PROBABILITY OF EXISTENCE OF LIMIT CYCLES FOR A FAMILY OF PLANAR SYSTEMS

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ABSTRACT. The goal of this work is the study of the probability of occurrence of limit cycles for a family of planar differential systems that are a natural extension of linear ones. To prove our results we first develop several results of non-existence, existence, uniqueness and non-uniqueness of limit cycles for this family. They are obtained by studying some Abelian integrals, via degenerate Andronov-Hopf bifurcations or by using the Bendixson-Dulac criterion. To the best of our knowledge, this is the first time that the probability of existence of limit cycles for a non-trivial family of planar systems is obtained analytically. In particular, we give vector fields for which the probability of having limit cycles is positive, but as small as desired.

1. INTRODUCTION AND MAIN RESULTS

Many efforts have been devoted to study the existence, non-existence, uniqueness or number of limit cycles of planar autonomous systems, see for instance [8, 19, 22, 27, 28] and their references. Many of these results involve polynomial differential systems due to the big interest on the celebrated Hilbert's XVI-th problem. Nevertheless, as far as the authors know, the problem of knowing which is the probability of existence of limit cycles for a given family of such vector fields has been seldom analytically studied. In this work we propose a quite natural family of planar vector fields for which we face this problem. Before stating our results we introduce and formalize this question in more detail.

Consider the planar linear differential systems

$$\dot{x} = ax + by, \quad \dot{y} = cx + dy, \tag{1}$$

where $(a, b, c, d) \in \mathbb{R}^4$. To know the probability of occurrence of each of its possible phase portraits (saddle, node, focus, center, ...) there is a well-established way to mathematize this problem. Take the planar random linear systems

$$\dot{x} = Ax + By, \quad \dot{y} = Cx + Dy,$$

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where A, B, C and D are independent and identically distributed (iid) random variables, with Gaussian distribution $N(0,1)$. Then, the probability that system (1) has a given phase portrait, say for instance of being a saddle, is exactly

$$P\left(\omega \,:\, A(\omega)D(\omega) - B(\omega)C(\omega) < 0\right),$$

that for short we will write as $P(AD - BC < 0)$ and, in this case, it is $1/2$. The reason for the suitability of this probability calculation model is explained for instance in [23, 25] or [10, Thm 2.1]. In a few words, what happens is that if for some values (a, b, c, d) a linear system (1) has a given phase portrait, then the system associated with the parameters $\lambda(a, b, c, d)$, $\lambda \in \mathbb{R}^+$, also has the same phase portrait. Therefore we want to assign the same probability to each half-straight line of systems starting at the origin. Under previous hypotheses on A, B, C and D, this is the case, because the random vector

$$\left(\frac{A}{M}, \frac{B}{M}, \frac{C}{M}, \frac{D}{M}\right), \quad \text{where} \quad M = \sqrt{A^2 + B^2 + C^2 + D^2},$$

has uniform distribution on the sphere $\mathbb{S}^3$ and each point of this sphere can be identified with one of these half-straight lines. By using previous approach it is proved in [26], and also in [9, 10], that the only phase portraits with positive probability are saddles, nodes or foci and that their respective probabilities are $1/2$, $(\sqrt{2} - 1)/2$ and $1 - \sqrt{2}/2$. This result is extended in different directions: in [10] to higher dimensions, in [9] to planar homogeneous quadratic and cubic vector fields, and in [12] to some planar quasi-homogenous vector fields.

The aim of this work is to address the same problem for a non-linear generalization of system (1) that also admits phase portraits with limit cycles, and then compute the probability of existence of them. At this point it is worth to comment that this question has been also addressed in [3] for planar quadratic vector fields. In that paper, because of the difficulty of that family, the probability of limit cycles was obtained by Monte Carlo simulation, together with a numerical study of the solutions of the differential equation. The authors obtained that the probability of existence of limit cycles in the quadratic family is around 0.0323.

We will consider the family of random vector fields

$$\dot{x} = A\, f(x) + By, \quad \dot{y} = C\, f(x) + Dy, \tag{2}$$

where f is a fixed smooth function such that $f(0) = 0$ and, as above, A, B, C and D are iid random variables with $N(0,1)$ distribution. For some f we will compute, using only analytic tools, the probability that system (2) has limit cycles.

Of course, prior to compute this probability we need to study conditions on $(a, b, c, d) \in \mathbb{R}^4$ and f that allow us to control the number

of limit cycles of the corresponding deterministic system

$$\dot{x} = af(x) + by, \quad \dot{y} = cf(x) + dy, \qquad (3)$$

that can be seen as a realization of system (2). It should be noticed that system (3) is equivalent to a class of generalized Liénard equations, see Lemma 2.5. Therefore, all the criteria proved about limit cycles for the generalized Liénard systems can be applied to get properties about system (3), see for instance [7, 17, 27, 28]. We will also use this approach in one of our results, see Proposition 2.13.

Section 2 contains the proof of our results on system (3) and on the natural extension considered in the next theorem. We obtain results on non-existence, existence or uniqueness of limit cycles, as well as examples with several limit cycles. Our results are collected in the following three theorems.

Theorem 1.1. *Consider system*

$$\dot{x} = af(x) + bg(y), \quad \dot{y} = cf(x) + dg(y), \qquad (4)$$

being $a, b, c, d \in \mathbb{R}$, and where f and g are smooth real functions such that $f(0) = g(0) = 0$. Then:

   (i) If $abcd \leq 0$, system (4) has no limit cycles.
   (ii) Assume that f and g are analytic, $f(x) = x^{2l-1} + O(x^{2l})$ and $g(y) = y^{2k-1} + O(y^{2k})$, for some positive integer numbers k and l, with $k \neq l$. Then, there exist a, b, c, d such that system (4) has at least one limit cycle surrounding the origin which, whenever it exists, is hyperbolic.
  (iii) There exist f and g such that for some values of a, b, c and d, system (4) has more than one limit cycle surrounding the origin. Moreover, the same holds using $g(y) \equiv y$, that is for system (3).

We remark that our aim in item (iii) is simply to prove that, in general, there is no uniqueness of limit cycles for systems (3) or (4). As we will see, our proof indicates that there is no upper bound for the number of limit cycles for none of these two families.

Theorem 1.2. *Consider system* (3),

$$\dot{x} = af(x) + by, \quad \dot{y} = cf(x) + dy,$$

with $ad \neq 0$. Let $f(x)$ be the polynomial $f(x) = \alpha x^k + \sum_{k<i<m} f_i x^i + \beta x^m$, with $\alpha\beta \neq 0$, $k \leq m$ odd integers and $m > 1$. Assume moreover that $x = 0$ is the unique real root of $f(x) = 0$.

   (i) If $\beta(ad - bc) \leq 0$, then it has no periodic orbits.
   (ii) If $\beta(ad - bc) > 0$ and, either $k = 1$ and $\beta a(a\alpha + d) > 0$, or $k > 1$ and $\beta ad > 0$, then it has zero or an even number of limit cycles.

(iii) If $\beta(ad - bc) > 0$ and, either $k = 1$ and $\beta a(a\alpha + d) < 0$, or $k > 1$ and $\beta ad < 0$, then it has an odd number of limit cycles.

In all the cases, each limit cycle is counted with its multiplicity.

Next result gives an upper bound for the number of limit cycles for some families of systems given by (3). We will prove it by using the Bendixon-Dulac theorem for non simply connected regions, see for instance [15, 16] for more information on this theorem and other applications. In particular, next theorem provides a criterion on uniqueness of limit cycles for systems with a unique equilibrium point. For instance, as we will see, it proves the uniqueness for systems with $f(x) = x^{2n-1}$, $n > 1$, already established in [18] by transforming this particular system into a Liénard one.

Theorem 1.3. *Consider system (3),*

$$\dot{x} = af(x) + by, \quad \dot{y} = cf(x) + dy,$$

where f is smooth and $f(0) = 0$. Assume that $M(x) = 2af'(x)F(x) - a(f(x))^2 - dxf(x) + 2dF(x)$ does not change sign and vanishes at isolated points, where $F' = f$ and $F(0) = 0$. Let K be the number of bounded intervals (counting also intervals degenerated to a point as intervals) of the closed set

$$\{x \in \mathbb{R} : \Delta(x) = (af(x) + dx)^2 - 8(ad - bc)F(x) \geq 0\}.$$

Then the system has at most K limit cycles, all of them hyperbolic.

