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LIMIT CYCLES OF LINEAR VECTOR FIELDS ON $(\mathbb{S}^2)^m \times \mathbb{R}^n$

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It is well known that linear vector fields defined in $\mathbb{R}^n$ cannot have limit cycles, but this is not the case for linear vector fields defined in other manifolds. We study the existence of limit cycles bifurcating from a continuum of periodic orbits of linear vector fields on manifolds of the form $(\mathbb{S}^2)^m \times \mathbb{R}^n$ when such vector fields are perturbed inside the class of all linear vector fields. The study is done using averaging theory. We also present an open problem about the maximum number of limit cycles of linear vector fields on $(\mathbb{S}^2)^m \times \mathbb{R}^n$.

1. Introduction and statement of the main results

The study of periodic orbits of differential systems plays an important role in the qualitative theory of ordinary differential equations and their applications. A *limit cycle* is defined as a periodic orbit of a differential system which is isolated in the set of all periodic orbits of the system. Among the many works devoted to limit cycles and their applications, we mention [Christopher and Lloyd 1996; Giacomini et al. 1996; Han and Li 2012; Ilyashenko 2002].

It is well known that linear vector fields in $\mathbb{R}^n$ cannot have limit cycles, but this is not the case if one considers linear vector fields in other manifolds different from $\mathbb{R}^n$. The objective of this paper is to study the existence of limit cycles of linear vector fields defined on the manifolds $(\mathbb{S}^2)^n \times \mathbb{R}^n$.

The problem of studying limit cycles of linear vector fields on manifolds different from $\mathbb{R}^n$ was already treated in [Llibre and Zhang 2016], where the authors consider linear vector fields on $\mathbb{S}^m \times \mathbb{R}^n$, and they conjecture that such vector fields may have at most one limit cycle.

Linear autonomous differential systems, namely, systems of the form $\dot{x} = Ax + b$, where A is a $n \times n$ real matrix and b is a vector in $\mathbb{R}^n$, are the easiest systems to study because their solutions can be completely determined (see [Arnold 2006; Sotomayor 1979]), but still they play an important role in the theory of differential

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