilibrium points.** System (7) allows three three-dimensional finite equilibrium points  $p_0 = (0, 0, 0)$ ,  $p_1 = (1, 0, 0)$  and  $p_2 = (-1, 0, 0)$ ,  $p_0$  has eigenvalues  $-3, 0, 0$ ,  $p_1$  and  $p_2$  have the same eigenvalues  $6, 3, 0$ . Here  $p_1$  and  $p_2$  are the intersection points of  $x^2 - z^2 = 1$ ,  $s = 0$  and  $z = 0$  that were just studied in the previous subsections

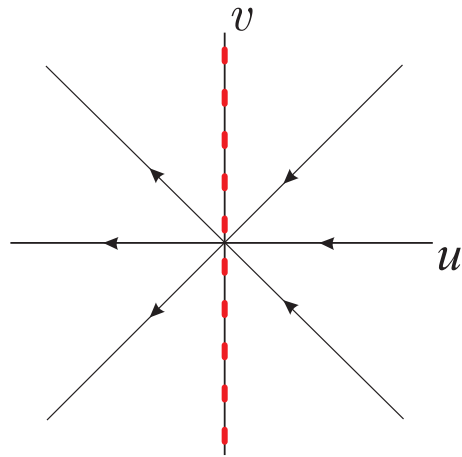


FIGURE 4. The local phase portrait at the origin of  $U_2$  for  $x^2 - z^2 = 1$ .

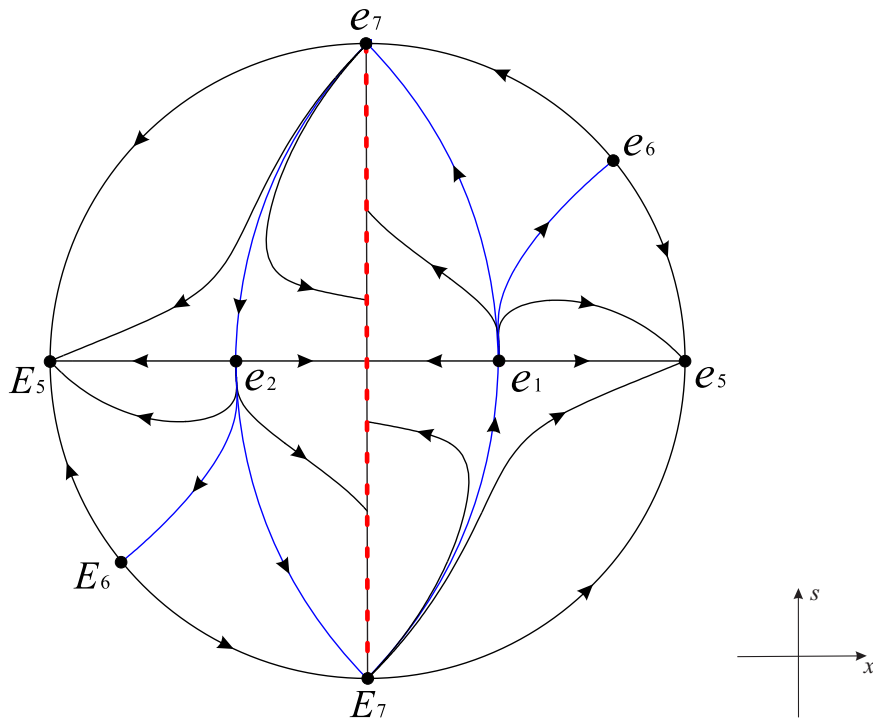


FIGURE 5. Phase portrait on the invariant surface  $x^2 - z^2 = 1$ .

3.1-3.3, that is,  $p_1$  and  $p_2$  are the equilibrium points  $e_1$  and  $e_2$ , respectively. The origin  $p_0$  is located in the middle of the intersection of  $z = 0$  and  $s = 0$ , and it is the equilibrium point  $e_0$  studied in the previous subsection 3.1.

#### 4. PHASE PORTRAIT ON THE SURFACE OF POINCARÉ BALL AT INFINITY

The three-dimensional Poincaré compactification (see Appendix or [30] for more details) is used to study the dynamics of the system (7) near infinity in this section. So we have  $x = 1/z_3$ ,  $z = z_1/z_3$ ,  $s = z_2/z_3$  on the local chart  $U_1$ , and then system (7) on the  $U_1$  is reduced to

$$(20) \quad \begin{aligned} \frac{dz_1}{dN} &= z_1 [3z_3^2 - \sqrt{6}z_2(-1 + z_1^2 + z_3^2)], \\ \frac{dz_2}{dN} &= z_2 \left[ 3(z_3^2 - 1) + \sqrt{6}z_2 \left( \frac{1+n}{n} - z_1^2 - z_3^2 \right) \right], \\ \frac{dz_3}{dN} &= z_3 [3(z_3^2 - 1) + \sqrt{6}z_2(1 - z_1^2 - z_3^2)]. \end{aligned}$$

In different local charts,  $z_3 = 0$  corresponds to the infinity of  $\mathbb{R}^3$ . The equilibrium points of the system (20) are listed in Table 1, where the equilibrium point  $u_{31}$  represents the origin of the local chart  $U_3$ , and the other equilibrium points lie in the local chart  $U_1$ . Additionally, for any constant  $a$ ,  $u_{a0}$  means that  $s = 0$  on local chart  $U_1$  is filled with equilibrium points.

For the case  $z_3 = 0$  system (20) becomes

$$(21) \quad \begin{aligned} \frac{dz_1}{dN} &= \sqrt{6}z_1z_2(1 - z_1^2), \\ \frac{dz_2}{dN} &= z_2 \left[ -3 + \sqrt{6}z_2 \left( \frac{1+n}{n} - z_1^2 \right) \right]. \end{aligned}$$

TABLE 1. The equilibrium points on the local charts of the surface of the Poincaré ball at  $\mathbb{R}^3$  infinity.