We want to highlight the role played by the closed set $\Delta(x) \geq 0$ in obtaining an upper bound for the number of limit cycles. From the proof of the theorem we will see that the periodic orbits of system (3) are contained in the connected components of $\mathbb{R}^2 \backslash \{V(x, y) = 0\}$ and that its number depends on the shape of this set. Here the function $V(x, y)$, which is quadratic on y and appears also in the proof, is such that its discriminant with respect to y is precisely $\Delta(x)$. Moreover, each bounded interval of the set $\Delta(x) \geq 0$ gives rise to an oval or a bounded component of $\{V(x, y) = 0\}$.

Corollary 1.4. *Assume that system (3) is under the hypothesis on M given in Theorem 1.3, and consider that the origin is the only equilibrium point. Then, it has at most one limit cycle and, when it exists, it is hyperbolic.*

Section 3 collects our results about the probability of existence of limit cycles for the family of random vector fields (2). To prove them we have applied all the previous results, together with some simple tools of probability theory.

Theorem 1.5. *Consider the random system (2),*

$$\dot{x} = Af(x) + By, \quad \dot{y} = Cf(x) + Dy,$$

where $f(x) = \alpha x^k + \sum_{k<i<m} f_i x^i + \beta x^m$, with $\alpha\beta \neq 0$, $k \leq m$ odd integers, $m > 1$, and A, B, C, D iid $N(0,1)$ random variables. Assume also that $x = 0$ is the unique real root of $f(x) = 0$. Then:

    (i) When $k > 1$, the probability of having an odd number of limit cycles is $1/8$, and the probability of not having limit cycles or having an even number of them is $7/8$.

    (ii) When $k = 1$ and $\beta > 0$, the probability of having an odd number of limit cycles is $P^+(\alpha) \leq 1/2$, and the probability of not having limit cycles or having an even number of them is $1 - P^+(\alpha)$. Here $P^+ : \mathbb{R} \to (0, 1/2)$ is a decreasing function that satisfies

$$\lim_{\alpha \to -\infty} P^+(\alpha) = 1/2, \quad P^+(0) = 1/8, \quad \lim_{\alpha \to +\infty} P^+(\alpha) = 0,$$

    given by

$$P^+(\alpha) = \frac{1}{4\pi^2} \iiiint_{T(\alpha)} \mathrm{e}^{-\frac{a^2+b^2+c^2+d^2}{2}} \, \mathrm{d}a \, \mathrm{d}b \, \mathrm{d}c \, \mathrm{d}d,$$

    where $T(\alpha) = \{(a, b, c, d) : ad - bc > 0, a(a\alpha + d) < 0\}$.

    (iii) When $k = 1$ and $\beta < 0$, the same results as in item (ii) hold but changing P^+ by P^-, where $P^-(\alpha) = P^+(-\alpha)$.

In all the cases, each limit cycle is counted with its multiplicity.

As we have already commented, for the families (2) with an f such that they have at most one limit cycle, the results of Theorem 1.5 can be refined by using Corollary 1.4. As a concrete example we give next consequence of Theorem 1.5 and forthcoming Corollary 2.11.

Corollary 1.6. *Consider random system*

$$\dot{x} = Ax^k + By, \quad \dot{y} = Cx^k + Dy, \tag{5}$$

where $k > 1$ is an odd integer and A, B, C, D are iid random variables with distribution $N(0,1)$. Then, the probability of having one limit cycle is $1/8$, and the probability of not having limit cycles is $7/8$.

We also prove the following natural result, which we think is interesting. It will be a consequence of a result on a deterministic generalized Liénard system, given in Proposition 2.13.

Corollary 1.7. *For any $\varepsilon > 0$, there exists a polynomial f of the form given in Theorem 1.5 with $k = 1$ and degree 3, such that the corresponding random system (2) has at most one limit cycle and it exists with a positive probability smaller than ε.*

The example given in Proposition 2.13, that will allow us prove the above corollary, is similar in spirit, but different, to the ones provided by the family of examples of planar system given in [16, Thm. 3.7]. In that paper the authors prove that for each $\varepsilon > 0$ there is an one-parametric family of systems for which the limit cycles do exist only when the parameter is inside an interval of length smaller than ε.

α	-100	-10	-1	0	1	10	100
MC-10^6	0.4984	0.4814	0.3127	0.1255	0.0624	0.0128	0.0016
MC-10^8	0.49829	0.48129	0.31247	0.12498	0.06254	0.01303	0.00155

TABLE 1. Some approximated values of $P^+(\alpha)$ obtained by Monte Carlo (MC) simulation taking 10^6 and 10^8 random systems. We know that $P^+(0) = 1/8 = 0.125$.

Unfortunately, we have not been able to obtain a simpler analytic expression for $P^+(\alpha)$ than the one given in Theorem 1.5. In fact, the only explicit value that we have found is $P^+(0) = P^-(0) = 1/8$. For this reason we include in Table 1 some approximations of $P^+(\alpha)$ for several values of α, which we have computed by using Monte Carlo simulation. In Subsection 3.1 we explain how we have obtained these values and the reason why the expected errors of these approximations are, respectively, of order 10^{-3} or 10^{-4} in the second and third row of Table 1. In Figure 3 of that subsection we also plot a numerical approximation of $P^+(\alpha)$.

To end this section, we remark that there are many mathematical models involving differential systems for which the parameters come from sources with uncertainty, see for instance [4, 5, 6, 24]. This uncertainty can be approached by assuming them to be random variables, and it is commonly supposed that these parameters follow a Gaussian distribution. Hence, apart for their own theoretical interest, our results also give methods to approach this type of applied problems.

2. DETERMINISTIC SYSTEMS

We split this section into two subsections. In the first we prove our results about system (4), while in the second we collect our results on system (3).

For the sake of shortness, when we say that the stability of some object (point, periodic orbit, infinity) is given by the sign of some quantity, say ρ, we mean that when $\rho > 0$ the object is a repellor and when $\rho < 0$ it is an attractor.

2.1. **Results on system** (4) **and proof of Theorem 1.1.** The proof of Theorem 1.1 will follow from several previous lemmas and propositions.

Next lemmas give conditions for the non-existence of limit cycles or periodic orbits for system (4).

Lemma 2.1. *Consider system* (4) *with* $f, g \in C^1(\mathbb{R})$ *and such that* $f(0) = g(0) = 0$.

(i) *If* $abcd < 0$, *then it has no periodic orbits.*

(ii) *If $abcd = 0$ and either $ac \neq 0$ or $db \neq 0$, then it has no periodic orbits.*

(iii) *If $ac = bd = 0$, then it has no limit cycles.*

(iv) *If $ad - bc = 0$, then it has no periodic orbits.*

Proof. Let us consider the functions F and G such that $F'(x) = f(x)$ and $G'(y) = g(y)$. Taking $H(x, y) = cF(x) - bG(y)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} H(x, y) = \dot{H}(x, y) = cf(x)\dot{x} - bg(y)\dot{y} = acf^2(x) - bdg^2(y).$$

Under the hypotheses of cases (i) or (ii) we have that $\dot{H} \neq 0$ and, hence, $H(x, y)$ does not change sign on any trajectory. This fact implies that the function $t \to H(x(t), y(t))$ is a monotonous function on the orbits of system (4), increasing when $ac > 0$ and decreasing when $ac < 0$. Then, no periodic orbits can exist in any of these two cases. Concerning (iii), we get that $\dot{H}$ vanishes identically and, then, since system (4) is an integrable system, although periodic orbits are possible, no limit cycles can exist. In the case (iv), i.e. when $ad - bc = 0$, we consider the function $W(x, y) = cx - ay$. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t} W(x, y) = \dot{W}(x, y) = (bc - ad)g(y) = 0.$$

Then, as in the third case, system (4) is integrable and neither limit cycles, nor periodic orbits exist, because of the shape of the level sets of W. $\qquad\square$

This second lemma is simply a consequence of the classical Dulac criterion, because the divergence of the vector field associated to system (4) is $af'(x) + dg'(y)$.

