NILPOTENT BI-CENTERS IN CONTINUOUS PIECEWISE $\mathbb{Z}_2$-EQUIVARIANT CUBIC POLYNOMIAL HAMILTONIAN VECTOR FIELD: CUSP-CUSP TYPE

TING CHEN[1] AND JAUME LLIBRE[2]

ABSTRACT. In this paper we study the global dynamics for a class of continuous piecewise $\mathbb{Z}_2$-equivariant cubic Hamiltonian vector fields with nilpotent bi-centers at $(\pm 1, 0)$. We consider these polynomial vector fields with a challenging case where the bi-centers $(\pm 1, 0)$ come from the combination of two nilpotent cusps separated by $y = 0$. We call it a cusp-cusp type. We use the Poincaré compactification, the blow-up theory, the index theory and the theory of discriminant sequence for determining the number of distinct or negative real roots of a polynomial, to classify the global phase portraits of these vector fields in the Poincaré disc.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

From the works of Poincaré [34] and Dulac [17] the *center-focus problem*, i.e. the problem of distinguishing between a focus and a center, has been one of the important problems in the qualitative theory of planar differential vector fields. To overcome this classical problem, many methods have been developed, such as Poincaré-Liapunov method, Melnikov function method, Poincaré compactification method, and so on, see [3, 4, 5, 7, 18, 23, 33]. Thus, for instance, in the articles [14, 15, 25, 38, 39, 40] these methods have been used for studying the center-focus problem of the quadratic and cubic polynomial differential vector fields.

In recent years in order to model better some natural phenomena many authors started to analyze the non-smooth vector fields see for instance [1, 2, 6, 19, 32]. In this paper we deal with the following family of piecewise smooth vector fields

$$(1)\quad (\dot{x}, \dot{y}) = \left\{ \begin{array}{l} (-y,\ x) \\ (0,\ x) \\ (0,\ 0) \end{array} \right\} + \big(F^{\pm}(x,y), G^{\pm}(x,y)\big) = \big(P^{\pm}(x,y), Q^{\pm}(x,y)\big) \quad \text{for} \quad \pm\,\Gamma(x,y) \geq 0,$$

where $\Gamma : \mathbb{R}^2 \to \mathbb{R}$ is a $\mathcal{C}^\infty$ function, $F^{\pm}(x,y)$ and $G^{\pm}(x,y)$ are real polynomials without constant and linear terms. In fact, the vector field of (1) has two different regions $\Gamma^+ = \{(x,y) \in \mathbb{R}^2 : \Gamma(x,y) > 0\}$ and $\Gamma^- = \{(x,y) \in \mathbb{R}^2 : \Gamma(x,y) < 0\}$ separated by the line $S = \Gamma^{-1}(0)$. We say that an equilibrium point q of the piecewise smooth vector field (1) is a *center* if there is a neighborhood U of this equilibrium point, such that $U \setminus \{q\}$ is filled with periodic orbits. When the origin of the piecewise smooth vector field (1) is a center, it is called a *linear type center*, a *nilpotent center*, or a *degenerate center* if in (1) we have $(-y, x)$, $(0, x)$, or $(0, 0)$, respectively. It is well known that the center-focus problem of a piecewise smooth vector field (1) becomes much more difficult than for smooth vector fields. The classical methods have been developed for studying the center-focus problem of a differential vector field (1) with the linear type, see [8, 16, 21, 22, 36]. But this problem for the piecewise smooth quadratic polynomial differential vector fields still remains open.

As far as we know the center-focus problem for a non-elementary equilibrium point of a polynomial vector field is much more challenging compared with the study for an elementary equilibrium point. In order to overcome this type problem the authors of [20, 30, 31, 35] developed some computationally efficient methods for planar smooth vector fields with a nilpotent

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equilibrium point. However, there are not many papers for studying piecewise smooth polynomial vector fields with a nilpotent center. The possible local phase portraits of nilpotent equilibrium points are more rich than the ones of elementary equilibrium points. So it is natural to ask how a nilpotent equilibrium point of piecewise smooth vector fields can be a center or a focus? Recently Chen *el at.* [9, 13] developed the Poincaré-Lyapunov and Poincaré compactification methods for studying piecewise smooth vector fields with a nilpotent equilibrium point.

Planar $\mathbb{Z}_n$-equivariant smooth vector fields are important systems for studying the famous Hilbert's 16th problem. The phase portraits of these vector fields are unchanged by a rotation of $2\pi/n$ ($n \in \mathbb{Z}^+$) radians around one point, see [10, 11, 24, 26, 27, 28, 29]. Although this symmetry helps in order to find more limit cycles, the study of piecewise smooth vector fields with many parameters is very difficult. The main goal of this paper is to study the global dynamics of some piecewise $\mathbb{Z}_n$-equivariant polynomial vector fields, in particular with nilpotent centers, for which there are almost no results in the qualitative theory of differential vector fields.

The general piecewise smooth vector field (1) is $\mathbb{Z}_2$-equivariant if and only if it satisfies

$$(P^+(-x,-y), Q^+(-x,-y)) = (-P^-(x,y), -Q^-(x,y)),$$

where also $\Gamma(-x,-y) = -\Gamma(x,y)$. Without loss of generality a piecewise $\mathbb{Z}_2$-equivariant cubic polynomial vector field separated by the straight line $y = 0$ can be written as

$$(2)\qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 \\ + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 \\ + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3, \end{pmatrix} & \text{for } \; y \geq 0, \\ \begin{pmatrix} -a_{00} + a_{10}x + a_{01}y - a_{20}x^2 - a_{11}xy - a_{02}y^2 + a_{30}x^3 \\ + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\ -b_{00} + b_{10}x + b_{01}y - b_{20}x^2 - b_{11}xy - b_{02}y^2 + b_{30}x^3 \\ + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3, \end{pmatrix} & \text{for } \; y \leq 0, \end{cases}$$

where a_{ij} and b_{ij} are real parameters. Assume that system (2) is Hamiltonian with two isolated nilpotent equilibrium points at $(\pm 1, 0)$, this implies that

$$(3)\qquad \begin{array}{l} a_{01} = -a_{21}, \quad a_{12} = -3b_{03}, \quad b_{10} = -b_{30}, \quad b_{12} = -a_{21}, \\ a_{00} = a_{10} = a_{20} = a_{11} = a_{30} = b_{00} = b_{01} = b_{20} = b_{11} = b_{02} = b_{21} = 0. \end{array}$$

The Jacobian matrices at $(\pm 1, 0)$ are

$$\begin{pmatrix} 0 & 0 \\ 2b_{30} & 0 \end{pmatrix}.$$

After doing convenient rescalings in the time and the parameters in system (2) it transforms the above Jacobian matrix into the canonical one given in (1), i.e. $b_{30} = \frac{1}{2}$.

With conditions (3) and $b_{30} = \frac{1}{2}$ the vector field (2) becomes the following continuous Hamiltonian vector field

$$(4)\qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -a_{21}y + a_{02}y^2 + a_{21}x^2y - 3b_{03}xy^2 + a_{03}y^3, \\ -\dfrac{1}{2}x + \dfrac{1}{2}x^3 - a_{21}xy^2 + b_{03}y^3, \end{pmatrix} & \text{for } \; y \geq 0, \\ \begin{pmatrix} -a_{21}y - a_{02}y^2 + a_{21}x^2y - 3b_{03}xy^2 + a_{03}y^3, \\ -\dfrac{1}{2}x + \dfrac{1}{2}x^3 - a_{21}xy^2 + b_{03}y^3, \end{pmatrix} & \text{for } \; y \leq 0, \end{cases}$$

because

$$(P^+(x,0), Q^+(x,0)) = (P^-(x,0), Q^-(x,0)).$$

The Hamiltonian functions for system (4) are

$$H^{+}(x,y)=\frac{1}{4}x^2-\frac{1}{8}x^4-\frac{1}{2}a_{21}y^2+\frac{1}{3}a_{02}y^3+\frac{1}{2}a_{21}x^2y^2-b_{03}xy^3+\frac{1}{4}a_{03}y^4,$$

for the Hamiltonian vector field in $y\geq 0$, and

$$H^{-}(x,y)=\frac{1}{4}x^2-\frac{1}{8}x^4-\frac{1}{2}a_{21}y^2-\frac{1}{3}a_{02}y^3+\frac{1}{2}a_{21}x^2y^2-b_{03}xy^3+\frac{1}{4}a_{03}y^4,$$

for the Hamiltonian vector field in $y\leq 0$.

Now we consider the case when the nilpotent equilibrium points $(\pm 1,0)$ of the vector field (4) are bi-centers. By the transformation $x\to x+1$ the vector field (4) becomes

(5)

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{cases}\begin{pmatrix}2a_{21}xy+a_{21}x^2y+(a_{02}-3b_{03})y^2-3b_{03}xy^2+a_{03}y^3=P^{+}(x,y),\\ x+\dfrac{3x^2}{2}+\dfrac{x^3}{2}-a_{21}y^2-a_{21}xy^2+b_{03}y^3=x+G^{+}(x,y),\end{pmatrix} & \text{for } y\geq 0,\\ \begin{pmatrix}2a_{21}xy+a_{21}x^2y-(a_{02}+3b_{03})y^2-3b_{03}xy^2+a_{03}y^3=P^{-}(x,y),\\ x+\dfrac{3x^2}{2}+\dfrac{x^3}{2}-a_{21}y^2-a_{21}xy^2+b_{03}y^3=x+G^{-}(x,y),\end{pmatrix} & \text{for } y\leq 0,\end{cases}$$

and so the equilibrium point $(1,0)$ of the vector field (4) is moved to the origin of the vector field (5). We assume that

$$x=f^{\pm}(y)=\sum_{k=2}^{\infty}f_k^{\pm}y^k$$

are the unique solutions of $x+G^{\pm}(x,y)=0$ in a neighborhood of the origin, respectively. For studying the local phase portrait at the origin we write

$$P^{\pm}(f^{\pm}(y),y)=\sum_{k=2}^{\infty}\lambda_k^{\pm}y^k,$$

$$\left[\frac{\partial P^{\pm}}{\partial x}+\frac{\partial G^{\pm}}{\partial y}\right]_{(f^{\pm}(y),y)}=\sum_{k=1}^{\infty}\mu_k^{\pm}y^k,$$

where

$$\lambda_2^{\pm}=\pm a_{02}-3b_{03},\qquad \lambda_3^{\pm}=a_{03}+2a_{21}^2.$$

Let m be the smallest k for which $\mu_k\neq 0$, and let n be the smallest k for which $\lambda_k\neq 0$. It follows from Theorem 3.5 of [18] that if all the μ_k are zero and $\lambda_n\neq 0$ this nilpotent equilibrium point is a

$$\begin{cases}\text{center or focus if } n=2l+1 \text{ and } \lambda_n<0,\\ \text{saddle if } n=2l+1 \text{ and } \lambda_n>0,\\ \text{cusp if } n=2l.\end{cases}$$

If $\mu_m\neq 0$, $\lambda_n\neq 0$, $\kappa=\mu_m^2+4(m+1)\lambda_n$ this nilpotent equilibrium point is a

$$\begin{cases}\text{cusp if } n=2l,\ l\leq m,\\ \text{saddle-node if } n=2l,\ l>m,\\ \text{saddle if } n=2l+1,\ \lambda_n>0,\\ \text{center or focus if } n=2l+1,\ \lambda_n<0,\ l<m, \text{ or } l=m \text{ and } \kappa<0\ ,\\ \text{E-H point if } m \text{ is odd, } n=2l+1,\ \lambda_n<0,\ l>m, \text{ or } l=m \text{ and } \kappa\geq 0\ ,\\ \text{node if } m \text{ is even, } n=2l+1,\ \lambda_n<0,\ l>m, \text{ or } l=m \text{ and } \kappa\geq 0,\end{cases}$$

where the local phase portrait of an E-H equilibrium point is formed by one elliptic sector and one hyperbolic sector.

