Central configurations in the five-body problem: Rhombus plus one

J. Lino Cornelio, Martha Alvarez-Ramírez and Josep M. Cors

Abstract. We show the existence of central configurations in the planar five-body problem where four bodies are located at the vertices of a rhombus, called rhombus plus one central configurations. Concretely we prove analytically their existence when one diagonal is nearly equal to the sides of the rhombus and when the two diagonals are either equal or nearly equal. In addition, we prove that given a rhombus plus one configuration, the corresponding vector of positive masses that makes the configuration central, if exists, is unique.

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1. Introduction

In the n-body problem a configuration is *central* if the acceleration vector of every body is proportional, same common scalar, to its position vector with respect to the center of mass. There are many reasons why such configurations are of special importance, and regardless of everything many fundamental questions about them are still open. For instance, they allow to obtain explicit solutions of the n-body problem, concretely, solutions where the shape of the configuration is preserved along the orbit up to rescaling and rotations. In [15, 16, 18] and references therein, the reader can found a detailed introduction about central configurations, together with new ideas and techniques which have been developed recently in their study.

The problem of the finiteness of central configurations for every choice of the positive masses, also known as the Chazy-Wintner conjecture, was included in Smale's list as a challenge question for the 21st century [20]. It is known that for $n = 3$ there are exactly five classes of central configurations

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for each choice of positive masses, counting up rotations and translations in the plane. Namely, three collinear, when the three bodies lie on the same line, and two equilateral, when the three bodies are located at the vertices of an equilateral triangle. When $n = 4$, Hampton and Moeckel [10] proved by computer-assisted that for each choice of masses there is a finite number of central configurations. Recently, for the planar five-body problem the conjecture has been solved by Albouy and Kaloshin [1], except for masses in some codimension two variety of the mass space. Hampton and Jensen [12] showed that in the five-body problem the number of spatial central configurations is finite, except for some special cases. So, the finiteness conjecture is wide open for $n \geq 6$.

The determination of possible shapes for central configurations, at least, the ones possessing some type of symmetry or defining a geometric property, is a relevant issue analogous to the finiteness conjecture. In spite of the fact that the finiteness conjecture is now settled for $n = 5$, in the planar case, there are still many unsolved problems regarding shapes of five-body central configurations. Williams ([23]) and Chen and Hsiao ([2]) studied the existence of convex central configurations in the five-body problem and gave some geometric properties. Apart from these fundamental results, little more is known, in contrast with the planar four-body problem, where recently, Corbera, Cors and Roberts [3] have classified the full set of convex central configurations.

In this paper we will investigate symmetric central configurations in the five-body problem where three bodies lie on the axis of symmetry of the configuration. Similar symmetric configurations were previously studied in [4, 5, 7, 13]. Works when only one of the five bodies lie on the axis of symmetry are in [9, 14]. Notice that in all the previous cited works the central configurations studied are staked, that is, a proper subset of the configuration is also a central configuration. See [9], where the concept of stacked central configurations was introduced. Since in the present work we consider configurations that, in addition to the axial symmetry, four of the five bodies lies at the vertices of a rhombus, such a central configurations will not be stacked anymore due to a result of Fernandes and Mello [6]. They proved that the only stacked central configuration of the five-body problem when one body is removed is the square with equal positive masses at its vertices and the removed body, with any positive mass, located at its center.

Among previous cited works on the five-body problem considering three bodies on the axis of symmetry, we distinguish the article of Shoaib et al. [21] for its similitude with the present paper. The authors, as in our set up, in addition to the axial symmetry, they impose that four of the five bodies have to be located at the vertices of a rhomboid, although a careful reading suggest that the use of term kite instead a rhomboid should be more accurate, see Fig. 1 in [21].

Besides of we consider four bodies at the vertices of a rhombus is worthy to mention that in our work any constraint on the masses is considered. We present in Figure 2 a complete classification of all central configurations of the five-body problem where in addition to place three bodies on the axis of symmetry of the configuration, four of the five bodies are at the vertices of a rhombus, that we call *rhombus plus one* configuration. Concretely, in Section 2, we introduce the equations for central configurations and state the two main theorems that insure analytically the existence of rhombus plus one central configurations in the five-body, when one diagonal is nearly equal to the sides of the rhombus and when the two diagonal are equal. Section 3 is devoted to the unicity question of what is called the inverse problem. We prove that, for a given rhombus plus one configuration, the corresponding vector of positive masses that makes the configuration central, if exists, is unique. Finally, the two existence theorems are proved in sections 4 and 5.