Equilibrium points	Eigenvalues
$u_{11} = (0, 0, 0)$	$-3, -3, 0$
$u_{12} = \left(-1, \frac{\sqrt{6}}{2}n, 0\right)$	$-3, 3, -6n$
$u_{13} = \left(1, \frac{\sqrt{6}}{2}n, 0\right)$	$-3, 3, -6n$
$u_{14} = \left(0, \frac{\sqrt{6}n}{2(n+1)}, 0\right)$	$-\frac{3}{n+1}, \frac{3n}{n+1}, 3$
$u_{a0} = (a, 0, 0)$	$-3, -3, 0$
$u_{31} = (0, 0, 0)$	$0, 0, 0$

Rescaling the time  $d\tau_5 = z_2 dN$ , system (20) is reduced to

$$(22) \quad \begin{aligned} \frac{dz_1}{d\tau_5} &= \sqrt{6}z_1(1 - z_1^2), \\ \frac{dz_2}{d\tau_5} &= -3 + \sqrt{6}z_2 \left( \frac{1+n}{n} - z_1^2 \right). \end{aligned}$$

Then system (22) allows equilibrium points  $e_{i,1}$ ,  $e_{i,2}$  and  $e_{i,3}$ , the coordinates of which are  $(\mp 1, \sqrt{6}n/2)$  and  $(0, \sqrt{6}n/(2n+2))$ , respectively. Note that the subscript “i” here has no special meaning. Although the corresponding equilibrium points are indeed on the infinite Poincaré sphere, they are just to distinguish them from the equilibrium points on the invariant planes and surface. The equilibrium points  $e_{i,1}$  and  $e_{i,2}$  are hyperbolic unstable saddles with eigenvalues 3 and  $-6n$ , and the equilibrium point  $e_{i,3}$  is a hyperbolic unstable node with eigenvalues  $3n/(1+n)$  and 3. The phase portrait on local chart  $U_1$  of the Poincaré sphere at infinity is shown in Figure 6.

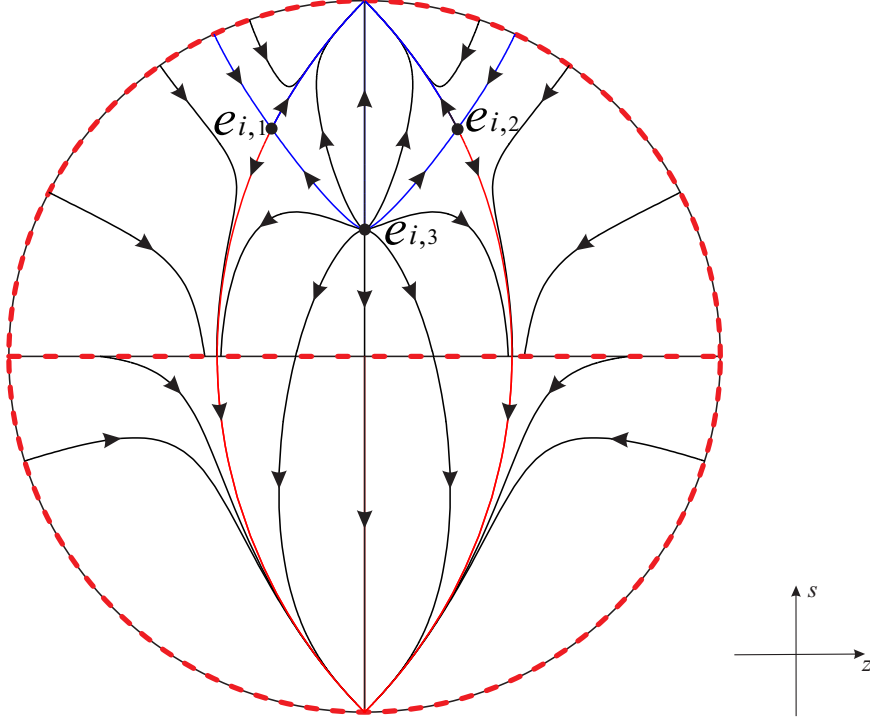


FIGURE 6. The phase portrait of system (7) on the local chart  $U_1$  at infinity.

On the local chart  $U_2$  we have Poincaré compactification  $x = z_1/z_3$ ,  $z = 1/z_3$ ,  $s = z_2/z_3$ , the system (7) becomes

$$(23) \quad \begin{aligned} \frac{dz_1}{dN} &= -z_1 z_3^2 + \sqrt{6} z_2 (1 - z_1^2 + z_3^2), \\ \frac{dz_2}{dN} &= z_1 z_2 \left( -3z_1 + \frac{\sqrt{6}}{n} z_2 \right), \\ \frac{dz_3}{dN} &= -3z_1^2 z_3. \end{aligned}$$



To study the phase portrait at infinity we take  $z_3 = 0$ , and changing the time  $d\tau_6 = z_2 dN$  system (23) is equivalent to

$$(24) \quad \begin{aligned} \frac{dz_1}{d\tau_6} &= \sqrt{6}(1 - z_1^2), \\ \frac{dz_2}{d\tau_6} &= z_1 \left( -3z_1 + \frac{\sqrt{6}}{n}z_2 \right). \end{aligned}$$

Since  $(0, 0)$  is the non-equilibrium point of system (24), there is no need to continue to investigate the equilibrium points at infinity in  $U_2$ . These have been discussed in the chart  $U_1$ .