Lemma 2.2. *Consider system (4) with $f, g \in C^1(\mathbb{R})$ and such that $f(0) = g(0) = 0$. If $af'(x) + dg'(y)$ does not change sign and vanishes on a set of zero measure, then it has no periodic orbits.*

Next, we prove that there exist many functions f and g such that system (3), for some values of its parameters a, b, c and d, has at least one limit cycle. In fact, the family considered is a perturbed Hamiltonian system such that its associated Abelian integral has at least one simple zero near the origin and as a consequence, at least one limit cycle bifurcates from the period annulus. See [8, 13] for an introduction to this subject.

Proposition 2.3. *Consider system*

$$\dot{x} = \varepsilon\alpha f(x) + g(y), \quad \dot{y} = -f(x) + \varepsilon\delta g(y), \tag{6}$$

with f and g analytic such that $f(x) = x^{2l-1} + O(x^{2l})$ and $g(y) = y^{2k-1} + O(y^{2k})$, for some positive integer numbers k and l, $k \neq l$. Then, for ε small enough, there exist α and δ, with $\alpha\delta < 0$, such that system (6)

has at least one limit cycle surrounding the origin, which whenever it exists, is hyperbolic.

Proof. Let F and G be such that $F'(x) = f(x)$ and $G'(x) = g(x)$ and with $F(0) = G(0) = 0$. Note that when $\varepsilon = 0$, system (6) is a Hamiltonian system with Hamiltonian function $H(x,y) = F(x) + G(y)$ where $H(0,0) = 0$. We denote by $\gamma_h = \{F(x) + G(x) = h\}$ the oval of the level curve $H(x,y) = h$ surrounding the origin. Then the Abelian integral that controls the number of limit cycles bifurcating from the periodic orbits γ_h is given by

$$
\begin{aligned}
I(h) &= -\delta \int_{\gamma_h} g(y)\mathrm{d}x + \alpha \int_{\gamma_h} f(x)\mathrm{d}y \\
&= \delta \iint_{\mathrm{Int}(\gamma_h)} g'(y)\mathrm{d}x\,\mathrm{d}y + \alpha \iint_{\mathrm{Int}(\gamma_h)} f'(x)\mathrm{d}x\,\mathrm{d}y,
\end{aligned}
\tag{7}
$$

where this last equality comes from the Green theorem and $\mathrm{Int}(\gamma_h)$ denotes the region surrounded by γ_h. Let us define

$$
U(h) = \iint_{\mathrm{Int}(\gamma_h)} f'(x)\mathrm{d}x\,\mathrm{d}y, \quad V(h) = \iint_{\mathrm{Int}(\gamma_h)} g'(y)\mathrm{d}x\,\mathrm{d}y. \tag{8}
$$

Let us prove that there exist real constants $C_1, C_2 > 0$ such that

$$
\begin{aligned}
U(h) &= \iint_{\mathrm{Int}(\gamma_h)} f'(x)\mathrm{d}x\,\mathrm{d}y \sim C_1\, h^{1+(l-k)/(2kl)}, \\
V(h) &= \iint_{\mathrm{Int}(\gamma_h)} g'(y)\mathrm{d}x\,\mathrm{d}y \sim C_2\, h^{1+(k-l)/(2kl)},
\end{aligned}
\tag{9}
$$

in a neighborhood of $h = 0$ for $h > 0$.

We prove the above fact for the Abelian integral U and the same steps apply for V. We introduce the change of variables $u = x \sqrt[2l]{2l\frac{F(x)}{x^{2l}}} = x \sqrt[2l]{1 + O(x)}$ which is regular in a neighborhood of $x = 0$ and admits inverse $x = \phi(u) = u + O(u^2)$. Observe that $u^{2l} = 2lF(x)$. Similarly, in a neighborhood of $y = 0$, $v = y \sqrt[2k]{2k\frac{G(y)}{y^{2k}}} = y \sqrt[2k]{1 + O(y)}$ with regular inverse $y = \psi(v) = v + O(v^2)$ and it holds that $v^{2k} = 2kG(y)$. It is straightforward to see that by applying this change of variables the integral $U(h)$ of (8) becomes

$$
U(h) = \iint_{D_h} (2l - 1)u^{2l-2}\left(1 + O(u,v)\right)\mathrm{d}u\,\mathrm{d}v, \tag{10}
$$

where $D_h = \left\{\frac{u^{2l}}{2l} + \frac{v^{2k}}{2k} \le h\right\}$. Let us introduce a second change of variables given by $u = w^k X$, $v = w^l Y$ and $h = w^{2kl}$. Then, if $\tilde{U}(w) = U(h) = U(w^{2kl})$, it holds that

$$
\tilde{U}(w) = \iint_D (2l - 1)w^{2k(l-1)}X^{2l-2}\left(1 + O(w^k X, w^l Y)\right)w^{k+l}\mathrm{d}X\,\mathrm{d}Y,
$$

where $D = \left\{ \frac{X^{2l}}{2l} + \frac{Y^{2k}}{2k} \leq 1 \right\}$. Now we compute the following limit

$$\lim_{h \to 0^+} \frac{U(h)}{h^{1+(l-k)/(2kl)}} = \lim_{w \to 0} \frac{\tilde{U}(w)}{w^{2kl+l-k}}$$
$$= \iint_D (2l - 1) X^{2l-2} \mathrm{d}X \, \mathrm{d}Y =: C_1 > 0.$$

In a similar way we can approximate $V(v)$ by $C_2 h^{1+(k-l)/(2kl)}$, for a constant $C_2 > 0$. Therefore (9) is proved.

To end the proof, notice that when $k \neq l$ the Wronskian with respect to h of the two functions $h^{1+(l-k)/(2kl)}$ and $h^{1+(k-l)/(2kl)}$ does not vanish in $(0, \infty)$. Hence the same holds with the two functions $U(h)$ and $V(h)$ for h in a small interval of the form $(0, \overline{h})$, for some $\overline{h} > 0$. Therefore, these two functions form an extended completed Chebyshev system (ECT-system) on this interval, see for instance [20]. Then any linear combination of them has at most one zero in $(0, \overline{h})$, taking into account its multiplicity, and there are combinations having one zero. Since, from (7),

$$I(h) = \alpha U(h) + \delta V(h),$$

for some suitable α and δ this function has a simple zero $h^* \in (0, \overline{h})$. As a consequence, a hyperbolic limit cycle bifurcates from the oval γ_{h^*} when ε is small enough. $\square$

The following result proves that in systems of type (3) and (4) we can find subfamilies with more than one limit cycle.

Proposition 2.4. *(i) There are systems* (3)*, with f a polynomial of degree 4, having at least 3 hyperbolic limit cycles surrounding the origin.*

(ii) There are systems (4)*, with f and g polynomials of degree 3, having at least 3 hyperbolic limit cycles surrounding the origin.*

Proof. The key point of our proof will be the computation of the Lyapunov constants of the system. Although this approach is rather standard we face it from a slightly different point of view, as it was suggested to us by our colleague Joan Torregrosa. We first recall the classical one due to Lyapunov and afterward this small variation.