2. Statement of the problem and main results

Let $\mathbf{q}_i \in \mathbb{R}^2$ and $m_i > 0$ denote the position and mass, respectively, of the i-th body. Let $r_{ij} = \|\mathbf{q}_i - \mathbf{q}_j\|$ be the Euclidean distance between the i-th and j-th bodies. If $M = \sum_{i=1}^5 m_i$ denotes the sum of the masses, then the center of mass is given by $\mathbf{c} = \frac{1}{M}\sum_{i=1}^5 m_i\mathbf{q}_i$. The planar five-body problem is governed by the equations

$$m_i\ddot{\mathbf{q}}_i = \frac{\partial U}{\partial \mathbf{q}_i}, \qquad i = 1, \ldots, 5 \tag{2.1}$$

where U denotes the Newtonian potential given by

$$U(\mathbf{q}) = \sum_{1\le i<j\le 5} \frac{m_i m_j}{r_{ij}}.$$

The system (2.1) is smooth except on the collision set given by

$$\triangle = \{\mathbf{q} \in \mathbb{R}^{10} : \mathbf{q}_i = \mathbf{q}_j \quad \text{for some} \quad i \neq j\}.$$

A non-collision configuration $\mathbf{q} = (\mathbf{q}_1, \ldots, \mathbf{q}_5) \in \mathbb{R}^{10} \setminus \triangle$ of five bodies form a *central configuration* for the positive masses $m_1, m_2, \ldots, m_5$ if there exists a constant $\lambda \neq 0$ such that

$$\sum_{i\neq j} \frac{m_j}{r_{ij}^3}(\mathbf{q}_i - \mathbf{q}_j) = -\lambda(\mathbf{q}_i - \mathbf{c}). \qquad i = 1, 2, \ldots, 5 \tag{2.2}$$

An equivalent set of equations for central configurations (2.2), in terms of the mutual distances r_{ij}, is given by the *Dziobek/Laura/Andoyer equations* (see page 241 in [8])

$$f_{ij} = \sum_{\substack{k=1\\k\neq i,j}}^{5} m_k(R_{ik} - R_{jk})\Delta_{ijk} = 0, \tag{2.3}$$

for $1 \le i < j \le 5$. Here, $R_{ij} = 1/r_{ij}^3$ and $\Delta_{ijk} = (\mathbf{q}_i - \mathbf{q}_j) \wedge (\mathbf{q}_i - \mathbf{q}_k)$. Thus, Δ_{ijk} gives twice the signed area of the triangle with vertices $\mathbf{q}_i$, $\mathbf{q}_j$ and $\mathbf{q}_k$.

Consider now a five-body configuration where four bodies are located at the vertices of a rhombus. It follows easily from the perpendicular bisector theorem [15], that the remaining body only can be placed along one of the two diagonals of the rhombus. Then, without loss of generality, we consider a configuration where fives bodies m_1, m_2, m_3 are collinear and m_4, m_5 are placed symmetrically with respect to the line that contains the first three bodies. Finally we impose that m_2, m_3, m_4 and m_5 are at the vertices of a rhombus. We distinguish two cases depending on where is located the body m_1, outside or inside the rhombus. See Figure 1 (a) and (b), respectively.

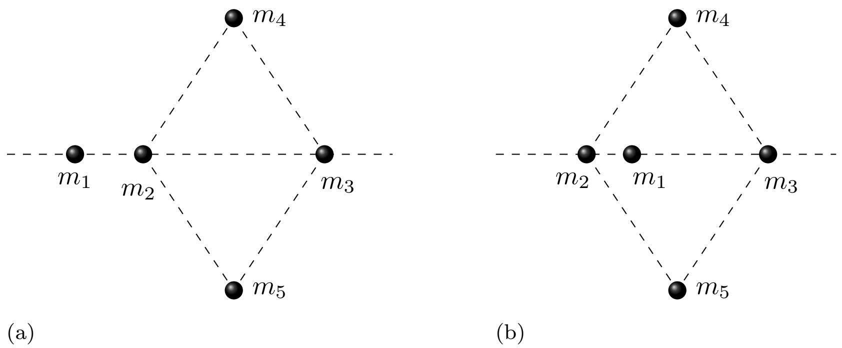

Figure 1. Configuration of five-body problem with four masses at the vertices of a rhombus and the fifth mass lying on the axis of symmetry outside (a) or inside (b) the rhombus.