Since the smallest subindex of a nilpotent monodromic equilibrium point (i.e. a center or a focus) of a smooth differential vector field is an odd positive integer greater than one, in [12] we study one class of vector fields (4) when the equilibrium point $(1,0)$ of the first vector field of (4) is a monodromic equilibrium point with subindex 3, that is $\lambda_2^{+}=0$ and $\lambda_3^{+}<0$.

In this paper we study the case $\lambda_2^+ \neq 0$, i.e. $a_{02} \neq 3b_{03}$ then the equilibrium point $(1,0)$ in the first smooth vector field of (4) is a cusp. If $\lambda_2^- = 0$ and $\lambda_3^- \neq 0$, this case is included in [12]. Hence we assume that $\lambda_2^- \neq 0$, then the equilibrium point $(1,0)$ in the second smooth vector field of (4) is also a cusp. Recall that the equilibrium points ($\pm$1,0) cannot be monodromic equilibrium points when $\lambda_2^+ > 0$ or $\lambda_2^- < 0$, so we only study the class of continuous piecewise $\mathbb{Z}_2$-equivariant cubic Hamiltonian vector fields (4) with $\lambda_2^+ = a_{02} - 3b_{03} < 0$ and $\lambda_2^- = -a_{02} - 3b_{03} > 0$, i.e.

$$a_{02} < \min\{-3b_{03}, 3b_{03}\}.$$

By the symmetry both nilpotent equilibrium points $(\pm 1, 0)$ come from the combination of two cusps. From Proposition 2.1 of [8] we have that the Hamiltonians of the first and second vector fields of (4) satisfy $H^+(x,0) \equiv H^-(x,0)$. Hence these two equilibrium points $(\pm 1, 0)$ of the vector field (4) are bi-centers.

We remark that the vector field (4) depends on four real parameters a_{02}, a_{21}, a_{03} and b_{03}, therefore it is very hard to study the global dynamics of these piecewise smooth vector fields. Thus the explicit expressions of the finite equilibrium points and infinite equilibrium points are in general complicated, and it is not easy to obtain their existence and local phase portraits. The tools used to overcome these difficulties also can be used to analyze more complicated piecewise smooth differential vector fields. It should be noted that the vector field (4) is invariant under the transformation

$$(x, y, t, a_{02}, a_{21}, a_{03}, b_{03}) \to (-x, y, -t, a_{02}, a_{21}, a_{03}, -b_{03}),$$

hence the global dynamics of the vector field (4) with $b_{03} < 0$ and $b_{03} > 0$ are topologically equivalent. But we will show all results with $b_{03} < 0$ and $b_{03} > 0$.

Theorem 1.1. *Assume that $a_{02} < min\{-3b_{03}, 3b_{03}\}$. In the Poincaré disc the global phase portraits of the piecewise $\mathbb{Z}_2$-equivariant cubic Hamiltonian vector field* (4) *with cusp-cusp type and bi-centers at* $(\pm 1, 0)$ *are topologically equivalent to one of the following* 52 *phase portraits of Figures* 1 *and* 2.

In section 2 we study the infinite equilibrium points of the piecewise smooth vector field (4). In section 3 we analyze the finite equilibrium points and characterize the global phase portraits of the vector field (4) in the Poincaré disc, that is we prove Theorem 1.1.

2. Infinite equilibrium points of the vector field (4)

In this section first we summarize the results on the Poincaré compactification that we need for presenting the phase portraits of the vector field (4) in the Poincaré disc, for more details see Chapter 5 of [18]. In order to study the infinity of the piecewise polynomial differential vector fields we present the Poincaré compactification of the piecewise differential vector fields. This tool identifies the plane $\mathbb{R}^2$ in $\mathbb{R}^3$ defined with the point $\mathbf{x} = (s_1, s_2, s_3) = (x_1, x_2, 1)$. The plane $\mathbb{R}^2$ is identified with the interior of the closed unit disc $\mathbb{D}^2$ centered at the origin of the plane, and extends analytically the piecewise differential vector fields to the boundary of $\mathbb{D}^2$, which is called the circle $\mathbb{S}^1$ of the infinity for piecewise polynomial differential vector fields. The sphere $\mathbb{S}^2 = \{\mathbf{x} \in \mathbb{R}^3 : s_1^2 + s_2^2 + s_3^2 = 1\}$ is called the Pointcaré sphere. The closed disc $\mathbb{D}^2$ is called the Poincaré disc. The equilibrium points in the interior of the disc $\mathbb{D}^2$ are called *finite equilibrium points*, and the ones on the boundary of $\mathbb{D}^2$ are called *infinite equilibrium points*. The dash line "$---$" in the phase portrait of the Poincaré disc is the separated line of piecewise differential vector fields.

In the plane $\mathbb{R}^2$ we consider the piecewise polynomial differential vector field separated by $x_2 = 0$, described by

$$(\dot{x_1}, \dot{x_2}) = (P^\pm(x_1, x_2), Q^\pm(x_1, x_2)), \quad \text{for} \quad \pm x_2 \geq 0, \tag{6}$$

where the degree of the real vector fields $(P^\pm(x_1, x_2), Q^\pm(x_1, x_2))$ are $d^\pm$, respectively.

For studying the neighborhood of the infinity of the piecewise polynomial differential vector field (6) in $\mathbb{R}^2$, we use four local charts $U_i = \{(x_1, x_2) \in \mathbb{D}^2 : \ x_i > 0\}$ and $V_i = \{(x_1, x_2) \in$

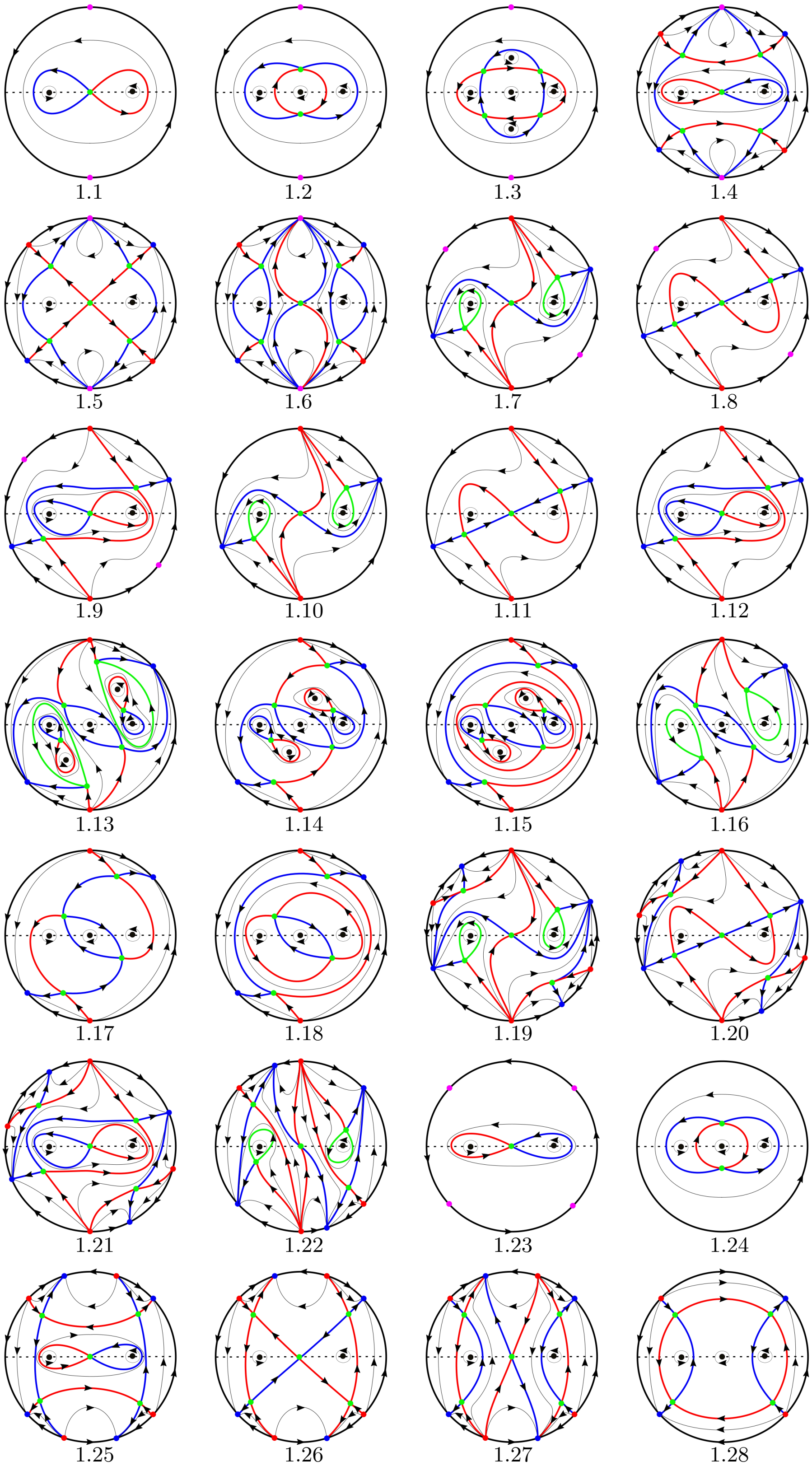


FIGURE 1. The 1st to 28th topological phase portraits of the vector field (4) in the Poincaré disc.

