INDICES OF POLYNOMIAL VECTOR FIELDS WITH APPLICATIONS

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Certifico que la present memoria ha estat realitzada per Anna Cima i Mollet, i dirigida per mi, al Departament de Matemàtiques de la Universitat Autònoma de Barcelona.

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INTRODUCTION

The global study of the curves defined by differential equations begin with the paper of H.Poincaré "Integral curves defined by differential equations" in 1881. With this work it is open a line of thought known by the Qualitative Theory of Differential Equations.

Given an autonomous differential equation $\dot{x} = X(x)$ with $X : U \longrightarrow \mathbf{R}^n$, where U is an open subset of $\mathbf{R}^n$ the Qualitative Theory study the behaviour of the orbits. In the case that X has polynomials as a components (in what follows we shall say that X is polynomial), the Index Theory has an special signification. It is due to the fact that the polynomial vector field X can be extended in an analytical way to a vector field defined on $\mathbf{S}^n$, and on $\mathbf{S}^n$ it is possible to apply de Poincaré-Hopf Theorem. On the other hand, if X is polynomial we can use Bezout Theorem by studying its critical points.

In *CHAPTER 1* by using Poincaré -Hopf Theorem and Bezout Theorem for a planar polynomial vector field X we obtain *the inequalities*:

(a) $\mid \sum_f i \mid \leq min\{deg(P), deg(Q)\}$,

(b) $\sum_f \mid i \mid \leq deg(P)deg(Q)$,

where P and Q are the components of X, $\sum_f i$ denotes the sum of the indices at the critical points of X and $\sum_f \mid i \mid$ denotes the sum of the absolute values of the indices at the critical points of X. In *CHAPTER 2* as an application of these results we obtain *a bound for the number of critical points of X with index $\pm$ 1.*

By using the Poincaré-Bendixson theory in the plane, it is known that the α and ω-limit set of the orbits are either the empty set, or a critical point, or a periodic orbit, or a graphic. The isolated periodic orbits or limit cycles were introduced by Poincaré and there are many open problems around them. One of these problems is the finiteness conjecture of Poincaré:"Each polynomial vector field in $\mathbf{R}^2$, has a finite number of limit-cycles". More ambitious is the conjecture given in the second half of the Hilbert's sixteenth problem:"What is the maximum number $N(n)$ of limit-cycles for a polynomial vector field $X = (P, Q)$ with $deg(P) \leq n$, $deg(Q) \leq n$ and what are their relative positions". Almost **nothing** is known about the function $N(n)$, only certain inferior bounds for an arbitrary n. Concerning at the finiteness problem, there are some results.Il'yasenko (1985) proves that a cycle of separatrices such that all this vertices are non degenerate saddles cannot be accumulation of limit cycles.Recently (1987), J.Ecalle, J.Martinet, R.Moussu and J.P.Ramis have presented a note in C.R. Acad.Sc. Paris where they claim that any graphic of an analytical vector field in the plane can be accumulation of limit cycles. In particular, it follows the finiteness conjecture.

By using the results obtained in Chapter 1, in *CHAPTER 3* we study *the distribution and the maximum number of nests of limit cycles.*A nest of limit cycles is an equivalence class under the following relation:two limit cycles C_1 and C_2 are equivalent if and only if $s(C_1) = s(C_2)$, where $s(C)$ denotes the set of all the critical points of X which are inside the limit cycle C.

Among the polynomial vector fields in the plane, the quadratic systems (i.e., $X = (P, Q)$ with $\max\{deg(P), deg(Q)\} = 2$) have been specially studied. For such vector fields there are several classification theorems and relations between the properties of the vector field and its limit cycles.At the same time, these vector fields provide a source of exemples and counterexemples. To advance in this line, in *CHAPTER 4* we give *an algebraic classification of the homogeneous vector fields in the plane;* this classification allow us to reduce the number of parameters in the study of the cubic systems and, at the same time, we know the critical points which the system has at infinity.After we give a topological classification of the homogeneous cubic vector fields, obtaining all possible phase-portraits. To do it, we use the algebraic classification and the result (a) given in Chapter 1.

CHAPTER 5 is devoted to study bounded polynomial vector fields. A vector field is bounded if the positive semiorbit lies inside some compact set. We prove that *every bounded polynomial vector field in the plane satisfies* $\sum_f i = 1$. By using the stereografic compactification, the index formula of Bendixson and the Poincaré- Hopf Theorem, we give a simple proof which is not valid in $\mathbf{R}^n$. *For bounded polynomial vector fields in* $\mathbf{R}^n$ *we prove generically that* $\sum_f i = (-1)^n$.

Finally in *CHAPTER 6 we clarify the relation between the index, the multiplicity and the intersection number at a critical point.* Each one of these positive integers are used in the local study at the isolated critical points. As an application of this study *we give a new proof of result* (b) *obtained in Chapter 1 for polynomial vector fields in* $\mathbf{R}^n$.

At the begining of each chapter there is a more detailed introduction.

M'agradaria expressar al Dr. Jaume Llibre el meu agraiment per l'estimul que hi rebut en tot moment i tambè per la dedicació amb que ha dirigit aquest treball. Al mateix temps agraiexo al Departament de Teoria e Historia Económica el suport que he rebut durant aquests anys.