Due to the symmetries of the configuration several of the mutual distances are equal. Concretely, $r_{14} = r_{15}$, $r_{24} = r_{25}$, $r_{34} = r_{35}$. Also happen with some of the signed areas, namely $\Delta_{125} = -\Delta_{124}$, $\Delta_{135} = -\Delta_{134}$ and $\Delta_{235} = -\Delta_{234}$. Moreover, since m_1, m_2, m_3 are collinear $\Delta_{123} = 0$. At last, the fact that m_2, m_3,m_4 and m_5 form a rhombus implies that $r_{24} = r_{34}$ and $\Delta_{345} = -\Delta_{245}$.

Using previous equalities, equation f_{12} in (2.3) becomes

$$(R_{14} - R_{24})(m_4 - m_5)\Delta_{124} = 0. \tag{2.4}$$

Clearly $\Delta_{124} \neq 0$ and $R_{14} \neq R_{24}$, otherwise collision occurs between m_1 and m_2. Then $m_4 = m_5$. Same result can be reached using equations f_{13} or f_{23} in (2.3). Since the masses can be scaled by any positive factor we fix $m_4 = m_5 = 1$ without any loss of generality.

Moreover, equation f_{45} in (2.3) holds trivially and $f_{i4} + f_{i5} = 0$, for $i = 1, 2, 3$. Then system (2.3) can be reduced to the following three equations

$$\begin{aligned} f_{14} &= (R_{24} - R_{12})\Delta_{124}m_2 + (R_{24} - R_{13})\Delta_{134}m_3 + (R_{14} - R_{45})\Delta_{145} = 0, \\ f_{24} &= (R_{12} - R_{14})\Delta_{124}m_1 + (R_{24} - 1)\Delta_{234}m_3 + (R_{24} - R_{45})\Delta_{245} = 0, \\ f_{34} &= (R_{13} - R_{14})\Delta_{134}m_1 + (1 - R_{24})\Delta_{234}m_2 - (R_{24} - R_{45})\Delta_{245} = 0. \end{aligned} \tag{2.5}$$

Notice that in equations (2.5) the value of the diagonal of the rhombus r_{23} has fixed to one, that is, $r_{23} = 1$, which specifies a particular choice of scaling of the configuration.

In the Section 3, we will be illustrating that the masses are easily obtained solving equations (2.5) and can be expressed as follows

$$\begin{aligned} m_1 &= \frac{\Delta_{234}(R_{24}-1)N_1}{\Delta_{124}\Delta_{134}\Delta_{234}(R_{24}-1)(R_{14}-R_{24})(R_{12}-R_{13})} = \frac{N_1}{\Delta_{124}\Delta_{134}(R_{14}-R_{24})(R_{12}-R_{13})}, \\ m_2 &= \frac{\Delta_{134}N_2}{\Delta_{124}\Delta_{134}\Delta_{234}(R_{24}-1)(R_{14}-R_{24})(R_{12}-R_{13})} = \frac{N_2}{\Delta_{124}\Delta_{234}(R_{24}-1)(R_{14}-R_{24})(R_{12}-R_{13})}, \\ m_3 &= \frac{\Delta_{124}N_3}{\Delta_{124}\Delta_{134}\Delta_{234}(R_{24}-1)(R_{14}-R_{24})(R_{12}-R_{13})} = \frac{N_3}{\Delta_{134}\Delta_{234}(R_{24}-1)(R_{14}-R_{24})(R_{12}-R_{13})}, \end{aligned} \tag{2.6}$$

where N_1, N_2 and N_3 can be written as a function of the mutual distances r_{ij}.