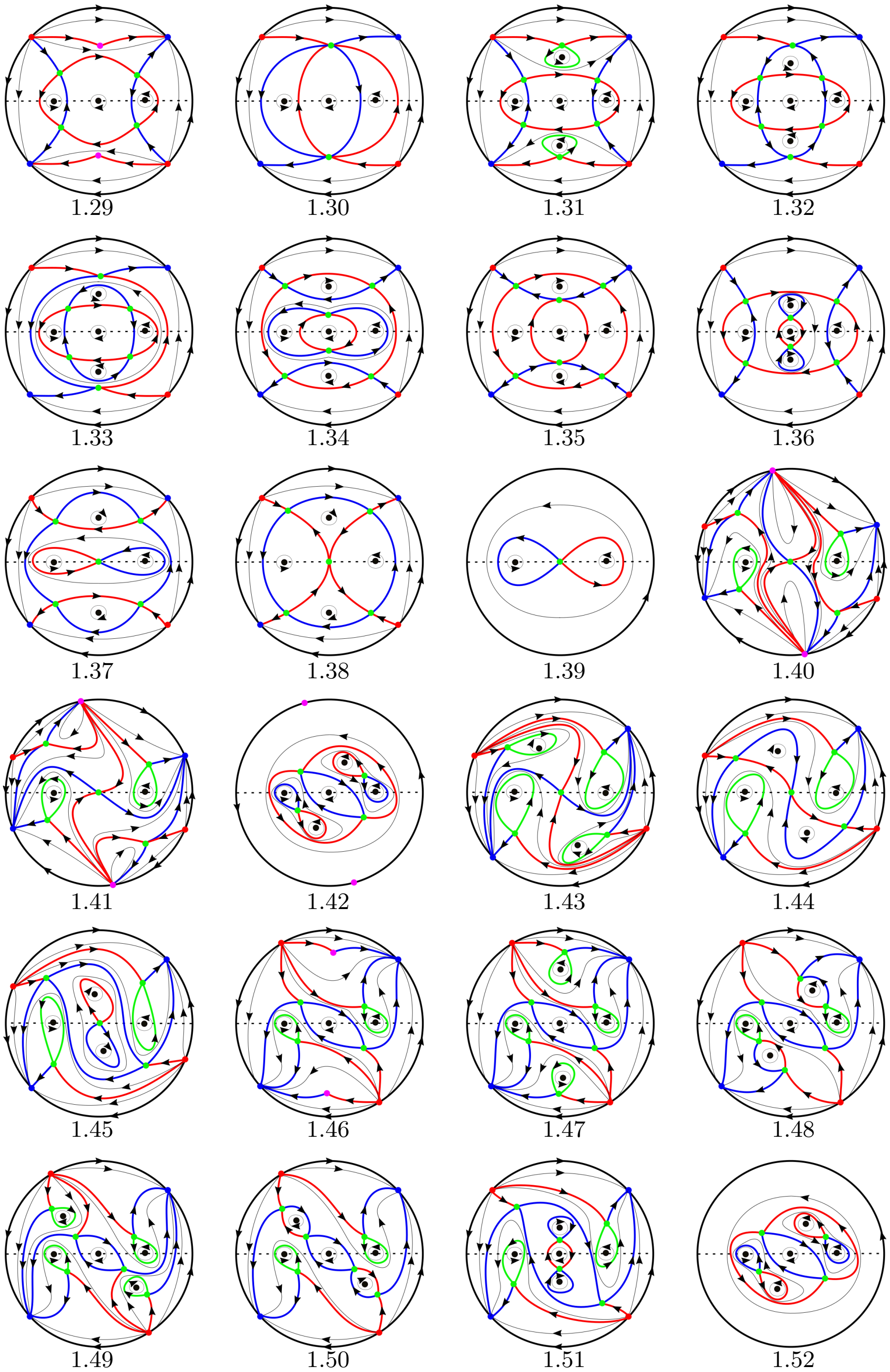


FIGURE 2. The 29th to 52th topological phase portraits of the vector field (4) in the Poincaré disc.

$\mathbb{D}^2: \ x_i < 0\}$, for $i = 1, 2$ to do the calculations. The corresponding diffeomorphisms of these charts are

$$\Phi_i: \ U_i \to \mathbb{R}^2, \qquad \Psi_i: \ V_i \to \mathbb{R}^2, \tag{7}$$

defined by $\Phi_1(x_1, x_2) = \Psi_1(x_1, x_2) = (\frac{x_2}{x_1}, \frac{1}{x_1}) = (u, v)$ and $\Phi_2(x_1, x_2) = \Psi_2(x_1, x_2) = (\frac{1}{x_2}, \frac{x_1}{x_2}) = (u, v)$, so the coordinates (u, v) will play distinct roles in the different local charts.

The expressions of the piecewise polynomial differential vector field (6) in the local chart U_1 with $u \geq 0$ and $u \leq 0$ are

$$(\dot{u}, \dot{v}) = \left(v^{d^{\pm}}(Q_1^{\pm} - uP_1^{\pm}), -v^{d^{\pm}+1}P_1^{\pm}\right), \quad \text{for} \quad \pm u \geq 0,$$

where $P_1^{\pm} = P^{\pm}(\frac{1}{v}, \frac{u}{v})$ and $Q_1^{\pm} = Q^{\pm}(\frac{1}{v}, \frac{u}{v})$. The expressions in V_1 are equal to the corresponding expressions in U_1 multiplied by $(-1)^{d^{\pm}-1}$. The expressions in U_2 and V_2 are

$$(\dot{u}, \dot{v}) = \left(v^{d^+}(P_2^+ - uQ_2^+), -v^{d^++1}Q_2^+\right)$$

and

$$(\dot{u}, \dot{v}) = \left((-1)^{d^-}v^{d^-}(uQ_2^- - P_2^-), (-1)^{d^-}v^{d^-+1}Q_2^-\right),$$

respectively, where $P_2^{\pm} = P^{\pm}(\frac{u}{v}, \frac{1}{v})$ and $Q_2^{\pm} = Q^{\pm}(\frac{u}{v}, \frac{1}{v})$.

We denote by A_i, B_i, $(i = 1, 2, \cdots)$, O_1 and O_2 the infinite equilibrium points in U_1, V_1, U_2 and V_2, respectively. Note that in the local charts U_2 and V_2 we only study if the origins O_1 and O_2 of these charts are infinite equilibrium points, because all the other infinite equilibrium points already have been studied in the local charts U_1 and V_1. Since the phase portraits of the piecewise $\mathbb{Z}_2$-equivariant cubic polynomial Hamiltonian vector field (4) is symmetry with respect to the origin, we just need to analyze the infinite equilibrium points in $U_1|_{v=0}$ and at the origin of U_2.

Now we recall two important theorems on the topological indices of the equilibrium points of the differential vector fields, which are extremely useful tools for studying the phase portraits of the equilibrium points, for more details about the following theorems see Chapter 6 of [18].

Theorem 2.1. *We denote by q an isolated equilibrium point with the finite sectorial decomposition property. Let h, e and p be the number of hyperbolic, elliptic and parabolic sectors in a small neighborhood of the equilibrium point q, respectively. Then the topological index of q equals $1 + (e - h)/2$, which is called Poincaré Index Formula.*

Corollary 2.2. *The topological indices of a center, a node, a saddle and a cusp equal 1, 1, -1 and 0, respectively.*

We can obtain a 2-dimensional Poincaré sphere $\mathbb{S}^2$ when we identify points to points the boundary $\mathbb{S}^1$ of the Poincaré disc $\mathbb{D}^2$. Then the flow defined by the Poincaré compactification on $\mathbb{D}^2$ can be extended to the 2-dimensional Poincaré sphere $\mathbb{S}^2$, which has a copy of the initial flow of the polynomial differential vector field in each one of the two components of $\mathbb{S}^2 \setminus \mathbb{S}^1$.

Theorem 2.3. *Let ψ be the extended flow of a Poincaré compactification on the Poincaré sphere $\mathbb{S}^2$ having finitely many equilibrium points, then the sum of the topological indices of all equilibrium points is 2.*

2.1. **Chart U_1.** Let $x = \frac{1}{v}$, $y = \frac{u}{v}$, $v > 0$, the first vector field of (4) becomes

$$\begin{aligned} \dot{u} &= \frac{1}{2}(1 - 4a_{21}u^2 - v^2 + 8b_{03}u^3 - 2a_{03}u^4 - 2a_{02}u^3v + 2a_{21}u^2v^2), \\ \dot{v} &= uv(-a_{21} + 3b_{03}u - a_{03}u^2 - a_{02}uv + a_{21}v^2). \end{aligned} \tag{8}$$

We compute the linear part of system (8) on $v = 0$ and obtain

$$\begin{pmatrix} -4uM_1 & -a_{02}u^3 \\ 0 & -uM_1 \end{pmatrix}, \tag{9}$$

where $M_1 = a_{21} - 3b_{03}u + a_{03}u^2$. The equilibrium points $(u, 0)$ of system (8) must satisfy

$$g(u) = 2\dot{u}|_{v=0} = 1 - 4a_{21}u^2 + 8b_{03}u^3 - 2a_{03}u^4 = 0. \tag{10}$$

We consider the following two cases $a_{03} = 0$ and $a_{03} \neq 0$.

Remark 2.4. *Although the infinite equilibrium points of the first vector field of (4) in U_1 with $u < 0$ are virtual points, due to the symmetry of the vector field (4) there are the corresponding infinite equilibrium points in V_1 with $u > 0$. Hence we study all real solutions of $g(u) = 0$, i.e. we study the infinite equilibrium points in U_1.*

Case 1: $a_{03}=0$. We define $\Delta_1 = 4a_{21}^3 - 27b_{03}^2$.