On the local chart  $U_3$  we have Poincaré compactification  $x = z_1/z_3$ ,  $z = z_2/z_3$ ,  $s = 1/z_3$ , then system (7) writes

$$(25) \quad \begin{aligned} \frac{dz_1}{dN} &= z_1 \left[ -\frac{\sqrt{6}(1+n)}{n}z_1 + 3z_1^2 - 3z_3^2 \right] + \sqrt{6}(z_2^2 + z_3^2), \\ \frac{dz_2}{dN} &= z_1 z_2 \left( -\frac{\sqrt{6}}{n} + 3z_1 \right), \\ \frac{dz_3}{dN} &= -\frac{\sqrt{6}}{n}z_1 z_3. \end{aligned}$$

It can be followed from system (25) that

$$(26) \quad \begin{aligned} \frac{dz_1}{dN} &= z_1 \left[ -\frac{\sqrt{6}(1+n)}{n}z_1 + 3z_1^2 \right] + \sqrt{6}z_2^2, \\ \frac{dz_2}{dN} &= z_1 z_2 \left( -\frac{\sqrt{6}}{n} + 3z_1 \right). \end{aligned}$$

Note that the origin  $(0, 0)$  is a linearly zero equilibrium point of the above system. According to the Poincaré-Hopf Theorem, the topological index is zero. The vertical blow-up technique will be applied to

investigate its local phase portrait. Then defining  $w = z_2/z_1$  we have

$$(27) \quad \begin{aligned} \frac{dz_1}{dN} &= z_1^2 \left[ \sqrt{6} \left( -\frac{1+n}{n} + w^2 \right) + 3z_1 \right], \\ \frac{dw}{dN} &= \sqrt{6} z_1 w (w^2 - 1). \end{aligned}$$

Rescaling the time  $d\tau_7 = z_1 dN$  and eliminating the common factor  $z_1$  of system (27), then we obtain

$$(28) \quad \begin{aligned} \frac{dz_1}{d\tau_7} &= z_1 \left[ \sqrt{6} \left( -\frac{1+n}{n} + w^2 \right) + 3z_1 \right], \\ \frac{dw}{d\tau_7} &= \sqrt{6} w (w^2 - 1). \end{aligned}$$

Since there are three hyperbolic equilibrium points  $e_{i,4} = (0, -1)$ ,  $e_{i,5} = (0, 1)$  and  $e_{i,6} = (0, 0)$  of system (28) on  $z_1 = 0$ , the previous two are stable nodes with eigenvalues of  $-2\sqrt{6}$  and  $-\sqrt{6}/n$ , the last one is an unstable saddle point with eigenvalues of  $-\sqrt{6}(n+1)/n$  and  $\sqrt{6}$ . We note that this is the same as the equilibrium points in the system (26) of reference [6]. So the local phase portraits of systems (28), (27) and (26) are shown in Figures 7(a), 7(b) and 7(c), respectively, and Figure 8 shows the phase portrait at the origin of the local chart  $U_3$ .

Combined with the previous discussion, Figure 9 shows the global phase portrait at infinity on the Poincaré sphere.

## 5. PHASE PORTRAITS WITHIN THE POINCARÉ SPHERE

CONDITIONED TO  $x^2 - z^2 \leq 1$

The dynamical system (7) is invariant under the two symmetry transformations about the origin and the  $z$ -axis, i.e.,  $(x, z, s) \mapsto (-x, -z, -s)$  and  $(x, z, s) \mapsto (-x, z, -s)$ . So we divide the Poincaré ball restricted