Given a weak focus of an analytic planar vector field, it is not restrictive, with an affine change of variables and a scaling of the time, to write its associated differential system as

$$\dot{x} = y + P(x, y), \quad \dot{y} = -x + Q(x, y), \tag{11}$$

where P and Q have neither constant nor linear terms. Consider $H(x, y) = \sum_{k \geq 2} H_k(x, y)$, where $H_2(x, y) = x^2 + y^2$ and H_k are homogeneous polynomials of degree k. Then Lyapunov's idea is to prove that there exist $H_k, k \geq 3$ (not unique), such that

$$\dot{H} = (y + P)\frac{\partial H}{\partial x} + (-x + Q)\frac{\partial H}{\partial y} = \sum_{m \geq 1} L_m (x^2 + y^2)^{m+1}. \tag{12}$$

Independently of the choice of the polynomials H_m, the values $L_m, m \geq 1$, are called Lyapunov constants. Indeed, for general vector fields with a focus at the origin, L_0 is by definition the real part of the eigenvalues associated to the equilibrium point, and it is a positive multiple of the divergence of the vector field at the origin. The sign of the first not null $L_m, m \geq 0$, gives the stability of the origin and if all them vanish the system is integrable and the origin is a center. Moreover, if we consider a parametric family of vector fields, with parameters $\lambda \in \Lambda \subset \mathbb{R}^j$, these constants are analytic[1] functions of the parameters λ of the family. Moreover, if these functions satisfy:

(c1) for some $m = M \geq 1$ and some $\lambda = \lambda^*$, $L_M(\lambda^*) \neq 0$ and $L_j(\lambda^*) = 0$ for all $j < M$,

(c2) the map defined on a neighborhood of $\lambda = \lambda^*$,

$$\lambda \to (L_0(\lambda), L_1(\lambda), \ldots, L_{M-1}(\lambda))$$

fills a complete neighborhood of the origin,

then there is a degenerate Andronov-Hopf bifurcation for $\lambda = \lambda^*$ at the origin for this family. In particular, there are differential systems in the family having at least M hyperbolic limit cycles in a small neighborhood of the origin and surrounding it, see for instance [11] and the references therein. In fact, to simplify the computations, the functions $L_m(\lambda)$ are usually reduced, giving expressions of them when all the previous ones $L_j(\lambda)$, $j < m$, vanish, because only in this situation they have a dynamical meaning.

As it can be easily seen, for our system the first step of this classical procedure, namely the application of the affine change of variables to write the differential system as in (11) entails some computational difficulties. The proposed small modification for a general system with a weak focus at the origin

$$\dot{x} = P_1(x,y) + P(x,y), \quad \dot{y} = Q_1(x,y) + Q(x,y), \quad \mathrm{div}(P_1, Q_1)(0,0) = 0,$$

where $\dot{x} = P_1(x,y)$, $\dot{y} = Q_1(x,y)$, has a center at the origin, consists simply on changing the function $H_2(x,y) = x^2 + y^2$ used above by the first integral of this linear system, which always exists, and is a positive definite quadratic form. Then the other terms $H_k, k \geq 3$, also exist and a similar formula to (12),

$$\dot{H} = \sum_{m \geq 1} \widehat{L}_m (x^2 + y^2)^{m+1},$$

also holds, where the only difference with (12) is the expression of H_2. The new values $\widehat{L}_m$, that by abuse of notation are also called Lyapunov constants, and again denoted as L_m, have exactly the same properties

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[1]Indeed polynomial functions when $L_0 = 0$.

that the above ones. We will apply this modified computation of the Lyapunov constant to systems of the form

$$\dot{x} = y + rx + P(x, y), \quad \dot{y} = -x - ry + Q(x, y), \quad |r| < 1,$$

for which $H_2(x, y) = x^2 + y^2 + 2rxy$.

More specifically, to prove item (i) we consider $f(x) = f_4 x^4 + f_3 x^3 + x^2 + x$, $a = 1/2$, $b = 1$, $c = -1$ and $d = -1/2 + e$. Then $L_0 = e$. Hence we take $e = 0$ and compute the next Lyapunov constants by using the above approach with $r = 1/2$. Then

$$L_1 = \frac{3f_3 - 2}{12}, \quad L_2 = -\frac{37}{704}f_3{}^2 - \frac{19}{176}f_3 - \frac{13}{44}f_4 + \frac{271}{1584},$$
$$L_3 = -\frac{22151}{639232}f_3{}^3 + \frac{634259}{719136}f_3{}^2 - \frac{26467}{119856}f_3 f_4 - \frac{21}{227}f_4{}^2$$
$$- \frac{3343861}{4314816}f_3 + \frac{27607}{44946}f_4 + \frac{23539}{3236112}.$$

By simplifying each L_m when $L_j = 0$, for $j < m$ we obtain

$$L_0 = e, \quad L_1 = \frac{3f_3 - 2}{12}, \quad L_2 = \frac{10 - 39f_4}{132}, \quad L_3 = -\frac{1610}{115089}.$$

Hence, we are under the hypotheses $(c1)$ and $(c2)$ to have a degenerate Andronov-Hopf bifurcation with $M = 3$ and, as a consequence, in this family we can generate three limit cycles bifurcating from the origin.

To prove item (ii), in this case we take $f(x) = f_3 x^3 + f_2 x^2 + x$, $g(y) = g_3 y^3 + y^2 + y$, $a = 1/2$, $b = 1$, $c = -1$ and $d = -1/2 + e$. Then $L_0 = e$ and when $e = 0$ we can apply again the above procedure to get the Lyapunov constants with $H_2(x, y) = x^2 + y^2 + xy$. For the sake of shortness we only give the simplified expressions of the L_m when all the previous L_j vanish. We get that

$$L_1 = \frac{1}{12}\left(2 - 3g_3 + 3f_3 - 2f_2{}^2\right),$$
$$L_2 = -\frac{(f_2^2 - 1)}{264}\left(18f_2{}^2 - 8f_2 - 57f_3 - 20\right),$$
$$L_3 = \frac{2(f_2^2 - 1)}{245841}\left(440f_2{}^4 + 599f_2{}^3 - 828f_2{}^2 + 599f_2 + 440\right),$$
$$L_4 = -\frac{7(f_2^2 - 1)}{735606432000}\left(570000621f_2{}^3 - 607497532f_2{}^2 \right.$$
$$\left. + 286980301f_2 + 210803560\right),$$

and $L_5 = L_6 = L_7 = 0$. In particular, it is easy to see that our vector field has a center at the origin if and only if $e = 0$, $f_2 = \pm 1$ and $g_3 = f_3$, implying in particular that $L_j = 0$ for all $j \geq 0$. This is so because in these cases it has a symmetry with respect to an straight line through the origin, and as a consequence, the origin is a reversible center. Moreover, take any of the two real roots of $L_3 = 0$ different

from $f_2 = \pm 1$, for instance,

$$f_2^* = \frac{-599 - 19\sqrt{9321} + \sqrt{626082 + 22762\sqrt{9321}}}{1760} \approx -0.4278353$$

and the values of f_3 and g_3, say f_3^* and g_3^*, obtained from the two conditions $L_1 = 0$ and $L_2 = 0$. In this case, the corresponding L_4 is non zero and it is not difficult to see that we are again under the hypotheses of the degenerated Andronov-Hopf bifurcation with $M = 3$. As a consequence, for values of f_j near f_j^*, the corresponding system (4) has at least three hyperbolic limit cycles surrounding the origin. $\square$

It is clear from the proof of the above proposition that the number of limit cycles surrounding the origin for systems (3) and (4) will increase with the degrees of f and g. We have presented examples with 3 limit cycles simply to show that, in general, there is no uniqueness of limit cycles for these families of systems.

Proof of Theorem 1.1. The proof of item (i) is a simple consequence of Lemma 2.1. Item (ii) follows from Proposition 2.3, while the proof of item (iii) is given in Proposition 2.4. $\square$

2.2. Results on system (3) and proof of Theorems 1.2 and 1.3.

Next lemma shows that system (3) can be transformed into a generalized Liénard systems.

Lemma 2.5. *System* (3) *can be transformed into the Liénard equation* $\ddot{x} - (d + af'(x))\dot{x} - (bc - ad)f(x) = 0$, *which is equivalent to each of the systems*

$$\dot{x} = y, \quad \dot{y} = (bc - ad)f(x) + (d + af'(x))y, \tag{13}$$

or

$$\dot{x} = y + dx + af(x), \quad \dot{y} = (bc - ad)f(x). \tag{14}$$

Proof. From $\dot{x} = af(x) + by$ we get $\ddot{x} = af'(x)\dot{x} + b\dot{y}$. By using now that $\dot{y} = cf(x) + dy$ we obtain that

$$\ddot{x} = af'(x)\dot{x} + b\big(cf(x) + dy\big) = af'(x)\dot{x} + bcf(x) + d\big(\dot{x} - af(x)\big)$$
$$= \big(d + af'(x)\big)\dot{x} + (bc - ad)f(x).$$

Then, system (13) appears simply taking $\dot{x} = y$ in this second order differential equation, where this new variable has the same name as the old y, although it is not the same. To get the second system we consider a new y as $y = \dot{x} - d\,x - af(x)$. $\square$

Next lemma characterizes the number of equilibrium points of system (3).