Our main goal is to prove the existence of rhombus plus one central configurations in the five-body problem. Concretely, we prove the following two theorems.

Theorem 2.1. *Consider a five-body configuration where four bodies form a rhombus. Then, there exist central configurations of the five-body problem when the diagonal containing the fifth body is nearly equal but bigger to the sides of the rhombus, either inside or outside.*

Theorem 2.2. *Consider a five-body configuration where four bodies form a square. Then, there exist central configurations of the five-body problem when the fifth body lies in one diagonal, either inside or outside the square.*

It would be natural to think that there exists a large set of rhombus plus one central configurations apart from those stated by the two theorems. In Figure 2, we show numerical evidence of, indeed, the set of rhombus plus one central configurations is much larger. In fact, fixing to one the length of the diagonal of the rhombus that contains the fifth body, the other diagonal of the rhombus can reach any value from $\sqrt{3}/3$ to $\sqrt{3}$. Moreover, for any value between $\sqrt{3}/3$ and $\sqrt{3}$ of the diagonal of the rhombus, that not contains the fifth mass is not equal to 1, there exist two segments along the axis of symmetry where the fifth body can be placed, one outside the rhombus and the other inside. Clearly, from the symmetry of the configuration, when

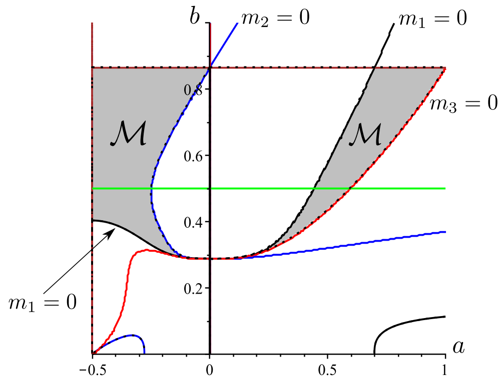

FIGURE 2. The open shaded region $\mathcal{M}$ in the (a, b)-plane. Any point belonging to $\mathcal{M}$ corresponds to a rhombus plus one central configuration of the five-body problem. Straight line $b = 1/2$ represents the case where the rhombus becomes a square.

the fifth body, is inside the rhombus, is enough to study when it is located between the closer vertex of the rhombus and the intersection of the two diagonals.

Let a be the signed distance between bodies with masses m_1 and m_2, that is, $a > 0$ when m_1 lies outside the rhombus and $a < 0$ when m_1 lies inside the rhombus, $|a| = r_{12}$, and let b be the half of the distance of the diagonal of the rhombus joining bodies with masses m_4 and m_5, that is $2b = r_{45}$. Then any point (a, b)-plane belonging to the open shaded region $\mathcal{M}$ in Figure 2 represents a rhombus plus one central configuration.

Boundary of $\mathcal{M}$ is given by curves where the values of the masses are zero, $m_1 = 0$ (black), $m_2 = 0$ (blue) and $m_3 = 0$ (red), and curves where the denominators in (2.6) are zero, the straight lines $b = \sqrt{3}/2$ ($R_{24} = 1$) and $a = -1/2$ ($R_{12} = R_{13}$), both in brown.

The particular case where the rhombus becomes a square is represented in Figure 2 by the green straight line $b = 1/2$ ($R_{45} = 1$) and has nonempty intersection with $\mathcal{M}$. In Figure 3, we show the values of the masses m_1 (black), m_2 (blue) and m_3 (red) along the set of square plus one central configurations. When the fifth mass is outside the square the admissible values of a go from $0.4402277051\ldots$ to $0.59267461067\ldots$. Specifically, at $a = 0.4402277051\ldots$ the value of the masses are $m_1 = 0$ and $m_2 = m_3 = 1$, and at the other end of the range $a = 0.5926746107\ldots$ we have $m_1 = 0.7298273415\ldots$, $m_2 = 0.7909651413\ldots$ and $m_3 = 0$. On the other hand, when the fifth mass is inside the rhombus the admissible values of a go from $-1/2$ to $-0.2495724426\ldots$.