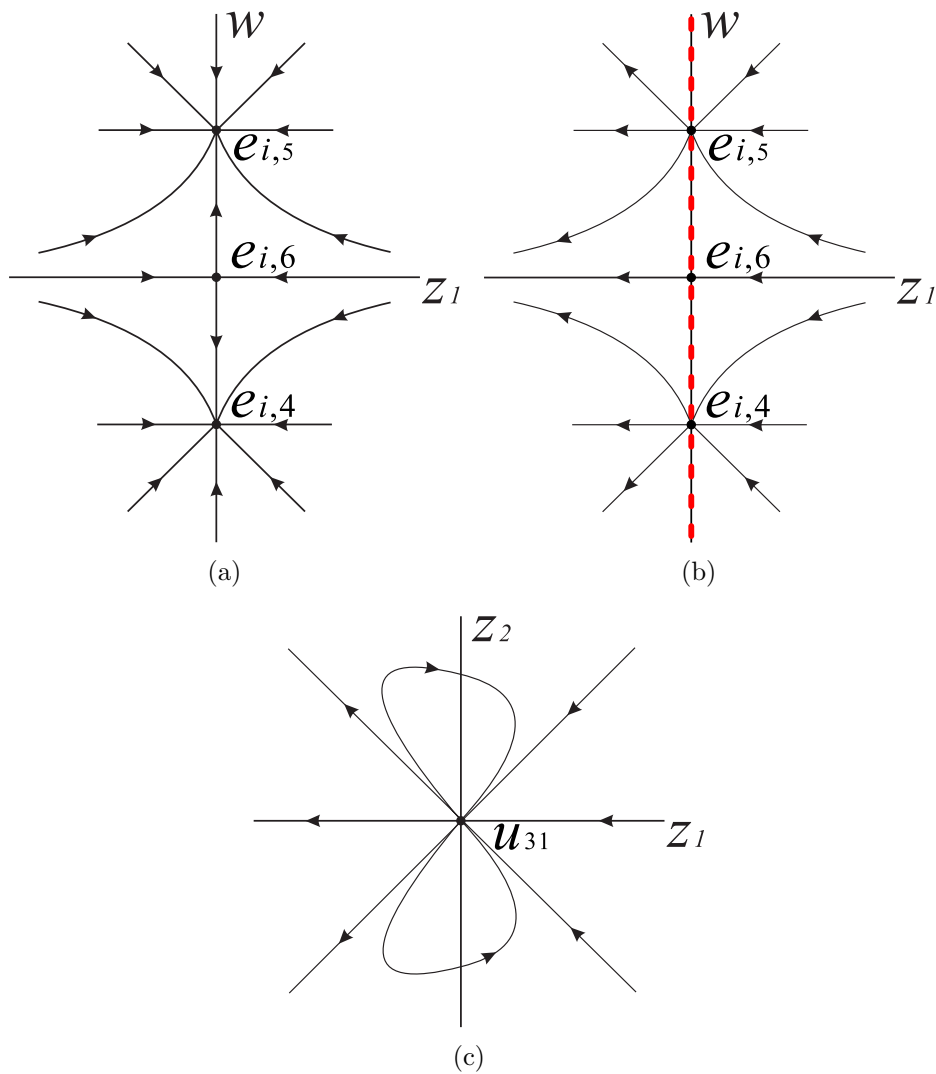


FIGURE 7. The local phase portraits (a), (b), and (c) of the above equilibrium points correspond to systems (28), (27), and (26), respectively.

to the region  $G$  into four regions as follows

$$R_1 : z \leq 0, s \geq 0. \quad R_3 : z \geq 0, s \geq 0.$$

$$R_2 : z \leq 0, s \leq 0. \quad R_4 : z \geq 0, s \leq 0.$$

In view of the aforementioned symmetries, we only need to focus on the phase portrait of system (7) in one region (such as  $R_1$ ).

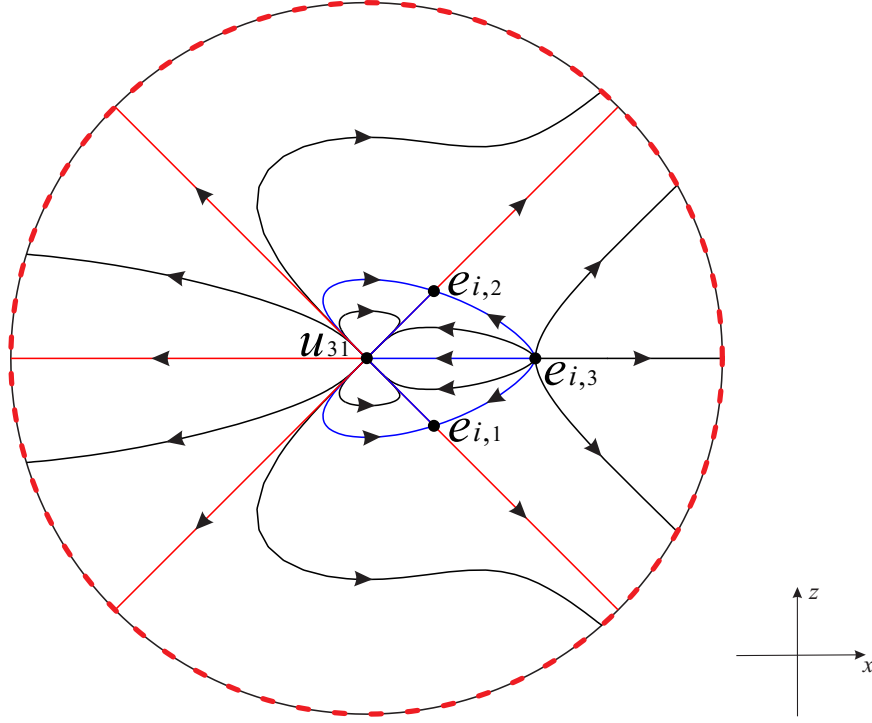


FIGURE 8. The phase portrait of system (7) in the local chart  $U_3$  at infinity.

Combining the phase portrait in the invariant planes  $s = 0$ , and  $z = 0$  with the phase portrait in the invariant surface  $x^2 - z^2 = 1$ , and the phase portrait at infinity, the phase portrait on the boundary surface of  $R_1$  is as shown in Figure 10. Now we divide the boundary surface of the region  $R_1$  into six parts (see Figure 11 for more details), to show the phase portrait of  $R_1$  more clearly. It can be found from Figures 10(a) and 10(b) that the equilibrium point  $u_{31}$  of the Poincaré ball is stable on the front boundary surfaces  $F_1$  and  $F_2$ , and there is a stable parabolic sector and an elliptic sector segment  $F_3$ . However, on the back boundary surfaces  $B_1$  and  $B_2$ , the north pole  $u_{31}$  is unstable.