Lemma 2.6. *The equilibrium points of system* (3) *are given by the following conditions:*

(i) If $ad - bc = 0$, then $af(x) + by = 0$ or $cf(x) + dy = 0$ is a curve of equilibrium points.

(ii) If $ad - bc \neq 0$, then the equilibrium points are the points $(x^*, 0)$, where $f(x^*) = 0$.

In the next lemma we study, in the generic cases, the behaviour of the orbits of system (3) around the origin when $f(0) = 0$ and f is analytic. The non-generic ones could be also analysed without major difficulties, so we will only give some comments in Remark 2.8. We omit their detailed analysis because these systems will correspond to situations of zero probability of occurrence when we deal with their associated random systems in the next section.

Lemma 2.7. *Consider system* (3) *with* $bd(ad - bc) \neq 0$. *Assume that* $f(x)$ *is analytic at the origin,* $f(x) = \alpha x^k (1 + O(x))$, *where* $\alpha \neq 0$ *and* $k \geq 1$. *Then*

(i) *Assume that* $k = 1$. *If* $\alpha(ad - bc) < 0$, *then the origin is a saddle. When* $\alpha(ad - bc) > 0$, *the origin is a node or a focus, and when* $a\alpha + d \neq 0$ *its stability is given by the sign of* $a\alpha + d$.

(ii) *If* $k \geq 3$ *is odd and* $\alpha(ad - bc) > 0$, *then the origin is a node. The node will be stable in the case* $d < 0$ *and unstable when* $d > 0$.

(iii) *If* $k \geq 3$ *is odd and* $\alpha(ad - bc) < 0$, *then the origin is a saddle.*

(iv) *If* k *is even, then the origin is a saddle-node.*

Proof. (i) When $k = 1$ the origin is a non degenerated singularity. Its associated differential matrix is

$$\begin{pmatrix} a\alpha & b \\ c\alpha & d \end{pmatrix},$$

hence the result is a consequence of the well-known classification of this type of equilibrium points. When $a\alpha + d = 0$ the origin is a weak focus and its stability could be found by computing its Lyapunov constants, by following, for instance, the same procedure that in the proof of Proposition 2.4.

$(ii) - (iv)$ When $k > 1$ and $d \neq 0$ the origin is a semi-elementary singularity, namely the eigenvalues associated to its linear part are 0 and d. Since $b \neq 0$ we can take the new coordinates $z = -dx + by$, $y = y$ and the new time $s = dt$; then, system (3) writes as

$$z' = \frac{bc - ad}{d} f\left(\frac{by - z}{d}\right), \quad y' = y + \frac{c}{d} f\left(\frac{by - z}{d}\right),$$

where the prime denotes derivative with respect to s. In these new coordinates the system is in the usual normal form

$$\dot{z} = X(z, y), \quad \dot{y} = y + Y(z, y),$$

where X and Y have expansions at the origin that begin at least with second order terms in z and y. In order to prove this fact, it is straightforward to know which type of equilibrium point is the origin. The method goes as follows, see [2, 13]. The first step is obtaining $y = h(z)$, the solution of $y + Y(z, y) = 0$ that satisfies $h(0) = 0$. Then one has to compute $X(z, h(z)) = uz^m + O(z^{m+1})$, where $u \neq 0$. One gets that if m is odd and $u > 0$, then the point is a node, whereas if m is odd and $u < 0$, then it is a saddle. Finally, when m is even, it is a saddle-node.

For our system $h(z) = O(z^k)$,

$$X(z, h(z)) = X(z, O(z^k)) = vz^k + O(z^{k+1}), \quad v = \frac{(bc - ad)(-1)^k \alpha}{d^{k+1}},$$

and as a consequence $m = k$, $u = v$, and the lemma follows. □

Remark 2.8. *For completeness of previous lemma we include some comments about the case $d = 0$ and $\alpha bc \neq 0$. By Lemma 2.5, after a change of variables, the system writes as the Liénard system*

$$\dot{x} = y, \quad \dot{y} = bcf(x) + af'(x)y.$$

Hence the origin is a nilpotent singularity and the above expression is already in the normal form to apply the theorem of classification of Andreev, see for instance [1, 13]. Skipping all the details we obtain that when k is odd and $\alpha bc > 0$ the origin is a saddle, when k is odd and $\alpha bc < 0$ it is a focus when $a \neq 0$ and a center when $a = 0$. Finally, when k is even the point is the union of two hyperbolic sectors.

In what follows we study the behaviour of the trajectories of system (3) near infinity to characterize whether infinity is repulsive or attractive provided that $f(x)$ is a polynomial.

Lemma 2.9. *Consider system (3) with $ab \neq 0$, f polynomial, $f(x) = \alpha x^k + \sum_{k<i<m} f_i x^i + \beta x^m$ with $\alpha\beta \neq 0$, $k \leq m$ positive integers and $m > 1$. Then the following statements hold.*

(i) *The infinity is a repellor if and only if m is odd, $(ad - bc)\beta > 0$ and $a\beta < 0$.*

(ii) *The infinity is an attractor if and only if m is odd, $(ad - bc)\beta > 0$ and $a\beta > 0$.*

Proof. By Lemma 2.5 the system can be transformed into the generalized polynomial Liénard system (13). All these Liénard systems are studied in [14] by using the so called Poincaré-Lyapunov compactification. See for instance [13] for more details about this compactification. In particular, all the possible phase portraits in a neighborhood of infinity are classified and listed in [14]. By using their results the lemma follows. □

After studying the behaviour of the phase portrait close to infinity and close to the origin of system (3), we are in a position to study the

existence of limit cycles for some families of system (3) having a unique finite equilibrium point and to prove Theorem 1.2.

Proof of Theorem 1.2. Let us prove (i). When $\beta(ad-bc) = 0$ the result is a consequence of item (iv) of Lemma 2.1.

Next, we note that, by item (ii) of Lemma 2.6, when $ad - bc \neq 0$, the hypothesis that the origin is the unique singularity, implies that f has no positive zeros and $\alpha\beta > 0$. Therefore, when $\beta(ad - bc) < 0$, since $\alpha\beta > 0$, it also holds that $\alpha(ad - bc) < 0$. Hence, the origin, which is the unique equilibrium point of the system, is of saddle type by items $(i), (iii)$ of Lemma 2.7. As a consequence, no periodic orbit can appear in this case due to the fact that a saddle can never be the only equilibrium point surrounded by a periodic orbit.

$(ii) - (iii)$ In all these situations $\beta(ad - bc) > 0$ and $\alpha\beta > 0$. The proof is based on the following facts:

- By Lemma 2.9, the stability of infinity is given by the sign of $a\beta$.
- By Lemma 2.7, when $k = 1$ the stability of the origin is given by the sign of $a\alpha + d$. When this value is 0, the origin is a weak focus with purely imaginary eigenvalues and it is either a focus or a center. By the analyticity of the vector field and the first fact, i.e. since infinity is either repellor or attractor, when $a \neq 0$ the origin can not be a center. See also Remark 2.8.
- Again by Lemma 2.7, when $k > 1$ the stability of the origin is given by the sign of d.
- The above three items imply that there is no accumulation of limit cycles neither to the origin nor to the infinity.
- By the Poincaré-Bendixson theorem, since the origin is the only equilibrium point of the system and because the vector field is analytic, there is no accumulation of limit cycles to a non isolated periodic orbit. Moreover all limit cycles have finite multiplicity.

By joining all the above facts we have that the parity of the total number of limit cycles, counting each one of them with its corresponding multiplicity, depends only on the stabilities of the origin and of infinity. More concretely, if both stabilities coincide, then an odd number of limit cycles appears; while, in the opposite case, either none or an even number of them appear. Hence the result is proved.   □

Remark 2.10. *When, for a given f, system* (3) *has at most one limit cycle, taking into account its multiplicity, Theorem 1.2 distinguishes between the cases of existence (and then uniqueness) or non-existence of limit cycles.*

To set up a family to which we can apply the previous remark we have stated Corollary 1.4 on existence of at most of one limit cycle. This

corollary is a consequence of Theorem 1.3. In turn, Theorem 1.3 will be a consequence of the Bendixon-Dulac theorem for multiply connected regions, see [15, 16] and their references for more details about this classical theorem and other applications. Let us prove both results.