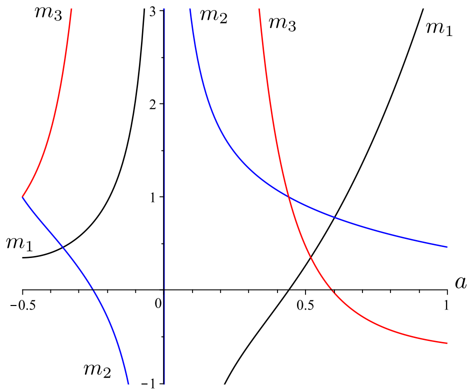

FIGURE 3. The value of the masses m_1 (black), m_2 (blue) and m_3 (red) when $b = 1/2$, that is, a square plus one configuration. The admissible values of a go from $0.4402277051\ldots$ to $0.5926746107\ldots$ when the fifth mass is outside the square, and from $-1/2$ to $-0.2495724426\ldots$ when the fifth mass is inside the square.

At $a = -0.2495724426\ldots$ the masses are $m_1 = 0.7268534679\ldots$, $m_2 = 0$ and $m_3 = 6.814939450\ldots$. Recall that when the fifth mass is located at the intersection of the two diagonals of the square ($a = -1/2$) the configuration is central for any value of m_1 and $m_2 = m_3 = 1$. See [7].

From Figure 3 the question about the number of central configurations for a given positive masses is also answered numerically when the rhombus becomes a square. Clearly, given a normalized mass vector $(m_1, m_2, m_3, 1, 1)$ of the five-body problem, if a square plus one central configuration exists, such a configuration is unique. In the next section we settle the inverse question in the rhombus plus one central configurations.

3. The uniqueness of rhombus plus one central configurations

In this section we prove that given a rhombus plus one configuration the normalized mass vector $(m_1, m_2, m_3, 1, 1)$ of the five-body problem that makes the configuration central is unique.

Theorem 3.1. *Given a configuration of the planar five-body problem, where four bodies m_2, m_3, m_4 and m_5 are located at the vertices of a rhombus and the fifth body m_1 is placed on the diagonal containing m_2 and m_3, then the positive normalized mass vector $(m_1, m_2, m_3, 1, 1)$ that makes the configuration central is unique, as long as the particular case where the fifth body is at the intersection of the two diagonals of the rhombus is excluded.*

Proof. Equations (2.5) can be written as a non homogeneous system $Bm = b$ where

$$B = \begin{pmatrix} 0 & (R_{24} - R_{12})\Delta_{124} & (R_{24} - R_{13})\Delta_{134} \\ (R_{12} - R_{14})\Delta_{124} & 0 & (R_{24} - 1)\Delta_{243} \\ (R_{13} - R_{14})\Delta_{134} & (1 - R_{24})\Delta_{234} & 0 \end{pmatrix},$$

$$b = \begin{pmatrix} (-R_{14} + R_{45})\Delta_{145} \\ (-R_{24} + R_{45})\Delta_{245} \\ (R_{24} - R_{45})\Delta_{245} \end{pmatrix}, \qquad m = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}.$$

The existence and uniqueness of m_1, m_2 and m_3 (positive or not) depends on whether the determinant of B, $\det(B)$ is zero or not, where

$$\det(B) = \Delta_{124}\Delta_{134}\Delta_{234}(R_{24} - 1)(R_{14} - R_{24})(R_{12} - R_{13}). \tag{3.1}$$

Clearly $\Delta_{234} \neq 0$, as well as the other two areas Δ_{124} and Δ_{134} in (3.1), otherwise there would be collision between m_1 and m_2, which is excluded. Again $R_{14} = R_{24}$ implies collision between m_1 and m_2, along with $R_{12} = R_{13}$ implies that m_1 is located at the intersection of the two diagonals of the rhombus. Last case is excluded and was studied by Gidea and Llibre in [7]. It corresponds to case (i) of the proof of their Theorem 1 (a).

Then the normalized mass vector that solve equations (2.5) will be unique, as long as $R_{24} \neq 1$, that is, the sides of the rhombus are equal to the diagonal containing the fifth body, that is, when the rhombus becomes a diamond.

Claim: if $r_{24} = 1$ the equations (2.5) has no solution.