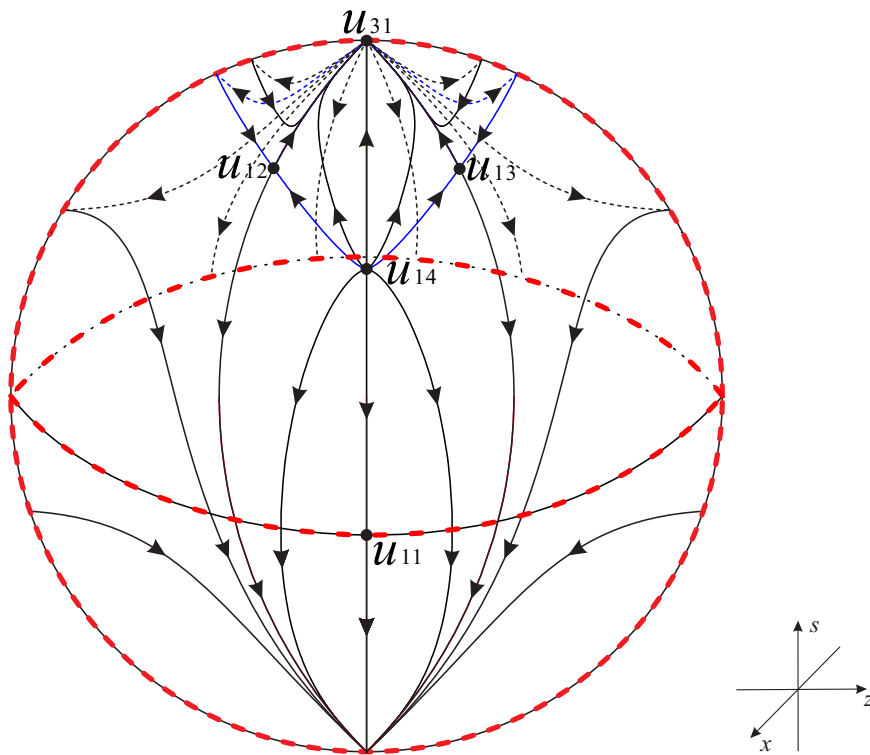


FIGURE 9. The global phase portrait of system (7) on the Poincaré sphere at infinity.

5.1. **Dynamics inside the region  $R_1$ .** System (7) has three finite equilibrium points  $p_0$ ,  $p_1$  and  $p_2$ . The dynamical behavior of system (7) in the Interior of the region  $R_1$  depends on the comprehensive performance of the flow in the surface and planes

$$h(x, z, s) = 0, \quad x = 0, \quad z = 0, \quad s = 0,$$

where

$$h(x, z, s) = \sqrt{6}s(z^2 - x^2 + 1) + 3x(x^2 - 1).$$

The above planes and surface cut the region  $R_1$  into four different subregions  $S_i$ ,  $i = (1, \dots, 4)$ , see Figures 12 and 13 for more details. It is noted that  $h < 0$  in the subregions  $S_1$  and  $S_4$ , and  $h > 0$  in the

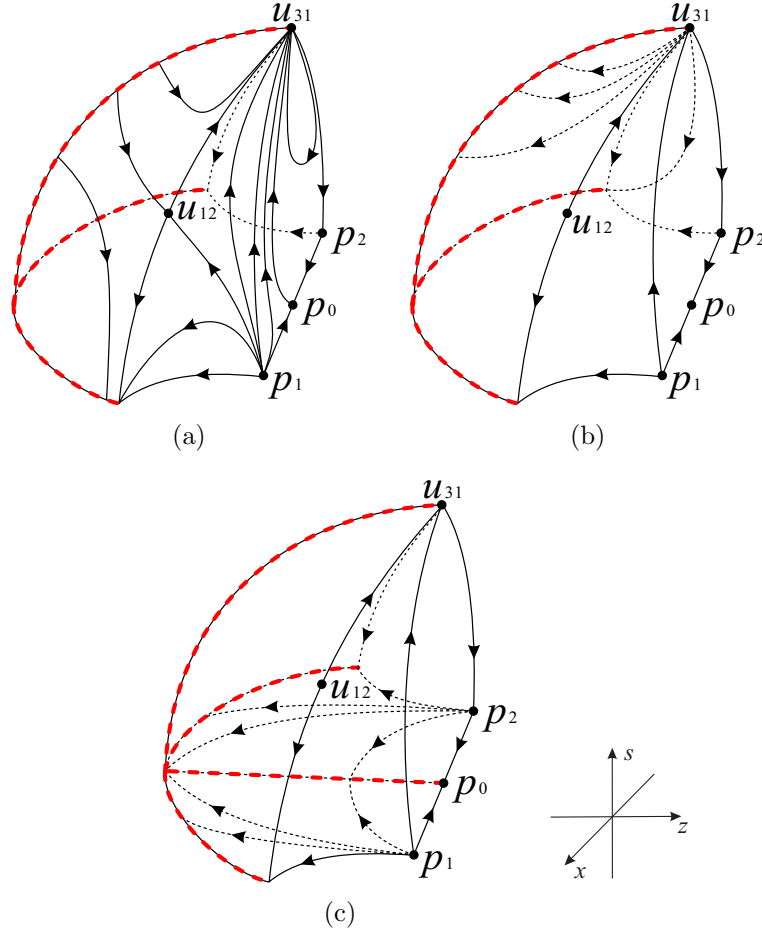


FIGURE 10. Phase portraits on the boundary surface  $R_1$ :  
 (a). The front boundary surface; (b). The back boundary surface; (c). The bottom plane.

subregions  $S_2$  and  $S_3$ . To avoid confusion, please note that the solid and dotted lines in Figure 13 is consistent with those in Figure 12.

As it is shown in  $S_1$  (see Figure 13(a)) the upper surface is included in the blue surface  $h = 0$ , and the bottom plane is included in the invariant plane  $s = 0$ . According to Table 2, the orbits monotonically decrease in the  $x$  and  $z$  directions. In the  $s$  direction, they monotonically increase, so the orbits in the subregion  $S_1$  can only start at the finite equilibrium point  $p_1$  and pass through the upper surface of  $S_1$  into the subregion