Proposition 2.5. *For system (8) with $a_{03}=0$ the following statements hold.*

(i) *If $b_{03}=0$ and $a_{21}\le 0$ system (8) has no infinite equilibrium points.*
(ii) *If $b_{03}=0$ and $a_{21}>0$ system (8) has two infinite equilibrium points $A_{1,2}=(\pm\frac{\sqrt{a_{21}}}{2a_{21}},0)$, which are an attracting and repelling node, respectively.*
(iii) *If $b_{03}\neq 0$ and $\Delta_1<0$ system (8) has one infinite equilibrium point $A_1=(u_1,0)$, which is a node.*
(iv) *If $b_{03}\neq 0$ and $\Delta_1=0$ system (8) has two infinite equilibrium points $A_1=(-\frac{9b_{03}^2}{8a_{21}^2},0)$ and $A_2=(\frac{a_{21}}{3b_{03}},0)$, which are a node and an E-H equilibrium point, respectively.*
(v) *If $b_{03}\neq 0$ and $\Delta_1>0$ system (8) has three infinite equilibrium points $A_{1,2,3}=(\widetilde{u}_{1,2,3},0)$, which are three nodes.*

Here

$$u_1=\frac{4a_{21}-\sqrt[3]{Y_1}-\sqrt[3]{Y_2}}{24b_{03}},\quad Y_{1,2}=-64a_{21}^3+12b_{03}(72b_{03}\pm 8\sqrt{3\Delta_1}),$$
$$\widetilde{u}_1=\frac{a_{21}-2a_{21}\cos\frac{\theta}{3}}{6b_{03}},\quad \widetilde{u}_{2,3}=\frac{a_{21}+a_{21}(\cos\frac{\theta}{3}\pm\sqrt{3}\sin\frac{\theta}{3})}{6b_{03}},$$
$$\theta=\arccos\frac{27b_{03}^2-2a_{21}^3}{2a_{21}^3}.$$

Proof. Since statements (i) and (ii) are easy to prove we only analyze the case $b_{03}\neq 0$.

If $b_{03}\neq 0$, by the discriminant of cubic polynomial equation, the equation $1-4a_{21}u^2+8b_{03}u^3=0$ has one, two and three real roots when $\Delta_1<0$, $\Delta_1=0$ and $\Delta_1>0$, respectively. Then we obtain the corresponding infinite equilibrium points of system (8).

Now we consider the local phase portraits of the equilibrium points of system (8). By computing the Gröbner basis of the two polynomials $1-4a_{21}u^2+8b_{03}u^3$ and $\widetilde{M}_1=a_{21}-3b_{03}u$ we obtain the four polynomials

$$-64b_{03}\Delta_1,\ -a_{21}+3b_{03}u,\ -9b_{03}+4a_{21}^2u,\ -3+4a_{21}u^2.$$

Hence system (8) has one nilpotent equilibrium point when $\Delta_1=0$. From (9) we obtain that the other elementary infinite equilibrium points are nodes.

Assume that $\Delta_1=0$ (i.e. $b_{03}^2=\frac{4}{27}a_{21}^3\neq 0$), then system (8) has two equilibrium points $A_1=(-\frac{9b_{03}^2}{8a_{21}^2},0)$ and $A_2=(\frac{a_{21}}{3b_{03}},0)$, where A_1 is a node and A_2 is a nilpotent equilibrium point when $\widetilde{M}_1=0$. We do the change $u\to U+\frac{a_{21}}{3b_{03}}$, $t=-\frac{4a_{21}b_{03}}{a_{02}}\tau$, then system (8) becomes

$$\begin{aligned}U' =& v-\frac{8a_{21}b_{03}}{a_{02}}U^2+\frac{9b_{03}}{a_{21}}Uv-\frac{b_{03}}{a_{02}}v^2-\frac{64a_{21}^3}{27a_{02}}U^3+4a_{21}U^2v\\ &-\frac{8a_{21}^2}{3a_{02}}Uv^2-4b_{03}U^3v-\frac{4a_{21}b_{03}}{a_{02}}U^2v^2,\\ v' =& -\frac{b_{03}}{a_{02}a_{21}^3}v(a_{21}+3b_{03}U)(4a_{21}^3U-3a_{02}a_{21}v-9a_{02}b_{03}Uv+9a_{21}b_{03}v^2),\end{aligned}\tag{11}$$

where $\{'\}:=\mathrm{d}/\mathrm{d}\tau$. Since we obtain $\mu_1=-\frac{20a_{21}b_{03}}{a_{02}}\neq 0$, $\lambda_3=-\frac{32a_{21}^2b_{03}^2}{a_{02}^2}<0$ and $\kappa=\mu_1^2+8\lambda_3=\frac{144a_{21}^2b_{03}^2}{a_{02}^2}>0$, we can deduce that the origin of system (11) is an E-H equilibrium point. Hence the nilpotent equilibrium point A_2 is an E-H equilibrium point. □

Case 2: $a_{03}\neq 0$. We analyze the nilpotent equilibrium points of U_1. By computing the Gröbner basis for the polynomials $g(u)$ and M_1 we obtain ten polynomials, the following four polynomials

$$
(12)\qquad
\begin{aligned}
M_2 &= a_{03}(a_{03}+2a_{21}^2)^2 - 36a_{03}a_{21}b_{03}^2 - 8a_{21}^3b_{03}^2 + 54b_{03}^4,\\
M_3 &= -a_{03}(a_{03}+2a_{21}^2) + 6a_{21}b_{03}^2 + 2b_{03}(4a_{03}a_{21} - 9b_{03}^2)u,\\
M_4 &= -a_{21}(a_{03}+2a_{21}^2) + 9b_{03}^2 - b_{03}(3a_{03} - 2a_{21}^2)u,\\
M_5 &= a_{21} - 3b_{03}u + a_{03}u^2,
\end{aligned}
$$

are enough for our analyzing. Since the above four polynomials must be zero we obtain that system (8) has at most two nilpotent equilibrium points. Then we consider the following two subcases $b_{03}=0$ and $b_{03}\neq 0$.

Assume that $b_{03}=0$ we have $a_{03}=-2a_{21}^2\neq 0$ from $M_{2,3,4}=0$. Since $M_5=a_{21}+a_{03}u^2=0$ system (8) has no nilpotent equilibrium points when $a_{21}<0$, and it has two nilpotent equilibrium points $(\pm\frac{\sqrt{2a_{21}}}{2a_{21}},0)$ when $a_{21}>0$. Now we analyze the local phase portraits of these nilpotent equilibrium points. By doing the change $u\to U+\frac{\sqrt{2a_{21}}}{2a_{21}}$, $t=-\frac{2\sqrt{2a_{21}}a_{21}}{a_{02}}\tau$ system (8) becomes

$$
(13)\qquad
\begin{aligned}
U' =& v - \frac{8\sqrt{2a_{21}}a_{21}^2}{a_{02}}U^2 + 3\sqrt{2a_{21}}Uv - \frac{16a_{21}^3}{a_{02}}U^3 + 6a_{21}U^2v - \frac{4a_{21}^2}{a_{02}}Uv^2\\
&- \frac{4\sqrt{2a_{21}}a_{21}^3}{a_{02}}U^4 + 2\sqrt{2a_{21}}a_{21}U^3v - \frac{2\sqrt{2a_{21}}a_{21}^2}{a_{02}}U^2v^2,\\
v' =& -\frac{4\sqrt{2a_{21}}a_{21}^2}{a_{02}}Uv + \sqrt{2a_{21}}v^2 - \frac{12a_{21}^3}{a_{02}}U^2v + 4a_{21}Uv^2 - \frac{2a_{21}^2}{a_{02}}v^3\\
&- \frac{4\sqrt{2a_{21}}a_{21}^3}{a_{02}}U^3v + 2\sqrt{2a_{21}}a_{21}U^2v^2 - \frac{2\sqrt{2a_{21}}a_{21}^2}{a_{02}}Uv^3,
\end{aligned}
$$

and the equilibrium point $(\frac{\sqrt{2a_{21}}}{2a_{21}},0)$ move to the origin of system (13). From

$$
\mu_1 = -\frac{20\sqrt{2a_{21}}a_{21}^2}{a_{02}},\quad \lambda_3 = -\frac{64a_{21}^5}{a_{02}^2}<0,\quad \mu_1^2+8\lambda_3 = \frac{288a_{21}^5}{a_{02}^2}>0,
$$

we have that the origin of system (13) is an E-H equilibrium point. Due to the symmetry, these two nilpotent equilibrium points $(\pm\frac{\sqrt{2a_{21}}}{2a_{21}},0)$ are both E-H equilibrium points.

Assume that $b_{03}\neq 0$. If the coefficient $4a_{03}a_{21}-9b_{03}^2$ of first order term of M_3 is zero, then we have $-8a_{21}^3+27b_{03}^2=0$ and $a_{03}=\frac{2}{3}a_{21}^2$ from $M_3=0$. So we obtain $M_4=0$ and $M_5=\frac{(2a_{21}-3b_{03}u)^2}{4a_{21}}$. Hence system (8) has only one nilpotent equilibrium point $(\frac{2a_{21}}{3b_{03}},0)$. Furthermore we have

$$
\mu_2 = -\frac{3b_{03}^4}{a_{02}^2},\quad \lambda_5 = \frac{15b_{03}^2}{2a_{02}}<0,\quad \mu_2^2+12\lambda_5 = \frac{81b_{03}^4}{4a_{02}^2}>0,
$$

so this nilpotent equilibrium point is a node.