When $r_{24} = 1$, we have $r_{45} = \sqrt{3}$. Indeed, by adding equations f_{24} and f_{34}, from (2.5), the resulting equation is

$$(R_{12} - R_{14})\Delta_{124} + (R_{13} - R_{14})\Delta_{134} = 0, \tag{3.2}$$

that can be written as a function of a.

At this point we distinguish two cases. The case where m_1 is outside the rhombus ($a > 0$), and where is inside ($a < 0$).

Equation (3.2) when $a > 0$ becomes

$$\frac{(1/2 + a)(1 + a)^2 a^2 - (a^2 + a + 1/2)(a^2 + a + 1)^{\frac{3}{2}}}{2a^2(1 + a)^2(a^2 + a + 1)^{\frac{3}{2}}} = 0. \tag{3.3}$$

Trivially the denominator in this last expression do not vanish when $a > 0$. So we have to prove that the numerator have the same sign for all positive values of a. In order to eliminate the square root of the numerator in (3.3), we equals the numerator to zero and then we square it conveniently to obtain a polynomial equation of degree 8,

$$15a^8 + 60a^7 + 115a^6 + 135a^5 + 109a^4 + 63a^3 + 26a^2 + 7a + 1 = 0.$$

Since all coefficients are positive, from Descartes' rule of signs, see [22], there are no positive roots, and consequently equation (3.3) is not satisfied for any positive value of a.

We now compute (3.2) for the case where m_1 is inside the rhombus ($a < 0$), and is found to be

$$\frac{(1/2+a)\big((a^2+a+1)^{\frac{3}{2}}+a^2(1+a)^2\big)}{2a^2(1+a)^2(a^2+a+1)^{\frac{3}{2}}}=0. \tag{3.4}$$

To obtain the zeros of this equation, if any, we proceed as in the previous case, arriving at the polynomial of degree 8 given by

$$-a^8-4a^7-5a^6-a^5+5a^4+7a^3+6a^2+3a+1.$$

Now using the Sturm's theorem in the interval $(-1/2,0)$ we check that has no root on it. Remember that from the symmetry of the configuration is enough to study when the fifth body, m_5, is located between the body m_2 and the intersection of the two diagonals of the rhombus. Also notice that the denominator of equation (3.2) when $a < 0$ is equal to one obtained in the case $a > 0$, and as before is different from zero in the interval $(-1/2,0)$. □

4. Proof of Theorem 2.1

The sides of the rhombus are nearly equal to the diagonal of the rhombus, $r_{23} = 1$, containing the fifth mass, m_1, when $r_{24} \approx 1$. From the proof of Theorem 3.1 we know that the determinant (3.1), that appears at the denominator of the expressions of the masses (2.6), is different from zero unless $r_{24} = 1$. Also is easy to check that such a determinant is negative when r_{24} is sufficiently close to one and $r_{24} - 1 < 0$.

We are going to prove that when $r_{24} = 1$ there exists an open interval of values of r_{12} such that the numerators N_1, N_2 and N_3 of m_1, m_2 and m_3, respectively, in (2.6) have constant sign. Therefore there will exist values of r_{12} such that $m_1 > 0$ and the other two masses, m_2 and m_3, are unbounded with positive sign. Finally, using the continuity of the values of the masses with respect to the mutual distances, for values of r_{24} sufficiently close to one and less than one, there will exist an open set of dimension two in a neighborhood of the $r_{24} = 1$ where the three masses are positive.

Setting $r_{24} = 1$ and $r_{12} = |a|$, all other mutual distances and signed areas that appear in N_1, N_2 and N_3 in equations (2.6) can be only written as a function of real variable a. From now on, we will distinguish when the fifth mass is outside ($a > 0$) or inside ($-1/2 < a < 0$) the rhombus, as usual.

- Case $a > 0$

Numerator, N_1, of m_1 in equations (2.6) is equal to

$$N_1 = -\frac{(9-\sqrt{3})(2a^5+5a^4+4a^3-a^2-2a-1)}{48a^2(1+a)^2}. \tag{4.1}$$

Let $P(a)$ be the above polynomial of degree 5. Applying the Descartes's rule of signs, $P(a)$ has only one positive root. Moreover, $P_5(6/10) < 0$ and $P_5(7/10) > 0$. Therefore $P_5(a) > 0$, and consequently $N_1 < 0$, for all $a > 7/10$.