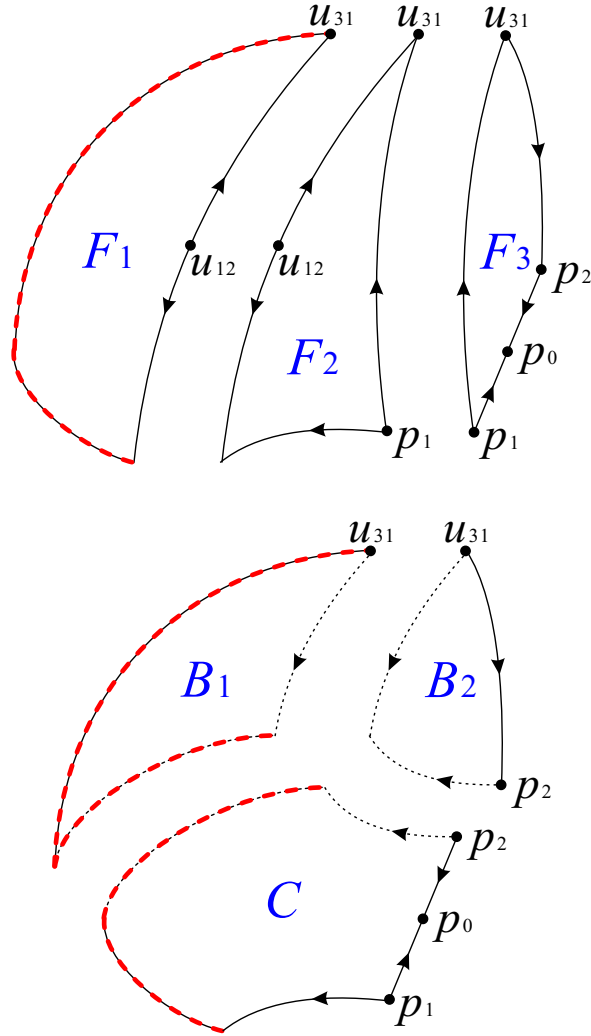


FIGURE 11. The six boundaries of  $R_1$ .

$S_2$ . However in subregion  $S_2$  (see Figure 13(b), i.e. the remaining part after  $S_1$  is extracted from the region  $R_1$  when  $x > 0$ , the orbits monotonically increase in the  $x$  and  $s$  directions but monotonically decrease in the  $z$ -direction. So we find that the orbits in  $S_2$  can only start at  $p_1$  or the equilibrium points on the negative  $z$ -axis, and they eventually move towards the infinity equilibrium point  $u_{12}$  fixed on the surface of  $S_2$ . Therefore the dynamic behavior of the orbits on these

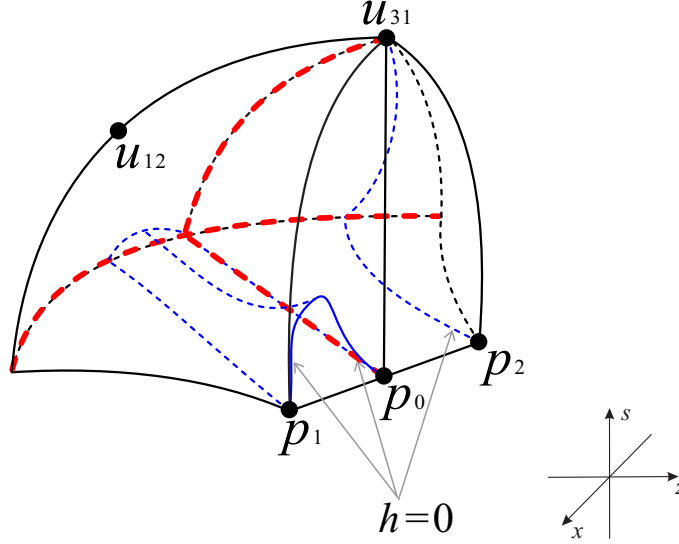


FIGURE 12. There are four subregions inside the region  $R_1$  of the Poincaré ball.

two subregions can be represented in the following form

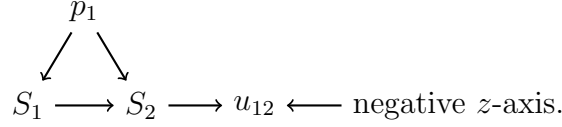


TABLE 2. The dynamics of the four subregions.

Subregions	Associated Region	Monotonicity
$S_1$	$h < 0, x > 0, z < 0, s > 0$	$\dot{x} < 0, \dot{z} < 0, \dot{s} > 0$
$S_2$	$h > 0, x > 0, z < 0, s > 0$	$\dot{x} > 0, \dot{z} < 0, \dot{s} > 0$
$S_3$	$h > 0, x < 0, z < 0, s > 0$	$\dot{x} > 0, \dot{z} < 0, \dot{s} < 0$
$S_4$	$h < 0, x < 0, z < 0, s > 0$	$\dot{x} < 0, \dot{z} < 0, \dot{s} < 0$

In subregion  $S_3$  (see Figure 13(c)) the left vertical plane facing us is contained in the plane  $x = 0$ , and the right vertical plane is contained