If $4a_{03}a_{21}-9b_{03}^2\neq 0$ and $M_2=0$ system (8) has the nilpotent equilibrium point

$$
(u_*,0) = \left(\frac{a_{03}^2+2a_{03}a_{21}^2-6a_{21}b_{03}^2}{2b_{03}(4a_{03}a_{21}-9b_{03}^2)},0\right).
$$

We do the change $u \to U + u_*$, $t = -\frac{1}{a_{02}u_*^3}\tau$ and system (8) becomes

$$
\begin{aligned}
U' =& v - \frac{8b_{03}^3(4a_{03}a_{21}-9b_{03}^2)M_6}{a_{02}(a_{03}^2+2a_{03}a_{21}^2-6a_{21}b_{03}^2)^3}U^2 + \frac{6b_{03}(4a_{03}a_{21}-9b_{03}^2)}{a_{03}^2+2a_{03}a_{21}^2-6a_{21}b_{03}^2}Uv \\
&+ \frac{4b_{03}^3(4a_{03}a_{21}-9b_{03}^2)^2(a_{03}a_{21}+2a_{21}^3-9b_{03}^2)}{a_{02}(a_{03}^2+2a_{03}a_{21}^2-6a_{21}b_{03}^2)^3}v^2 \\
&+ \frac{16b_{03}^2(4a_{03}a_{21}-9b_{03}^2)^2(a_{03}^3+2a_{03}^2a_{21}^2-4a_{03}a_{21}b_{03}^2+18b_{03}^4)}{a_{02}(a_{03}^2+2a_{03}a_{21}^2-6a_{21}b_{03}^2)^3}U^3 \\
&+ \frac{12b_{03}^2(4a_{03}a_{21}-9b_{03}^2)^2}{(a_{03}^2+2a_{03}a_{21}^2-6a_{21}b_{03}^2)^2}U^2v - \frac{8a_{21}b_{03}^2(4a_{03}a_{21}-9b_{03}^2)^2}{a_{02}(a_{03}^2+2a_{03}a_{21}^2-6a_{21}b_{03}^2)^2}Uv^2 \\
&+ \frac{8a_{03}b_{03}^3(4a_{03}a_{21}-9b_{03}^2)^3}{a_{02}(a_{03}^2+2a_{03}a_{21}^2-6a_{21}b_{03}^2)^3}U^4 + \frac{8b_{03}^3(4a_{03}a_{21}-9b_{03}^2)^3}{(a_{03}^2+2a_{03}a_{21}^2-6a_{21}b_{03}^2)^3}U^3v \\
&- \frac{8a_{21}b_{03}^3(4a_{03}a_{21}-9b_{03}^2)^3}{a_{02}(a_{03}^2+2a_{03}a_{21}^2-6a_{21}b_{03}^2)^3}U^2v^2, \\
v' =& - \frac{4b_{03}^3(4a_{03}a_{21}-9b_{03}^2)M_6}{a_{02}(a_{03}^2+2a_{03}a_{21}^2-6a_{21}b_{03}^2)^3}Uv + \frac{2b_{03}(4a_{03}a_{21}-9b_{03}^2)}{a_{03}^2+2a_{03}a_{21}^2-6a_{21}b_{03}^2}v^2 \\
&+ \frac{12b_{03}^2(4a_{03}a_{21}-9b_{03}^2)^2(a_{03}^3+2a_{03}^2a_{21}^2-14a_{03}a_{21}b_{03}^2+18b_{03}^4)}{a_{02}(a_{03}^2+2a_{03}a_{21}^2-6a_{21}b_{03}^2)^3}U^2v \\
&+ \frac{8b_{03}^2(4a_{03}a_{21}-9b_{03}^2)^2}{(a_{03}^2+2a_{03}a_{21}^2-6a_{21}b_{03}^2)^2}Uv^2 - \frac{4a_{21}b_{03}^2(4a_{03}a_{21}-9b_{03}^2)^2}{a_{02}(a_{03}^2+2a_{03}a_{21}^2-6a_{21}b_{03}^2)^2}v^3 \\
&+ \frac{8a_{03}b_{03}^3(4a_{03}a_{21}-9b_{03}^2)^3}{a_{02}(a_{03}^2+2a_{03}a_{21}^2-6a_{21}b_{03}^2)^3}U^3v + \frac{8b_{03}^3(4a_{03}a_{21}-9b_{03}^2)^3}{(a_{03}^2+2a_{03}a_{21}^2-6a_{21}b_{03}^2)^3}U^2v^2 \\
&- \frac{8a_{21}b_{03}^3(4a_{03}a_{21}-9b_{03}^2)^3}{a_{02}(a_{03}^2+2a_{03}a_{21}^2-6a_{21}b_{03}^2)^3}Uv^3,
\end{aligned}
\tag{14}
$$

where

$$M_6 = 88a_{03}^2a_{21}^3 + 48a_{03}a_{21}^5 + 27a_{03}^2b_{03}^2 - 594a_{03}a_{21}^2b_{03}^2 - 96a_{21}^4b_{03}^2 + 810a_{21}b_{03}^4.$$

By computing the resultant of M_2 and M_6 with respect to a_{03} we have $-4b_{03}^4(8a_{21}^3-27b_{03}^2)^5 \neq 0$, hence $M_6 \neq 0$. Then we obtain

$$
\begin{aligned}
\mu_1 &= -\frac{20b_{03}^3(-4a_{03}a_{21}+9b_{03}^2)M_6}{a_{02}(-a_{03}^2-2a_{03}a_{21}^2+6a_{21}b_{03}^2)^3} \neq 0, \\
\lambda_3 &= -\frac{32b_{03}^6(-4a_{03}a_{21}+9b_{03}^2)^2M_6^2}{a_{02}^2(-a_{03}^2-2a_{03}a_{21}^2+6a_{21}b_{03}^2)^6} < 0, \\
\mu_1^2 + 8\lambda_3 &= \frac{144b_{03}^6(-4a_{03}a_{21}+9b_{03}^2)^2M_6^2}{a_{02}^2(-a_{03}^2-2a_{03}a_{21}^2+6a_{21}b_{03}^2)^6} > 0.
\end{aligned}
$$

Further we obtain that this infinite nilpotent equilibrium point is an E-H equilibrium point.

Now we shall determine the phase portrait of the elementary infinite equilibrium points of the chart U_1 with $a_{03} \neq 0$. But it is not an easy work to calculate explicitly the coordinates of these infinite equilibrium points, which are complicated in terms of the parameters a_{21}, a_{03} and b_{03}. From (9) we see these infinite equilibrium points of system (8) must be nodes. Thus we just need to find the number of distinct real solutions of $g(u) = 0$, where the polynomial $g(u)$ is not identically zero. Now in order to study the number of the distinct real roots of the polynomial $f(z)$ with symbolic coefficients we shall use the method of the *discriminant sequence* associated to $f(z)$, for more details see [37].

We associate to the polynomial

$$f(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 \tag{15}$$

the following $(2n+1)\times(2n+1)$ matrix in terms of its symbolic coefficients

$$D(f)=\begin{pmatrix} a_n & a_{n-1} & a_{n-2} & \cdots & a_0 & & & & & \\ & na_n & (n-1)a_{n-1} & \cdots & a_1 & & & & & \\ & a_n & a_{n-1} & \cdots & a_{n-1} & a_0 & & & & \\ & & na_n & \cdots & 2a_2 & a_1 & & & & \\ & & & \cdots & \cdots & & & & & \\ & & & \cdots & \cdots & & & & & \\ & & & & a_n & a_{n-1} & \cdots & a_0 & & \\ & & & & & na_n & \cdots & a_1 & & \\ & & & & & a_n & \cdots & a_1 & a_0 & \end{pmatrix}.$$

This matrix is called the discriminant matrix of $f(z)$. We denote by d_i the determinant of the submatrix of $D(f)$ constructed with the first i columns and rows of $D(f)$ for $i=1,\ldots,2n+1$. Thus we have the principal minor sequence of $f(z)$,

$$\{d_1, d_2, \ldots, d_{2n+1}\}, \tag{16}$$

which is an important tool in our study.

Consider the called *sign list*

$$[\mathrm{sign}(d_1), \mathrm{sign}(d_2), \cdots, \mathrm{sign}(d_{2n+1})], \tag{17}$$

of the principal minor sequence (16), where as usual the sign function is

$$\mathrm{sign}(z) = \begin{cases} -1 & \text{if } z<0, \\ 0 & \text{if } z=0, \\ 1 & \text{if } z>0. \end{cases} \tag{18}$$

For any sign list $[\xi_1, \xi_2, \cdots, \xi_n]$ we define the associated *revised sign list* $[\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n]$ as follows:

1. If $\xi_k \neq 0$ we let $\varepsilon_k = \xi_k$.
2. If section $[\xi_i, \xi_{i+1}, \cdots, \xi_{i+j}]$ of the given list satisfies with $\xi_i \varepsilon_{i+j} \neq 0$ and $\xi_{i+1} = \xi_{i+2} = \cdots = \xi_{i+j-1} = 0$, we replace its subsection $[\xi_{i+1}, \xi_{i+2}, \cdots, \xi_{i+j-1}]$ with
$$[-\xi_i, -\xi_i, \xi_i, \xi_i, -\xi_i, -\xi_i, \xi_i, \xi_i, -\xi_i, \cdots]$$
keeping its number of terms.

Then the revised sign list $[\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n]$ has no zeros between two nonzero elements. We denote by RSL the revised sign list of a given sign list.

From Theorem 2.1 of [37] we obtain the following theorem.

Theorem 2.6. *Consider the polynomial* (15)*. If the number of nonzero elements of the RSL* $[d_2, d_4, \ldots, d_{2n}]$ *is equal to* ℓ*, and the number of the sign changes of this RSL is equal to* m*, then the number of the distinct real roots of* (15) *is* $\ell - 2m$*.*

Now we compute the sequence $\{d_2, d_4, d_6, d_8\}$ of $g(u)$ from (10) to analyze the number of distinct real roots of $g(u)=0$, and we have

$$\begin{aligned} d_2 &= 16a_{03}^2, \\ d_4 &= -256a_{03}^2(a_{03}a_{21} - 3b_{03}^2), \\ d_6 &= -2048a_{03}^2 M_7, \\ d_8 &= -8192a_{03}^2 M_2, \end{aligned} \tag{19}$$

where

$$M_7 = a_{03}^2 a_{21} + 2a_{03}a_{21}^3 - 3a_{03}b_{03}^2 - 4a_{21}^2 b_{03}^2.$$

From Theorem 2.6 we obtain that the polynomial $g(u)$ has four distinct real roots if and only if the RSL of (10) is $[1,1,1,1]$ or $[-1,-1,-1,-1]$, but the latter is impossible because we have $d_2>0$. We denote by $\mathrm{R}[f(z),i]$ the i-th real root of the polynomial $f(z)$, and these roots are ordered as follows $\mathrm{R}[f(z),i] < \mathrm{R}[f(z),j]$ with $i<j$. Then we obtain the corresponding condition of the RSL $[1,1,1,1]$ as shown in Table 1. Here we denote by N a node. Therefore

the polynomial $g(u)$ has at most four distinct real roots. Since $M_2 \neq 0$ the corresponding four infinite equilibrium points $A_{1,2,3,4}$ are nodes of U_1.

TABLE 1. The conditions of the RSL in order that (10) has four or three distinct real roots when $a_{03} \neq 0$.