Proof of Theorem 1.3. If $b = 0$ the first equation of system (3) is simply $\dot{x} = af(x)$ and hence it has no periodic orbits. Hence, from now one we assume that $b \neq 0$.

We follow the ideas developed in [15, 16] looking for a suitable Dulac function, $1/V$, to apply Bendixson-Dulac theorem to our system in a suitable region. We start with the well known formula

$$\operatorname{div}\left(\frac{P}{V}, \frac{Q}{V}\right) = \frac{V \operatorname{div}(P,Q) - V_x P - V_y Q}{V^2} =: \frac{R}{V^2}, \qquad (15)$$

where we have omitted the explicit dependence on (x, y) in all the functions. The first key idea is to search for a function

$$V(x, y) = y^2 + v(x)y + w(x),$$

for some v and w differentiable, such that when we compute R with $(P, Q) = (af(x) + by, cf(x) + dy)$, the vector field associated to system (3), we obtain that the above R depends only on x. Some computations give that

$$
\begin{aligned}
R(x, y) =& \big(af'(x) - bv'(x) - d\big)y^2 \\
&+ \big(af'(x)v(x) - af(x)v'(x) - bw'(x) - 2cf(x)\big)y \\
&+ af'(x)w(x) - af(x)w'(x) - cf(x)v(x) + dw(x).
\end{aligned}
$$

Hence, to achieve our goal, we can choose v and w as any solution of the differential equations obtained by equating the coefficients of R of y and y^2 to zero. More concretely, we take

$$v(x) = \frac{a}{b}f(x) - \frac{d}{b}x, \quad w(x) = \frac{2(ad - bc)}{b^2}F(x) - \frac{ad}{b^2}xf(x).$$

Then,

$$R(x, y) = \frac{(ad - bc)}{b^2}M(x),$$

where we recall that

$$M(x) = 2af'(x)F(x) - a(f(x))^2 - dxf(x) + 2dF(x),$$

and that, by hypothesis, M does not change sign and vanishes only at isolated points. Moreover, by item (iv) of Lemma 2.1 when $ad - bc = 0$ the system has no periodic orbits, so we can assume that $ad - bc \neq 0$. Then we have that R does not change sign and vanishes only at isolated points.

Notice that, precisely

$$M|_{V=0} = -(V_x P + V_y Q). \qquad (16)$$

Hence the hypothesis on the sign of M implies that the periodic orbits of system (3) cannot cut the set $\{V(x,y) = 0\}$. Therefore, all the periodic orbits are contained in one of the connected components of $\mathbb{R}^2 \setminus \{V(x,y) = 0\}$. Moreover notice that on each of these connected components

$$\mathrm{div}\left(\frac{P}{V}, \frac{Q}{V}\right) = \frac{R}{V^2},$$

does not change sign and vanishes at isolated points. Hence we can apply the Bendixson-Dulac theorem, with Dulac function $1/V$, and we get that the maximum number of periodic solutions of system (3) in each of these connected components depends only on the topology of each one of them. More specifically, applying the result given in [15, 16], if the connected component region where R does not change its sign is simply connected then it contains no periodic orbits; if this connected component region has $k \geq 1$ holes, then it has at most k periodic orbits, all of them hyperbolic.

In our case, because V is quadratic on y we have that the set $\{V(x,y) = 0\}$ is precisely

$$y = \frac{-v(x) \pm \sqrt{v^2(x) - 4w(x)}}{2} = \frac{dx - af(x) \pm \sqrt{\Delta(x)}}{2b}.$$

To illustrate the shape of $\{V(x,y) = 0\}$, independently of whether M changes sign or not, in Figure 1 we plot it in red when $(a,b,c,d) = (-1,-1,-1,1)$ and $f(x) = T_{11}(x)$, the 11-th Chebyshev polynomial. In that figure, in blue we also plot the curve $\mathcal{C} = \{(x,y) : y = \frac{dx-af(x)}{2b}\}$. From this special shape, notice that the set $\{V(x,y) = 0\}$ is symmetric
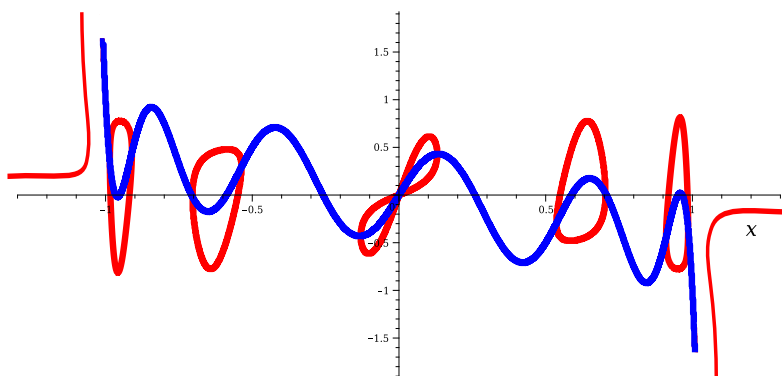

FIGURE 1. An example of set $\{V(x,y) = 0\}$, in red, together with its symmetry curve $\mathcal{C}$, in blue. It has 6 bubbles, with two colliding ones.

with respect the curve $\mathcal{C}$ and it is formed by several centered bubbles having $\mathcal{C}$ as a curve of symmetry in the following sense: the bubbles cut $\mathcal{C}$ at precisely two points whose x-coordinates are the extremes of each of the intervals defined by $\Delta(x) \geq 0$. These bubbles exist only in the zones where $\Delta(x) \geq 0$, they can collide between them, sharing one of their intersections with $\mathcal{C}$, or degenerate to a single point, also on $\mathcal{C}$. Finally, the set $\{V(x, y) = 0\}$ can contain one or two pieces, diffeomorphic to lines, each one of them associated to an unbounded interval where Δ is positive. These two pieces can eventually intersect with a bubble in a point of $\mathcal{C}$.

Notice that in this case, $\{V(x, y) = 0\}$ has 6 bubbles and two of them share a point. Hence the set $\Delta(x) \geq 0$ has 5 bounded closed intervals (plus 2 unbounded ones) and, consequently, $K = 5$.

In short, from all the given description, the set $\mathbb{R}^2 \setminus \{V(x, y) = 0\}$ has all its connected components simply connected but one, that has exactly K holes. Therefore only one of the components can contain limit cycles and it can contain at most K. Therefore, the Bendixson-Dulac theorem implies the existence of at most K limit cycles for our system. Let us prove that each existing limit cycle γ is hyperbolic. To do so it suffices to prove that its characteristic exponent

$$h(\gamma) = \int_0^T \operatorname{div}(P, Q)(x(t), y(t)) \, \mathrm{d}t \neq 0,$$

where $(x(t), t(t))$ is the time parameterization of γ and T its period, see [13]. The above condition holds because since $\gamma \cap \{V(x, y) = 0\} = \emptyset$, equality (15) can be written on γ as

$$\operatorname{div}(P, Q) = \frac{R}{V} + \frac{V_x P + V_y Q}{V},$$

and hence

$$h(\gamma) = \int_0^T \frac{R(x(t), y(t))}{V(x(t), y(t))} \, \mathrm{d}t + \ln\left(|V(x(t), y(t))|\right)\Big|_{t=0}^{t=T}$$

$$= \int_0^T \frac{R(x(t), y(t))}{V(x(t), y(t))} \, \mathrm{d}t \neq 0.$$

$\square$

Proof of Corollary 1.4. To prove the uniqueness of limit cycles we will use Theorem 1.3. We need to prove that, since the origin is the only equilibrium point of the system, we get that the value K in the statement of that theorem is at most one. This fact is a straightforward consequence of the following claim: each of the bubbles that constitute the set $\{V(x, y) = 0\}$ surrounds at least one critical point. Then the proof follows, because by assuming that this fact holds then only one bubble or two bubbles sharing a point can exist and so $K \leq 1$,

see the proof of Theorem 1.3 for more details about the shape of the $\{V(x, y) = 0\}$ set.