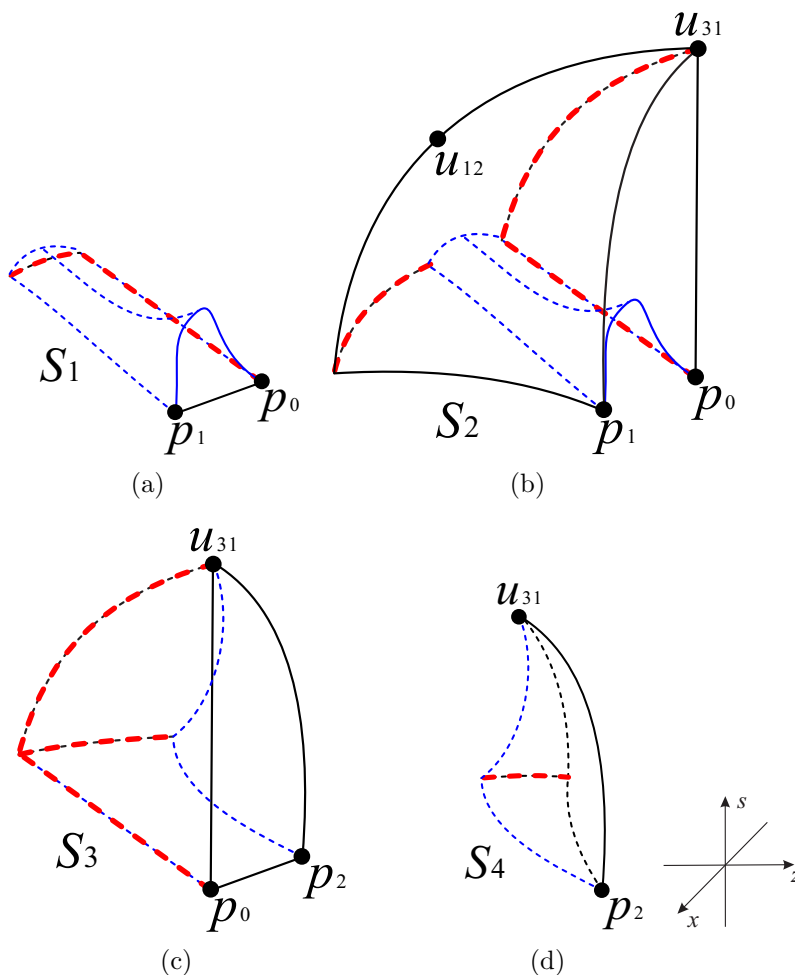
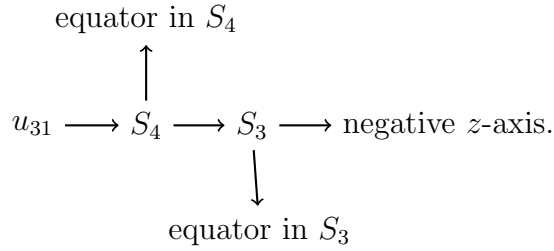


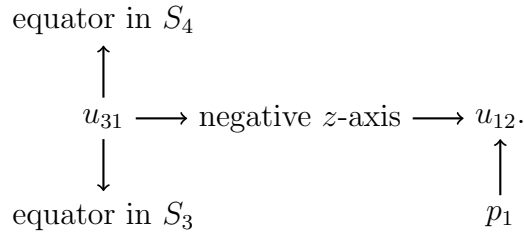
FIGURE 13. The four subregions  $S_i$  of  $R_1$  ( $i = 1 \cdots , 4$ ).

in the invariant plane  $z = 0$ . In the opposite surfaces the left one is included in the surface of the Poincaré sphere, and the right one is included in the surface  $h = 0$ . Since the orbits in  $S_3$  monotonically decrease in the  $z$  and  $s$  directions, and they increase monotonically in the  $x$  direction, it follows that the orbits will eventually go to the equilibrium point which lie in the equator or go to the negative  $z$ -axis in that region, and the orbits in this region may only come from the adjacent subregion  $S_4$  (see Figure 13(d)). The left and right surfaces

of subregion  $S_4$  belong to the surfaces  $h = 0$  and  $x^2 - z^2 = 1$  ( $x < 0$ ), respectively. Since all the orbits are monotonically decreasing in  $S_4$ , the orbits in this region arrive from the infinite equilibrium point  $u_{31}$ , i.e. north pole of the Poincaré ball, and tend to the infinite equilibrium points on the equator of this region at a future date, or enter into the subregion  $S_3$ . This dynamical behavior can be represented as follows



Therefore the orbits' dynamic behavior inside the four subregions of  $R_1$  investigated above can be condensed to



This flow chart states clearly that the orbits of system (7) included in  $R_1$  have  $\alpha$ -limit at  $p_1$  and the north pole  $u_{31}$  (also referred to as past attractors or negative attractors) of the Poincaré sphere. Additionally the orbits have  $\omega$ -limit either at  $u_{12}$  (also called future attractor or positive attractor), or other infinite equilibrium points on the equator of subregions  $S_3$  and  $S_4$ , which are located at the intersection of the Poincaré ball and at the infinity of the invariant planes  $s = 0$  in  $\mathbb{R}^3$  when  $x < 0$ , see Figure 12 or Figures 13(c) and 13(d).

Therefore all the global dynamical behavior of the system (7) is represented qualitatively and completely.

## 6. CONCLUSIONS

In the present paper we have fully described the global phase portrait of Hořava-Lifshitz cosmology in the presence of non-zero cosmological constant and zero curvature in the region of physical interest  $G$ . By taking the fact that the cosmological equations remains invariant under the two symmetries mentioned in section 5, the global phase portrait of the cosmological model in  $G$  is provided completely.

From the perspective of cosmology, combining the phase portrait analysis of system (7) in Sections 3-5 and the formulations of the density parameter and the equation-of-state parameter of dark energy in [10], we know that the unstable finite equilibrium points  $p_1$  and  $p_2$  are dominated by dark matter with negligible dark energy, and the finite equilibrium  $p_0$  located on the equilibrium point line may also be characterized by more physical dark matter if the initial conditions are not in the invariant plane  $z = 0$ . Besides the initial conditions in the invariant planes  $s = 0$  and  $z = 0$  as well as on the backside of the invariant surface  $x^2 - z^2 = 1$ , the phase portrait displays that the eventual evolution of the orbits of the cosmological model in  $G$  tends to the infinite equilibrium point  $u_{12}$ , which can be the late-time state of the universe and to other infinite equilibrium points placed at the equator of the Poincaré ball. For the Hořava-Lifshitz gravity in a FLRW space-time with  $k = 0$  and  $\Lambda \neq 0$ , equations (5) implies that the Hubble parameter  $H$  tends to zero in forward time in this cosmological model, and this is an important difference between the Hořava-Lifshitz model

and the standard  $\Lambda$ CDM model where the universe evolves to a purely  $\Lambda$ -dominated state with constant non-zero Hubble parameter.