RSL	infinite equilibria	Conditions
$[1,1,1,1]$	4 N	I:$a_{21} > 0$, $-\frac{2\sqrt{6a_{21}^3}}{9} < b_{03} < \frac{2\sqrt{6a_{21}^3}}{9}$, $\mathrm{R}[M_2,2] < a_{03} < \mathrm{R}[M_2,3]$;
$[1,1,1,0]$	2 N, 1 E-H	II: $a_{21} > 0$, $-\frac{2\sqrt{6a_{21}^3}}{9} < b_{03} < \frac{2\sqrt{6a_{21}^3}}{9}$, $b_{03} \neq 0$, $a_{03} = \mathrm{R}[M_2,2]$ or $a_{03} = \mathrm{R}[M_2,3]$.

As it is described in Table 1 the RSL of the associated discriminant sequence is $[1,1,1,0]$ when $g(u)$ has three distinct real roots, correspondly, there are three infinite equilibrium points in U_1. We compute the resultant of M_2 and the coefficient $4a_{03}a_{21} - 9b_{03}^2$ of the first order term of M_3 with respect to a_{03} and we obtain $b_{03}^2(8a_{21}^3 - 27b_{03}^2)^2 \neq 0$. From the previous analysis we have that these three infinite equilibrium points are two nodes $A_{1,3}$ and an E-H point $A_2 = (u_*, 0)$.

Furthermore we obtain the possible RSL of the associated discriminant sequences as we show in Table 2, when $g(u)$ of (10) has two distinct real roots, one real root and no real roots, respectively. Then system (8) has two, one and no equilibrium points in U_1.

TABLE 2. The conditions of the possible RSL in order that (10) has distinct real roots when $a_{03} \neq 0$.

RSL	infinite equilibria	Conditions
$[1,1,1,-1]$	2 N	III(a): $a_{21} > 0$, $b_{03} \geq \frac{2\sqrt{6a_{21}^3}}{9}$ or $b_{03} \leq -\frac{2\sqrt{6a_{21}^3}}{9}$, $\mathrm{R}[M_7,1] < a_{03} < \mathrm{R}[M_7,2]$; III(b): $a_{21} > 0$, $-\frac{2\sqrt{6a_{21}^3}}{9} < b_{03} < \frac{2\sqrt{6a_{21}^3}}{9}$, $b_{03} \neq 0$, $\mathrm{R}[M_7,1] < a_{03} < \mathrm{R}[M_2,2]$ or $\mathrm{R}[M_2,3] < a_{03} < \mathrm{R}[M_7,2]$; IV: $a_{21} = 0$, $b_{03} \neq 0$, $a_{03} > 0$; V: $a_{21} < 0$, $\mathrm{R}[M_7,2] < a_{03}$;
$[1,1,-1,-1]$	2 N	VI: $a_{21} > 0$, $b_{03} \neq 0$, $\mathrm{R}[M_2,1] < a_{03} \leq \mathrm{R}[M_7,1]$ or $\mathrm{R}[M_7,2] \leq a_{03} < \frac{3b_{03}^2}{a_{21}}$; VII: $a_{21} = 0$, $b_{03} \neq 0$, $\mathrm{R}[M_2,1] < a_{03} < 0$; VIII: $a_{21} < 0$, $b_{03} \neq 0$, $\mathrm{R}[M_2,1] < a_{03} \leq \mathrm{R}[M_7,2]$;
$[1,-1,-1,-1]$	2 N	IX: $a_{21} > 0$, $a_{03} \geq \frac{3b_{03}^2}{a_{21}}$; X: $a_{21} = 0$, $b_{03} = 0$, $a_{03} > 0$;
$[1,1,0,0]$	2 N 2 E-H	XI: $a_{21} > 0$, $b_{03} = \pm\frac{2\sqrt{6a_{21}^3}}{9}$, $a_{03} = \frac{2}{3}a_{21}^2$; XII: $a_{21} > 0$, $b_{03} = 0$, $a_{03} = -2a_{21}^2$;
$[1,1,-1,0]$	1 E-H	XIII: $a_{21} \in R$, $b_{03} \neq 0$, $a_{03} = \mathrm{R}[M_2,1]$;
$[1,1,-1,1]$	0	XIV: $a_{21} > 0$, $a_{03} < \mathrm{R}[M_2,1]$; XV: $a_{21} = 0$, $b_{03} \neq 0$, $a_{03} < \mathrm{R}[M_2,1]$; XVI: $a_{21} < 0$, $b_{03} \neq 0$, $\frac{3b_{03}^2}{a_{21}} < a_{03} < \mathrm{R}[M_2,1]$;
$[1,-1,1,1]$	0	XVII: $a_{21} < 0$, $b_{03} \neq 0$, $a_{03} \leq \mathrm{R}[M_7,1]$; or $b_{03} = 0$, $a_{03} < -2a_{21}^2$;
$[1,-1,-1,1]$	0	XVIII: $a_{21} = 0$, $b_{03} = 0$, $a_{03} < 0$; XIX: $a_{21} < 0$, $\mathrm{R}[M_7,1] < a_{03} \leq \frac{3b_{03}^2}{a_{21}}$;
$[1,-1,0,0]$	0	XX: $a_{21} < 0$, $b_{03} = 0$, $a_{03} = -2a_{21}^2$.

In summary we have the following result.

Proposition 2.7. *For system* (8) *with* $a_{03} \neq 0$ *the following statements hold.*

(i) *If condition I holds system* (8) *has four infinite equilibrium points* $A_{1,2,3,4}$*, which are four nodes.*
(ii) *If condition II holds system* (8) *has three infinite equilibrium points* $A_{1,2,3}$*, where* $A_{1,3}$ *are two nodes and* $A_2 = (u_*, 0)$ *is an E-H equilibrium point.*
(iii) *If one of the conditions III-XI holds system* (8) *has two infinite equilibrium points* $A_{1,2}$*, which are two nodes.*

(iv) *If condition XII holds system* (8) *has two infinite equilibrium points* $A_{1,2} = (\pm\frac{\sqrt{2a_{21}}}{2a_{21}}, 0)$*, which are two E-H equilibrium points.*
(v) *If condition XIII holds system* (8) *has one infinite equilibrium point* $A_1 = (u_*, 0)$*, which is an E-H point.*
(vi) *If one of the conditions XIV-XX holds system* (8) *has no infinite equilibrium points.*

2.2. **Chart U_2.** Let $x = \frac{1}{v}$, $y = \frac{u}{v}$, $v < 0$ then the first vector field of (4) becomes

$$\begin{aligned}\dot{u} =& \frac{1}{2}(2a_{03} - 8b_{03}u + 2a_{02}v + 4a_{21}u^2 - 2a_{21}v^2 - u^4 + u^2v^2),\\ \dot{v} =& -\frac{1}{2}v(2b_{03} - 2a_{21}u + u^3 - uv^2).\end{aligned} \tag{20}$$

We only need to study when the origin O_1 of system (20) is an equilibrium point and we have the following result.

Proposition 2.8. *For system* (20) *the following statements hold.*

(i) *If* $a_{03} \neq 0$ *the origin* O_1 *is not an infinite equilibrium point.*
(ii) *If* $a_{03} = 0$ *and* $b_{03} > 0$ *the origin* O_1 *is an infinite equilibrium point, which is an attracting node.*
(iii) *If* $a_{03} = 0$ *and* $b_{03} = 0$ *the origin* O_1 *is an infinite equilibrium point, which is an E-H equilibrium point.*
(iv) *If* $a_{03} = 0$ *and* $b_{03} < 0$ *the origin* O_1 *is an infinite equilibrium point, which is a repelling node.*

3. Finite equilibrium points and global phase portraits of the vector field (4)

Now we consider the finite equilibrium points of the vector field (4). The equilibrium points $q_{1,2} = (\pm 1, 0)$ are the bi-centers. And the origin q_3 is also an equilibrium point, whose Jacobian matrix is

$$\begin{pmatrix} 0 & -a_{21} \\ -\frac{1}{2} & 0 \end{pmatrix}. \tag{21}$$

From (21) we have that q_3 is a saddle when $a_{21} \geq 0$ (a nilpotent saddle when $a_{21} = 0$), or a center when $a_{21} < 0$. Now we need to study the remaining equilibrium points different from these three equilibrium points.

Since the phase portrait of the vector field (4) is symmetric with respect to the origin, and it has no equilibrium points different from $q_{1,2,3}$ on the straight line $y = 0$, we just need to study the extra equilibrium points of the first vector field of (4) for $y > 0$. The Jacobian matrix of the first vector field of (4) at a finite equilibrium point (x, y) is

$$\begin{pmatrix} y(2a_{21}x - 3b_{03}y) & M_8 \\ \frac{1}{2}(-1 + 3x^2 - 2a_{21}y^2) & -y(2a_{21}x - 3b_{03}y) \end{pmatrix}, \tag{22}$$

where

$$M_8 = -a_{21} + 2a_{02}y + a_{21}x^2 - 6b_{03}xy + 3a_{03}y^2.$$

As in the above section, we consider two cases $a_{03} = 0$ and $a_{03} \neq 0$ in the study the non-elementary and elementary equilibrium points.

3.1. **Case $a_{03} = 0$.** We analyze the non-elementary equilibrium points, which are different from the known equilibrium points. We compute the Gröbner basis for the polynomials $\dot{x}$, $\dot{y}$, $-1 + 3x^2 - 2a_{21}y^2$ and $2a_{21}x - 3b_{03}y$ and we obtain the following polynomials

$$b_{03},\ a_{02}^2 + 2a_{21}^3,\ -a_{21} + a_{02}y,\ a_{02} + 2a_{21}^2y,\ 1 + 2a_{21}y^2,\ x.$$

It means that the first vector field of (4) has one possible nilpotent equilibrium point $(0, \frac{a_{21}}{a_{02}})$ when the above polynomials are zero and $a_{21} < 0$. Applying Theorem 3.5 of [18] this nilpotent equilibrium point is a saddle.

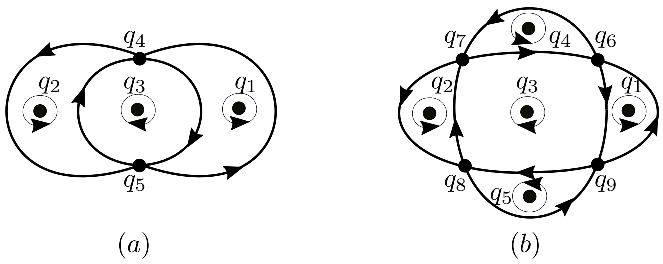


FIGURE 3. (a) A tri-heteroclinic loop; (b) A quadruple-heteroclinic loop.