To prove the claim we use expression (16). This equality shows that the region surrounded by any bubble is either a positive or negative invariant region and, as a consequence, by the Poincaré-Bendixson theorem, it contains an equilibrium point of the system, either in this bounded open region or on its boundary. From this fact the result follows. $\square$

To prove next corollary we apply Theorem 1.3 to a very particular family of systems (3). Next corollary gives a different proof of the result given in [18, Thm. 1.1 & Cor. 1].

Corollary 2.11. *Consider system*

$$\dot{x} = ax^{2n-1} + by, \quad \dot{y} = cx^{2n-1} + dy,$$

where $n > 1$ is an integer number. This system has at most one limit cycle and, when it exists, it is hyperbolic. Moreover, the limit cycle exists if and only if $ad - bc > 0$ and $ad < 0$, and its stability is given by the sign of $-d$.

Proof. When $a = d = 0$ the system is integrable and it has no limit cycles. Hence, from now on we assume that $a^2 + d^2 \neq 0$.

By Theorem 1.2 the number of limit cycles of this system is: zero when $ad - bc \leq 0$; an even number or zero when $ad > 0$; and an odd number when $ad < 0$.

Let us prove first that when $ad \geq 0$ it has no periodic orbits. The divergence of the vector field is $(2n - 1)ax^{2(n-1)} + d$ and it does not change sign, and vanishes at most at the origin when $d = 0$. Hence, the non-existence of periodic orbits follows from the classical Dulac criterion.

Let us consider the case $ad - bc > 0$ and $ad < 0$. By applying Theorem 1.3 when $f(x) = x^{2n-1}$, we have that

$$M(x) = \frac{(n-1)}{n} x^{2n} (ax^{2(n-1)} - d),$$

and hence this function does not change sign. Finally to prove the corollary we need to compute the number K of bounded intervals where $\Delta(x) \geq 0$. In this case

$$\Delta(x) = x^2 \Big(\frac{a^2}{b^2} x^{8(n-1)} + \frac{(4n-6)ad + 4bc}{(2n-1)b^2} x^{4(n-1)} + \frac{d^2}{b^2} \Big),$$

and it is not difficult to prove that $K = 1$ and, hence, the uniqueness of the limit cycle follows.

The shape of the set $\{V(x, y) = 0\}$ is depicted in Figure 2 when $(a, b, c, d) = (1, -1, -1.05, -1)$. In all cases the proof of uniqueness of the limit cycle is similar. Notice that if the set $\{V(x, y) = 0\}$ has two bubbles sharing a point, then $K = 1$. For some values of the parameters,

these two bubbles can degenerate to be the single isolated point $(0,0)$, that always belongs to $\{V(x,y) = 0\}$.
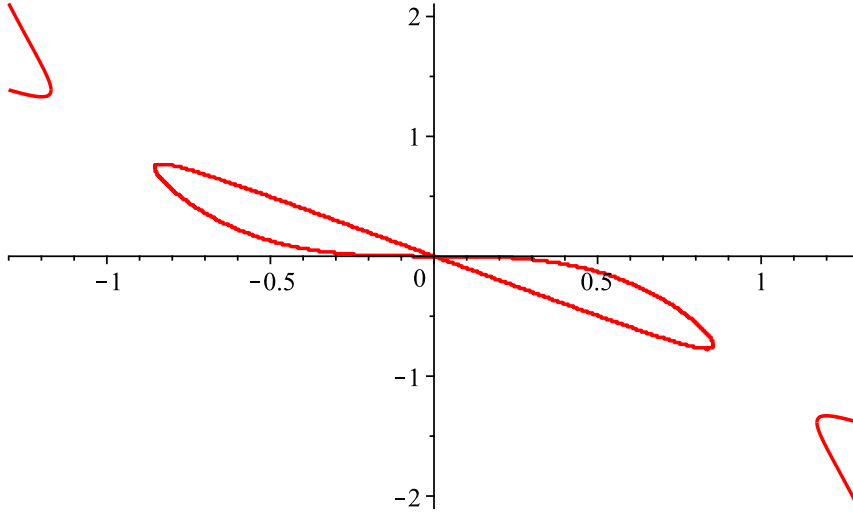

FIGURE 2. An example of set $\{V(x,y) = 0\}$ with two bubbles and sharing a common point.

As a consequence, using Theorem 1.2.(iii), the system in the statement of Corollary 2.11, when $ad - bc > 0$ and $ad < 0$, exactly has one limit cycle. Moreover, from Theorem 1.3, it is hyperbolic.

The stability of the origin is given by the sign of d and, since the limit cycle is hyperbolic, its stability is precisely the opposite of that of the origin. Using Lemmas 2.7 and 2.9, the proof of the corollary follows.   □

We remark that to get the uniqueness of the limit cycle in Corollary 2.11, instead of studying $\Delta(x)$, we also could have used Corollary 1.4 because the origin is the only equilibrium point of the system.

Next, we will use the following general result about Liénard systems, given in [28, Thm 4.1], to prove Proposition 2.13 and Corollary 2.14. This corollary characterizes the existence and uniqueness of limit cycles for a particular family of systems (3). It has also been used in [18, Lem. 2.5] to prove the uniqueness of the limit cycle of the system studied in Corollary 2.11.

Proposition 2.12 ([28, Thm 4.1]). *Consider the Liénard system*

$$\dot{x} = -y - F(x), \quad \dot{y} = g(x). \tag{17}$$

Assume that the next conditions hold:

   (a) g is an odd function, and $xg(x) > 0$ for $x \neq 0$;
   (b) F is an odd derivable function, and there exists $x_0 > 0$ such that $F(x) < 0$ for $0 < x < x_0$, and $F(x) \geq 0$ is monotonically increasing for $x \geq x_0$;

(c) $\int_0^\infty F'(x)\,\mathrm{d}x = \int_0^\infty g(x)\,\mathrm{d}x = +\infty;$

(d) F' and g satisfy the Lipschitz condition on any bounded interval.

Then, the system has a unique limit cycle which is stable.

A direct consequence of the above result is the next proposition.

Proposition 2.13. *The cubic generalized Liénard systems*

$$\dot{x} = y + px + qx^3, \quad \dot{y} = rx + sx^3, \tag{18}$$

with $rs \geq 0$, have at most one limit cycle. Moreover, it exists if and only if $r \leq 0$, $r^2 + s^2 \neq 0$ and $pq < 0$, and its stability is given by the sign of $-p$.

Proof. Since the divergence of the vector field is $p + 3qx^2$, by the classical Dulac theorem, the system has no limit cycles when $pq \geq 0$. Moreover since the origin is the unique equilibrium point and it is a saddle when $r > 0$, in this situation we know that it has no periodic orbits. Hence, for the rest of the proof we can assume $r \leq 0$ and $pq < 0$. By changing the sign of the time, the generalized Liénard system (18) writes as

$$\dot{x} = -y - (px + qx^3), \quad \dot{y} = -(rx + sx^3), \tag{19}$$

that is of the form (17) with $F(x) = px + qx^3$ and $g(x) = -rx - sx^3$. Changing (x, y, t) by $(-x, y, -t)$, if necessary, we get that it is not restrictive to consider $p < 0$. Hence system (19) is under the hypotheses of Proposition 2.12 and it has a unique limit cycle which is stable.

Due to the uniqueness and stability of the limit cycle we get that under these last conditions the stability of the limit cycle of the system (18) is given by the sign of $-p$ because it is the opposite of that of the stability of the origin. $\square$

A corollary of this proposition is the next result.

Corollary 2.14. *System*

$$\dot{x} = a(\alpha x + x^3) + by, \quad \dot{y} = c(\alpha x + x^3) + dy, \tag{20}$$

with $\alpha \geq 0$ has at most one limit cycle. Moreover, the limit cycle exists if and only if $ad - bc > 0$ and $a(a\alpha + d) < 0$, and its stability is given by the sign of $-(a\alpha + d)$.

Proof. By Lemma 2.5, system (20) can be written as the generalized Liénard system

$$\dot{x} = y + (a\alpha + d)x + ax^3, \quad \dot{y} = (bc - ad)(\alpha x + x^3).$$

Hence, by applying Proposition 2.13 the result follows. $\square$

Notice that the above result, when $\alpha = 0$, coincides with Corollary 2.11 when $n = 2$. Clearly, Corollary 2.14 could be extended to $f(x) = \alpha x + x^{2n-1}$, with $n > 1$.