Numerators N_2 and N_3 in equations (2.6) are equal to

$$N_2 = \frac{(9-\sqrt{3})(a^2+3a+3)F(a)}{24a(1+a)^5(a^2+a+1)^{\frac{3}{2}}},$$
$$N_3 = \frac{(9-\sqrt{3})(1-a)F(a)}{24a^5(1+a)^2(a^2+a+1)^{\frac{1}{2}}},$$

where $F(a) = (1/2+a)(1+a)^2a^2 - (a^2+a+\frac{1}{2})(a^2+a+1)^{\frac{3}{2}}$, which is equal to the numerator of the left hand side equation (3.3), and we know that has no positive roots. Since $F(0) < -1/2$, then $F(a) < 0$ and $N_2 < 0$ for all $a > 0$ and $N_3 < 0$ only for all $0 < a < 1$, due to the factor $(1-a)$ present in the expression of N_3.

- Case $-1/2 < a < 0$

Numerator, N_1, of m_1 in equations (2.6) is equal to

$$N_1 = -\frac{(9-\sqrt{3})(2a+1)(a^4+2a^3+a^2+1)}{48a^2(1+a)^2} < 0. \tag{4.2}$$

On the other hand N_2 and N_3 have similar expressions

$$N_2 = -\frac{(\sqrt{3}-9)(1/2+a)(a^2+3a+3)G(a)}{24a(1+a)^5(a^2+a+1)^{\frac{3}{2}}},$$
$$N_3 = \frac{(\sqrt{3}-9)(a^2-a+1)G(a)}{24a^5(1+a)(a^2+a+1)^{\frac{3}{2}}},$$

where $G(a) = a^2(1+a)^2 + (a^2+a+1)^{\frac{3}{2}}$, and again coincides with the numerator of the left hand side equation (3.2) when $a < 0$, and we know that has no roots in the interval $(-1/2, 0)$. Since $G(0) = 1 > 0$ then $G(a) > 0$ and $N_2 < 0$ in the interval $(-1/2, 0)$. Notice that $N_3 > 0$ when $-1/2 < a < 0$ and apparently the mass m_3 is unbounded, but negative. However, m_3 really has positive sign, since Δ_{124}, that has negative sign when the fifth mass is inside the rhombus and is the only negative factor in the determinant (3.1), do not appear in the denominator of m_3.

□

In summary we have proved that m_1, m_2 and m_3 are positives for all values of r_{12} between 7/10 and 1, when the fifth body, m_1, is outside the rhombus, and for all values of r_{12} between 0 and 1/2, when the fifth body,

m_1, is inside the rhombus, if r_{24} is sufficiently close to one but smaller. See Figure 2.

5. Proof of Theorem 2.2

Following the ideas in the previous section we are going to prove that when $r_{45} = 1$, that is, when the diagonals of the rhombus are equal, the masses m_1, m_2 and m_3 in (2.6) are positive for some values of r_{12}. In what follows we will show that in (2.6) when $r_{45} = 1$ the numerators and denominators have the same sign, respectively. Once again, we will distinguish when the fifth mass is outside ($a > 0$) or inside ($-1/2 < a < 0$) the square.

- Case $a > 0$

  Numerator, N_1, of m_1 in equations (2.6) is equal to

$$N_1 = -\frac{P_1(a)}{16a^2(1+a)^2(a^2+a+\frac{1}{2})\sqrt{4a^2+4a+2}},$$

  where

$$P_1(a) = (14a^5 + 35a^4 + 28a^3 + 9a^2 + 2a + 1 - (4a^2 + 4a + 2)\sqrt{2}\,)$$
$$(a^2 + a + 1/2)(4a^2 + 4a + 2)^{\frac{1}{2}} - 8(1+a)^2(1/2 + a)(\sqrt{2} - 1/2)a^2.$$

  In order to study the zeros of N_1 we use the same techniques as in Section 3. $P_1(a)$ can be written as a polynomial of degree 32, $P_{32}(a)$, with two number of sign changes. So from Descartes' rule of signs the number of zeros of $P_{32}(a)$ is zero or two. Since $P_{32}(3/10) < 0$, $P_{32}(2/5) > 0$, $P_{32}(1/2) < 0$ and $N_1(1/2) = \frac{5\sqrt{2}}{9} + \frac{2\sqrt{2}\sqrt{5}}{25} - \frac{83}{72} - \frac{\sqrt{5}}{25} < 0$, follows that $N_1(a) < 0$, and consequently $m_1 > 0$, for all $a > 1/2$.