Now we calculate the Gröbner basis of the four polynomials $\dot{x}$, $\dot{y}$, $2a_{21}x - 3b_{03}y$ and M_8, and we obtain twenty-nine polynomials. We can find that one polynomial a_{21}^5 must be zero when the first vector field of (4) has a nilpotent equilibrium point. We assume $a_{21} = 0$ and again compute the Gröbner for the above four polynomials and obtain

$$b_{03}y,\ a_{02}^2 y,\ (-1+x)x(1+x).$$

It means that there is no nilpotent equilibrium points different from $q_{1,2,3}$ when $a_{21} = 0$.

Now we consider the elementary finite equilibrium points and the global phase portraits of vector field (4). We separate the study in two subcases $b_{03} = 0$ and $b_{03} \neq 0$.

(i) Assume $b_{03} = 0$ then the vector field (4) is also symmetry respect to the x-axis and the y-axis.

(i.1) From Propositions 2.5 and 2.8 we obtain that if $a_{21} \leq 0$ the vector field (4) has only two infinite equilibrium points $O_{1,2}$, which are two E-H equilibrium points. And we have $\dot{u}|_{u=0,v=0} = \frac{1}{2} > 0$ in the chart U_1, showing that the flow at the neighborhood of the origin of U_1 is increasing in the direction u.

(i.1.1) If $a_{21} = 0$ the vector field (4) has no other finite equilibrium points different from $q_{1,2,3}$, where q_3 is a saddle. Since the finite equilibrium points of the Hamiltonian vector field (4) are either centers or saddles, there must be at least one saddle on the boundary of the period annulus of each center. Hence the saddle q_3 must be on the boundary of the region formed by the period annulus surrounding center q_1. Taking into account the symmetries, q_3 is on the boundary of period annulus of q_2, creating an eight-figure loop.

From the previous analysis such infinite equilibrium points $O_{1,2}$ of (4) are both formed by an elliptic sector and a hyperbolic sector, where the hyperbolic sector has its two separatrices contained in the straight line of the infinity. The elliptic sector must be outside the Poincaré disc, then the infinite and finite equilibrium points have total index 2 on Poincaré sphere. Further we obtain that the global phase portrait of the vector field (4) in the Poincaré disc is topologically equivalent to the phase portrait 1.1 of Figure 1.

(i.1.2) If $a_{21} < 0$ and $a_{02}^2 + 2a_{21}^3 \geq 0$, by the symmetry the vector field (4) has two equilibrium points $q_{4,5} = (0, \pm\frac{a_{21}}{a_{02}})$ different from $q_{1,2,3}$, where q_3 is a center and $q_{4,5}$ are two saddles. Since there are only two symmetric saddles, they must be on the boundary of the period annulus of the centers $q_{1,2,3}$, creating a tri-heteroclinic loop, see Figure 3 (a). The global phase portrait of the vector field (4) in this subcase is topologically equivalent to the phase portrait 1.2 of Figure 1.

(i.1.3) If $a_{02}^2 + 2a_{21}^3 < 0$ the vector field (4) has six equilibrium points $q_{4,5} = (0, \pm\frac{a_{21}}{a_{02}})$,

$$q_{6,7} = \left(\pm\sqrt{\frac{a_{02}^2 + 2a_{21}^3}{2a_{21}^3}}, -\frac{a_{02}}{2a_{21}^2} \right) = (\pm x^*, y^*)$$

and $q_{8,9} = (\mp x^*, -y^*)$ different from $q_{1,2,3}$, where $q_{3,4,5}$ are three centers and $q_{6,7,8,9}$ are four saddles. Since $H^+(x,y)|_{q_{6,7}} = H^-(x,y)|_{q_{8,9}} = -\frac{a_{02}^4 - 12a_{21}^6}{96a_{21}^6}$, these four saddles are in the same energy level. So one saddle at least be on the boundary of the period annulus of two centers. We assume that the saddle q_6 is on the boundary of the period annulus centers $q_{1,4}$, by the

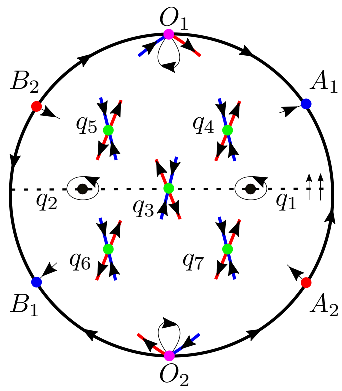


FIGURE 4. The local phase portraits at all finite and infinite equilibrium points of the vector field (4) when $a_{03} = b_{03} = 0$ and $a_{21} > 0$.

symmetry with respect to x-axis and the y-axis, and due to the existence of the center q_3 we obtain a quadruple-heteroclinic loop, see Figure 3 (b). The global phase portrait is topologically equivalent to the phase portrait 1.3 of Figure 1.

(i.2) If $a_{21} > 0$ the vector field (4) has six infinite equilibrium points $A_{1,2}$, $B_{1,2}$ and $O_{1,2}$, where $A_{1,2}$, $B_{1,2}$ are four nodes and $O_{1,2}$ are two E-H equilibrium points. The vector field (4) has four equilibrium points $q_{4,5,6,7} = (\pm x^*, \pm y^*)$ different from $q_{1,2,3}$, where $q_{3,4,5,6,7}$ are five saddles. The sum of the total indices of the infinite and finite equilibrium points except for $O_{1,2}$ is -2, so the elliptic sectors of the infinity equilibrium points $O_{1,2}$ must be inside the Poincaré disc and the hyperbolic ones are outside the Poincaré disc. By the symmetries we obtain the local phase portraits of all equilibrium points, see Figure 4.

(i.2.1) Assume that the saddle q_3 is on the boundary of the period annulus of the center q_1, by the symmetry it is also on the boundary of the period annulus of q_2, creating one eight-figure loop. Since $H^+(x, y)|_{q_{4,5}} = H^-(x, y)|_{q_{6,7}}$, the four saddles $q_{4,5,6,7}$ are in the same energy level. And since there are no other finite equilibrium points, these four saddles must connect to each other one by one. The phase portrait is topologically equivalent to the phase portrait 1.4 of Figure 1.

(i.2.2) Assume that the attracting and repelling separatrices of the saddle q_3 connect the infinite equilibrium points $O_{1,2}$. Then the saddles $q_{4,7}$ are on the boundary of the period annulus of q_1, creating one heteroclinic loop. By the symmetry the saddles $q_{5,6}$ are on the boundary of the period annulus of q_2. In this subcase the phase portrait is topologically equivalent to the phase portrait 1.6 of Figure 1.

(i.2.3) From the phase portrait 1.4 realized when $a_{02} = -1$ and $a_{21} = 0.5$ to the phase portrait 1.6 realized when $a_{02} = -1$ and $a_{21} = 1$, it follows by the continuity of the phase portraits with respect to the parameters the existence of the phase portrait 1.5 of Figure 1. In general it is not an easy work to obtain the explicit values a_{02} and a_{21} for this phase portrait. But in our case the Hamiltonians at these five saddles $q_{3,4,5,6,7}$ are $H^+(x, y)|_{q_{3,4,5}} = H^-(x, y)|_{q_{6,7}} = 0$, i.e. $a_{02}^4 - 12a_{21}^6 = 0$, so they are in the same energy level. Hence some of the separatrices of $q_{4,5,6,7}$ connect with the ones of the saddle q_3.

(ii) Assume that $b_{03} \neq 0$, from the first vector field of (4) we compute the Gröbner basis for $\dot{x}$ and $\dot{y}$ and we obtain sixteen polynomials, where the following two polynomials

$$y^2\left[x(a_{02}a_{21} + 2a_{21}^3 y - 9b_{03}^2 y) - b_{03}(3a_{21} - 3a_{02}y + 2a_{21}^2 y^2)\right]$$

and

$$\begin{aligned} & - y^2\Big[- a_{21}(a_{02}^2 - 9b_{03}^2) + a_{02}(a_{02}^2 - 4a_{21}^3 - 9b_{03}^2)y \\ & + 2a_{21}^2(2a_{02}^2 - 2a_{21}^3 + 15b_{03}^2)y^2 + 4a_{02}a_{21}(a_{21}^3 - 9b_{03}^2)y^3 - 2b_{03}^2\Delta_1 y^4\Big] \\ & = -y^2 f(y) \end{aligned} \tag{23}$$

are enough for our analysis.

(ii.1) The coefficient of the quartic term of $f(y)$ in (23) is zero, i.e. $\Delta_1 = 4a_{21}^3 - 27b_{03}^2 = 0$. We compute the resultant of $a_{21}(a_{21}^3 - 9b_{03}^2)$ and Δ_1 with respect to a_{21} and obtain $-34012224b_{03}^{10} \neq 0$. Thus the coefficient of y^3 in $f(y)$ is nonzero when $\Delta_1 = 0$. Further we have the constant term $a_{21}(a_{02}^2 - 9b_{03}^2) \neq 0$.

We assume that $a_{21}^3 = \frac{27}{4}b_{03}^2 > 0$ and $f(y)$ reduces to

$$f(y) = -a_{21}(a_{02}^2 - 9b_{03}^2) + a_{02}(a_{02}^2 - 36b_{03}^2)y + a_{21}^2(4a_{02}^2 + 3b_{03}^2)y^2 - 9a_{02}a_{21}b_{03}^2y^3.$$

Since $a_{02} < \min\{3b_{03}, -3b_{03}\}$ we have

$$M_9 = 4a_{02}^6 + 108a_{02}^4b_{03}^2 - 1026a_{02}^2b_{03}^4 - 243b_{03}^6 < 0,$$

then the discriminant of the cubic equation $f(y) = 0$ is

$$-\frac{27}{4}a_{21}b_{03}^2(4a_{02}^2 + 9b_{03}^2)^2 M_9 > 0.$$

Hence $f(y) = 0$ has three distinct real roots. Since the coefficients of the cubic and quadratic terms of $f(y)$ are positive and the constant one is negative, we obtain that $f(y) = 0$ has only one positive real root. By the symmetry the vector field (4) has two equilibrium points $q_{4,5}$ different from $q_{1,2,3}$ when $\Delta_1 = 0$. From Propositions 2.5 and 2.8 the vector field (4) has four nodes A_1, B_1, $O_{1,2}$ and two E-H points A_2, B_2, the known infinite and finite equilibrium points except $q_{4,5}$ have total index 6. Hence the total index of the elementary equilibrium points $q_{4,5}$ must be -4, by the symmetry they are two saddles.