3. Random systems

In this section we will study the probability of existence of limit cycles for the families of systems considered in the previous section. More concretely, we give a general result for the random system (4), Proposition 3.1, and as a consequence we prove Theorem 1.5 and Corollaries 1.6 and 1.7. Finally, in Theorem 1.5, we make some comments about how we have used Monte Carlo simulation to approach the function $P^+(\alpha)$.

Proposition 3.1. *Consider the following random system*

$$\dot{x} = Af(x) + Bg(y), \quad \dot{y} = Cf(x) + Dg(y),$$

where A, B, C, D are iid random variables with $N(0,1)$ Gaussian distributions and where g and f are smooth functions such that $f(0) = g(0) = 0$. Then, the probability that it does not have periodic orbits is greater than or equal to $1/2$. Equivalently, the probability of having some limit cycles is smaller than or equal to $1/2$.

Proof. By using Lemma 2.1, the probability that this system does not have periodic orbits is greater or equal to $P(ABCD < 0)$. Notice that the two random variables $ABCD$ and $-ABCD$ have the same distribution because, for instance, this happens with A and $-A$. Therefore

$$P(ABCD < 0) = P(-ABCD < 0) = P(ABCD > 0).$$

Since $P(ABCD = 0) = 0$, we get that $P(ABCD > 0) = P(ABCD < 0) = 1/2$. Thus, the probability for this system of having at least one limit cycle is $\leq 1/2$. $\qquad\square$

Proof of Theorem 1.5. Let us prove (i). We define the new iid random variables $X = AD$ and $Y = BC$. Note that the density function of the new variables X and Y is an even function. Then, the joint density function of (X, Y), $h(x, y)$, is symmetric respect to the origin, that is $h(x, y) = h(-x, -y)$.

From Theorem 1.2, to compute the probability of having an odd number of limit cycles we have to compute the $p = P(\beta(X - Y) > 0, \beta X < 0)$. Notice that $X - Y$ and $Y - X$ have the same distribution, and the same happens with X and $-X$. Thus, independently of the sign of β, $p = P(X - Y > 0, X < 0) = P(X - Y < 0, X > 0)$. Finally,

$$P(X - Y > 0, X < 0) = \iint_{\{(x,y)\in\mathbb{R}^2 \,:\, x-y>0, x<0\}} h(x, y)\,\mathrm{d}x\,\mathrm{d}y = \frac{1}{8}.$$

Again by Theorem 1.2, the probability of having none or an even number of limit cycles is the probability of the complementary set, modulus a set of zero measure, that is $7/8$.

(ii) By items (ii) and (iii) of Theorem 1.2 to compute $P^+(\alpha)$ we simply have to compute $P^+(\alpha) = P(AD - BC > 0, A(A\alpha + D) < 0)$

which is given by

$$P^+(\alpha) = \frac{1}{4\pi^2} \iiiint_{T(\alpha)} \mathrm{e}^{-\frac{a^2+b^2+c^2+d^2}{2}} \,\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c\,\mathrm{d}d,$$

where we recall that

$$T(\alpha) = \{(a,b,c,d) \,:\, ad - bc > 0,\ a(a\alpha + d) < 0\},$$

which is precisely the integral expression given in the statement. The fact that $P^+(0) = 1/8$ is simply because, when $\alpha = 0$, the set $T(0)$ coincides with the one considered in item (i) above. Moreover, the properties of the function $\alpha \to P^+(\alpha)$ are a consequence of the shape of the set $T(\alpha)$. It shrinks when α increases and, in fact,

$$\lim_{\alpha \to -\infty} T(\alpha) = \{ad - bc > 0\} \quad \text{and} \quad \lim_{\alpha \to +\infty} T(\alpha) = \emptyset.$$

Finally, to prove item (iii) we have to compute $P^-(\alpha)$ that is the probability of the event $\beta(AD - BC) > 0$ and $\beta A(A\alpha + D) < 0$. Therefore it is equivalent to prove that $AD - BC < 0$ and $A(-\alpha A - D) < 0$. Since $AD - BC$ has the same distribution as $BC - AD$ and $-D$ the same distribution as D, we obtain that $P^-(\alpha)$ can also be computed as the probability of occurrence of the event $AD - BC > 0$ and $A(-\alpha A + D) < 0$. This probability is precisely $P^+(-\alpha)$, as we wanted to show.

$\square$

Proof of Corollary 1.6. By Corollary 2.11 we know that each realization of the random system (5) has at most one limit cycle. Then the result is a straightforward consequence of Theorem 1.5 when $k > 1$. $\square$

Next we state a result that implies Corollary 1.7.

Proposition 3.2. *Consider the random system*

$$\dot{x} = A(\alpha x + x^3) + By, \quad \dot{y} = C(\alpha x + x^3) + Dy,$$

with $\alpha > 0$ and A, B, C, D iid random variables with $N(0,1)$ distribution. Then, for each $\varepsilon > 0$ there exists α big enough such that it has limit cycles with a positive probability, smaller than ε. Moreover, when the limit cycle exists it is unique.

Its proof is a straightforward consequence of Corollary 2.14 and of the property $\lim_{\alpha \to \infty} P^+(\alpha) = 0$ proved in Theorem 1.5.

3.1. **Some comments about Monte Carlo simulation.** To get our results in Table 1 we use Monte Carlo simulation. We take $N = 10^6$ or $N = 10^8$ samples of the random vector (A, B, C, D) where the four variables are iid, with distribution $N(0,1)$, and check how many of them, say J, satisfy $AD - BC > 0$ and $A(\alpha A + D) < 0$. Then simply $P^+(\alpha) \approx J/N$. Due to the law of large numbers and the law of iterated logarithm it is known that this approach gives an absolute error of

order $O(((\log \log N)/N)^{1/2})$, which is essentially $O(N^{-1/2})$, see [21, 25]. Hence, for $N = 10^6$ (resp. 10^8) this absolute error is expected to be of order 10^{-3} (resp. 10^{-4}). For this reason in Table 1 we only show 4 (resp. 5) digits of the Monte Carlo results.

To illustrate the above facts we have plotted in Figure 3, $P^+(\alpha)$ obtained with samples of either 10^4 or 10^6 points for 101 equidistributed values of α in $[-10, 10]$. Observe how increasing N, more regular the approximation is. Notice that from its expression it is clear that the actual $P^+(\alpha)$ is smooth. It is also apparent from the figure that 10^4 samples do not suffice to have a good approximation.
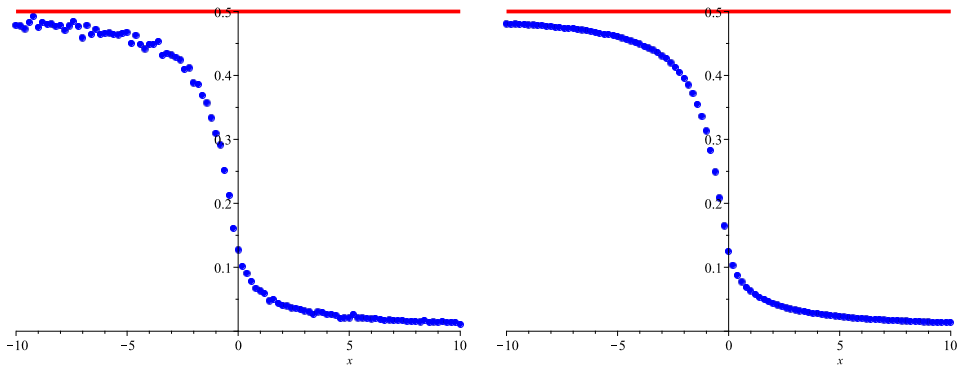

FIGURE 3. Numerical approximation of $P^+(\alpha)$ obtained with Monte Carlo simulation with samples of 10^4 (left panel) and 10^6 (right panel) points for 101 equidistributed values of α in $[-10, 10]$.

For some more comments concerning what can really be said about how small is $|P(\alpha) - J/N|$ see for instance [10, Sec. 3.2].

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