  Numerator, N_2, of m_2 in equations (2.6) is equal to

$$N_2 = -\frac{P_2(a)}{16a^2(1+a)^5(2a^2+2a+1)^3\sqrt{4a^2+4a+2}},$$

  where

$$\begin{aligned}P_2(a) = ((&-48a^{10} - 288a^9 - 808a^8 - 1352a^7 - 1516a^6 - 1248a^5 - 818a^4\\ &- 430a^3 - 164a^2 - 38a - 4)\sqrt{2} + 112a^{11} + 808a^{10} + 2720a^9\\ &+ 5668a^8 + 8124a^7 + 8430a^6 + 6504a^5 + 3769a^4 + 1622a^3\\ &+ 495a^2 + 96a + 9)(4a^2 + 4a + 2)^{\frac{1}{2}}\\ &- 224a^2(1+a)^5(1/2 + a)(a^2 + a + 1/2)^2.\end{aligned}$$

  Again, $P_2(a)$ can be written as a polynomial of degree 42 with no zeros in the interval $\in (0,1)$. Since $N_2(1/2) = \frac{15737\sqrt{2}}{30375} + \frac{7\sqrt{5}}{25} - \frac{66931}{30375} < 0$, then $N_2(a) < 0$, so that $m_2 > 0$ for all $0 < a < 1$.

Numerator, N_3, of m_3 in equations (2.6) is equal to

$$N_3 = -\frac{P_3(a)}{48a^5(1+a)^2(2a^2+2a+1)^3\sqrt{4a^2+4a+2}},$$

where

$$\begin{aligned} P_3(a) = ((&-144a^{10} - 576a^9 - 1128a^8 - 1224a^7 - 540a^6 + 288a^5 + 582a^4 \\ &+ 426a^3 + 186a^2 + 48a + 6)\sqrt{2} - 336a^{11} - 1272a^{10} - 2400a^9 \\ &- 2796a^8 - 2244a^7 - 1410a^6 - 816a^5 - 459a^4 - 234a^3 \\ &- 93a^2 - 24a - 3)(4a^2 + 4a + 2)^{\frac{1}{2}} \\ &+ 672a^5(1+a)^2(1/2+a)(a^2+a+1/2)^2. \end{aligned}$$

By using the same method as before, we proceed to find the zeros of $P_3(a)$. Thus we transform its expression to a polynomial of degree 42. Applying Sturm's theorem in the interval $(0,1)$, we obtain two zeros, one between $1/2$ and $51/100$ and another between $59/100$ and $3/5$. On the other hand, $N_3(55/100) = \frac{59238755508940350}{24506348699914631} - \frac{1247292558547000\sqrt{2}}{790527377416601} - \frac{7350\sqrt{541}}{292681} < 0$. Then $N_3(a) < 0$ and $m_3 > 0$ for all $\frac{51}{100} < a < \frac{59}{100}$.

In summary we have proved that masses m_1, m_2 and m_3 are positives as long as the values of a (the distance between m_1 and m_2) belong to the interval $(\frac{51}{100}, \frac{59}{100})$. So there exist square plus one central configurations in the five-body problem when the fifth mass is located outside the square and $r_{12} \in (\frac{51}{100}, \frac{59}{100})$.

- Case $-1/2 < a < 0$

  Using similar arguments to the previous case, ($a > 0$), it can be proved that m_1 and m_3 are always positive while m_2 is only positive for values less than $-1/4$. So there exist square plus one central configurations in the five-body problem when the fifth mass is located inside the square and $r_{12} \in (1/4, 1/2)$.

□

We would like to remark, first, that the bounds obtained along the proof of the theorem for the admissible values of r_{12} are not sharp. And secondly, that from the continuity of the values of the masses with respect to the mutual distances, the theorem also ensures the existence of rhombus plus one central configurations when the two diagonals are nearly equal.

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