#### APPENDIX: THE POINCARÉ COMPACTIFICATION IN $\mathbb{R}^3$ [28]

For a polynomial vector field  $X = (P_1, P_2, P_3)$ , and the degree  $n = \max\{\deg(P_i) : i = 1, 2, 3\}$ , its differential system is

$$\frac{dx}{dN} = P_1(x, y, z), \quad \frac{dy}{dN} = P_2(x, y, z), \quad \frac{dz}{dN} = P_3(x, y, z).$$

Defining the unit sphere in  $\mathbb{R}^4$  by  $\mathbb{S}^3 = \{y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4 : \|y\| = 1\}$ , we denote the northern hemisphere by  $\mathbb{S}_+ = \{y \in \mathbb{S}^3 : y_4 > 0\}$ , the southern hemisphere by  $\mathbb{S}_- = \{y \in \mathbb{S}^3 : y_4 < 0\}$ , the equator of  $\mathbb{S}^3$  by the 2-sphere  $\mathbb{S}^2 = \{y \in \mathbb{S}^3 : y_4 = 0\}$ , the tangent space at the point  $y$  of  $\mathbb{S}^3$  by  $T_y\mathbb{S}^3$ . Hence the tangent hyperplane  $T_{(0,0,0,1)}\mathbb{S}^3 = \{(x_1, x_2, x_3, 1) \in \mathbb{R}^4 : (x_1, x_2, x_3) \in \mathbb{R}^3\}$  is identified with  $\mathbb{R}^3$ . Moreover let

$$f_+ : \mathbb{R}^3 = T_{(0,0,0,1)}\mathbb{S}^3 \rightarrow \mathbb{S}_+, \quad f_+(x) = \frac{1}{\Delta x}(x_1, x_2, x_3, 1),$$

and

$$f_- : \mathbb{R}^3 = T_{(0,0,0,1)}\mathbb{S}^3 \rightarrow \mathbb{S}_-, \quad f_-(x) = -\frac{1}{\Delta x}(x_1, x_2, x_3, 1),$$

be the two central projections, where  $\Delta x = (\sum_{i=1}^3 x_i^2 + 1)^{1/2}$ . Then  $f_+$  and  $f_-$  are identified by  $\mathbb{R}^3$  with the two hemispheres of  $\mathbb{S}^3$ . These two central projections clarify two transcripts of  $X$ , i.e.  $Df_+ \circ X$  in  $\mathbb{S}_+$ , and  $Df_- \circ X$  in  $\mathbb{S}_-$ . Let  $\tilde{X}$  be the vector field on  $\mathbb{S}^3 \setminus \mathbb{S}^2 = \mathbb{S}_+ \cup \mathbb{S}_-$ .

Now we analytically continue the vector field  $\tilde{X}(y)$  to the entire sphere  $\mathbb{S}^3$  by  $p(X)(y) = y_4^{n-1}\tilde{X}(y)$ . The continued vector field  $p(X)$  is named *Poincaré compactification* of  $X$ . We note that the infinity

of  $\mathbb{R}^3$  represented by  $\mathbb{S}^2$  is invariant for the vector field  $p(X)$ . The compactification for polynomial vector fields in  $\mathbb{R}^2$  was introduced by Poincaré, and one can find its extension to  $\mathbb{R}^m$  from [30]. Next we shall study the orthogonal projection of the closed  $\mathbb{S}_+$  to  $y_4 = 0$ , which is a closed ball  $B$  (designated Poincaré ball) with radius 1, its inner area is diffeomorphic to  $\mathbb{R}^3$ , and its boundary surface  $\mathbb{S}^2$  is identified with  $\mathbb{R}^3$  at infinity.

Since  $\mathbb{S}^3$  is a differentiable manifold, we must consider eight local charts  $(U_i, F_i)$  and  $(V_i, G_i)$  in order to study the dynamics of  $p(X)$ , where

$$U_i = \{y \in \mathbb{S}^3 : y_i > 0\}, \quad V_i = \{y \in \mathbb{S}^3 : y_i < 0\},$$

and the diffeomorphisms

$$F_i : U_i \rightarrow \mathbb{R}^3, \quad G_i : V_i \rightarrow \mathbb{R}^3 \quad \text{for } i = 1, 2, 3, 4,$$

are the central projections' inverses from the origin to the center of the tangent planes at the points  $(\pm 1, 0, 0, 0)$ ,  $(0, \pm 1, 0, 0)$ ,  $(0, 0, \pm 1, 0)$  and  $(0, 0, 0, \pm 1)$ , respectively. Then the expression of  $p(X)$  in the local chart  $U_1$  is

$$\frac{z_3^n}{(\Delta z)^{n-1}}(-z_1 P_1 + P_2, -z_2 P_1 + P_3, -z_3 P_1),$$

where  $P_i = P_i(1/z_3, z_1/z_3, z_2/z_3)$ . In the local chart  $U_2$  we have

$$\frac{z_3^n}{(\Delta z)^{n-1}}(-z_1 P_2 + P_1, -z_2 P_2 + P_3, -z_3 P_2),$$

where  $P_i = P_i(z_1/z_3, 1/z_3, z_2/z_3)$ . In the local chart  $U_3$  we obtain

$$\frac{z_3^n}{(\Delta z)^{n-1}}(-z_1 P_3 + P_1, -z_2 P_3 + P_2, -z_3 P_3),$$

where  $P_i = P_i(z_1/z_3, z_2/z_3, 1/z_3)$ . In the local chart  $U_4$  we get

$$z_3^{n-1}(P_1, P_2, P_3),$$

where  $P_i = P_i(z_1, z_2, z_3)$ . Furthermore the demonstration of  $p(X)$  in the local chart  $V_i$  is the same as in the  $U_i$  multiplied by  $(-1)^{n-1}$ .

In addition the above factor  $1/(\Delta z)^{n-1}$  is omitted by doing a time rescaling when we use the expressions of the compactified vector field  $p(X)$  in the local charts.

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