If the saddle q_4 is on the boundary of the period annulus of the center q_1, then it creates a center-loop. And by the symmetry the saddle q_5 is on the boundary of the period annulus of the center q_2. Then the separatrices of the saddle q_3 connect with the nodes $O_{1,2}$, A_1 and B_1, we obtain the global phase portrait is topologically equivalent to the phase portrait 1.7 of Figure 1.

If the saddle q_3 is on the boundary of the period annulus of the center q_1, by the symmetry it is also on the boundary of the period annulus of q_2, creating one eight-figure loop. Then the saddles $q_{4,5}$ are on the boundary of the period annulus of this eight-figure loop. In this subcase the phase portrait is topologically equivalent to the phase portrait 1.9 of Figure 1.

From the phase portraits 1.7 to 1.9 it follows by the continuity of the phase portraits with respect to the parameters that there must exist the phase portrait 1.8 of Figure 1 that the saddles $q_{3,4}$ are on the boundary of the period annulus of the center q_1, and the saddles $q_{3,5}$ are on the boundary of the period annulus of the center q_2.

(ii.2) The coefficient of the quartic term of $f(y)$ in (23) is nonzero, i.e. $\Delta_1 \neq 0$. The explicit expressions of the finite equilibrium points different from q_k for $k = 1, 2, 3$, and their eigenvalues in terms of parameters a_{02}, a_{21} and b_{03} are complicated. Since on the straight line $y = 0$ there are no equilibrium points different from $q_{1,2,3}$ we just need to study the equilibrium points of the first vector field of (23) for $y > 0$. So we can determine the number of positive real roots of $f(y)$, and use the index theory to analyze the phase portrait of the remaining finite equilibrium points, which also is not an easy work.

We know that finding the number of the positive roots of $f(z)$ is equivalent to find the number of the negative roots of $-f(-z)$. Now we present one discriminant sequence to obtain the number of the negative roots of the polynomial $f(z)$ using the following theorem, for more detail see [37].

Theorem 3.1. *For the polynomial* (15) *with* $f(0) \neq 0$, *if the number of nonzero elements of the RSL* $[d_1d_2, d_2d_3, \cdots, d_{2k}d_{2k+1}]$ *is equal to* l, *and the number of the sign changes of this RSL is equal to* m, *then the number of the negative roots of* $f(z)$ *is* $l/2 - m$.

(ii.2.1) If $a_{21} = 0$, i.e. the constant term of $f(y)$ is zero, we obtain $f(y) = a_{02}(a_{02}^2 - 9b_{03}^2)y + 54b_{03}^4y^4$, which has only one positive real root $y = -\sqrt[3]{\frac{a_{02}(a_{02}^2-9b_{03}^2)}{54b_{03}^4}}$. Then the vector field (4) has two equilibrium points $q_{4,5}$ different from $q_{1,2,3}$. From Propositions 2.5 and 2.8 the vector field (4) has four infinite nodes A_1, B_1 and $O_{1,2}$, where $O_{1,2}$ are repelling. Then the known infinite and finite equilibrium points have total index 6, the total index of remaining finite equilibrium points $q_{4,5}$ must be -4, i.e. they are two saddles. In a similar way to the phase portraits 1.7–1.9 we obtain the phase portraits 1.10–1.12 of Figure 1.

(ii.2.2) If $a_{21} \neq 0$ we consider the discriminant sequence

$$\{\widehat{d_1}\widehat{d_2}, \widehat{d_2}\widehat{d_3}, \widehat{d_3}\widehat{d_4}, \widehat{d_4}\widehat{d_5}, \widehat{d_5}\widehat{d_6}, \widehat{d_6}\widehat{d_7}, \widehat{d_7}\widehat{d_8}, \widehat{d_8}\widehat{d_9}\} \tag{24}$$

associated to $-f(-y)$ of (23). And we have

$$\begin{aligned}
&\widehat{d_1} = 2b_{03}^2\Delta_1, \qquad \widehat{d_2} = 16b_{03}^4\Delta_1^2, \qquad \widehat{d_3} = 16a_{02}a_{21}b_{03}^4\Delta_1^2(a_{21}^3 - 9b_{03}^2),\\
&\widehat{d_4} = 64a_{21}^2b_{03}^4\Delta_1^2M_{10}, \qquad \widehat{d_5} = -32a_{21}b_{03}^4\Delta_1^2M_{11}, \qquad \widehat{d_6} = 64b_{03}^4\Delta_1^2M_{12},\\
&\widehat{d_7} = 16a_{02}b_{03}^4\Delta_1^2M_{13}, \qquad \widehat{d_8} = 16b_{03}^6\Delta_1^2M_{14}, \qquad \widehat{d_9} = 15a_{21}(a_{02}^2 - 9b_{03}^2)b_{03}^6\Delta_1^2M_{14},
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
M_{10} =& 3a_{02}^2a_{21}^6 - 38a_{02}^2a_{21}^3b_{03}^2 - 16a_{21}^6b_{03}^2 + 135a_{02}^2b_{03}^4 + 228a_{21}^3b_{03}^4 - 810b_{03}^6,\\
M_{11} =& 8a_{02}^4a_{21}^9 - 8a_{02}^2a_{21}^{12} - 92a_{02}^4a_{21}^6b_{03}^2 + 124a_{02}^2a_{21}^9b_{03}^2 + 64a_{21}^{12}b_{03}^2 + 405a_{02}^4a_{21}^3b_{03}^4\\
&- 552a_{02}^2a_{21}^6b_{03}^4 - 1392a_{21}^9b_{03}^4 - 729a_{02}^4b_{03}^6 - 405a_{02}^2a_{21}^3b_{03}^6 + 10080a_{21}^6b_{03}^6\\
&+ 6561a_{02}^2b_{03}^8 - 24300a_{21}^3b_{03}^8,\\
M_{12} =& 8a_{02}^6a_{21}^{12} + 32a_{02}^4a_{21}^{15} + 32a_{02}^2a_{21}^{18} + 104a_{02}^6a_{21}^9b_{03}^2 - 488a_{02}^4a_{21}^{12}b_{03}^2 - 1184a_{02}^2a_{21}^{15}b_{03}^2\\
&- 256a_{21}^{18}b_{03}^2 - 1584a_{02}^6a_{21}^6b_{03}^4 + 328a_{02}^4a_{21}^9b_{03}^4 + 14216a_{02}^2a_{21}^{12}b_{03}^4 + 7488a_{21}^{15}b_{03}^4\\
&+ 5832a_{02}^6a_{21}^3b_{03}^6 + 19764a_{02}^4a_{21}^6b_{03}^6 - 70512a_{02}^2a_{21}^9b_{03}^6 - 86688a_{21}^{12}b_{03}^6 - 6561a_{02}^6b_{03}^8\\
&- 90396a_{02}^4a_{21}^3b_{03}^8 + 113076a_{02}^2a_{21}^6b_{03}^8 + 496368a_{21}^9b_{03}^8 + 118098a_{02}^4b_{03}^{10}\\
&+ 166212a_{02}^2a_{21}^3b_{03}^{10} - 1405512a_{21}^6b_{03}^{10} - 531441a_{02}^2b_{03}^{12} + 1574640a_{21}^3b_{03}^{12},\\
&- 19683a_{02}^8b_{03}^8 - 51018336a_{21}^3b_{03}^{14},\\
M_{13} =& - 64a_{02}^6a_{21}^{15} - 256a_{02}^4a_{21}^{18} - 256a_{02}^2a_{21}^{21} + 512a_{02}^8a_{21}^9b_{03}^2 - 352a_{02}^6a_{21}^{12}b_{03}^2\\
&+ 1952a_{02}^4a_{21}^{15}b_{03}^2 + 7168a_{02}^2a_{21}^{18}b_{03}^2 + 2048a_{21}^{21}b_{03}^2 - 5184a_{02}^8a_{21}^6b_{03}^4 - 4008a_{02}^6a_{21}^9b_{03}^4\\
&+ 27728a_{02}^4a_{21}^{12}b_{03}^4 - 36768a_{02}^2a_{21}^{15}b_{03}^4 - 41472a_{21}^{18}b_{03}^4 + 17496a_{02}^8a_{21}^3b_{03}^6\\
&+ 95256a_{02}^6a_{21}^6b_{03}^6 - 240624a_{02}^4a_{21}^9b_{03}^6 - 328752a_{02}^2a_{21}^{12}b_{03}^6 + 207360a_{21}^{15}b_{03}^6\\
&- 411156a_{02}^6a_{21}^3b_{03}^8 + 128304a_{02}^4a_{21}^6b_{03}^8 + 3592728a_{02}^2a_{21}^9b_{03}^8 + 1007424a_{21}^{12}b_{03}^8\\
&+ 531441a_{02}^6b_{03}^{10} + 2519424a_{02}^4a_{21}^3b_{03}^{10} - 10118520a_{02}^2a_{21}^6b_{03}^{10} - 13328064a_{21}^9b_{03}^{10}\\
&- 4782969a_{02}^4b_{03}^{12} + 3542940a_{02}^2a_{21}^3b_{03}^{12} + 45244656a_{21}^6b_{03}^{12} + 14348907a_{02}^2b_{03}^{14},\\
M_{14} =& 8a_{02}^8a_{21}^3 + 48a_{02}^6a_{21}^6 + 96a_{02}^4a_{21}^9 + 64a_{02}^2a_{21}^{12} - 27a_{02}^8b_{03}^2 - 288a_{02}^6a_{21}^3b_{03}^2\\
&- 1424a_{02}^4a_{21}^6b_{03}^2 - 1952a_{02}^2a_{21}^9b_{03}^2 - 512a_{21}^{12}b_{03}^2 + 486a_{02}^6b_{03}^4 + 4536a_{02}^4a_{21}^3b_{03}^4\\
&+ 15728a_{02}^2a_{21}^6b_{03}^4 + 9600a_{21}^9b_{03}^4 - 2187a_{02}^4b_{03}^6 - 37152a_{02}^2a_{21}^3b_{03}^6 - 59904a_{21}^6b_{03}^6\\
&+ 124416a_{21}^3b_{03}^8.
\end{aligned}$$

(ii.2.2.1) If $\Delta_1 < 0$ and $a_{21} > 0$ we have $\widehat{d_1}\widehat{d_2} < 0$, $\widehat{d_2}\widehat{d_3} > 0$ and $\widehat{d_8}\widehat{d_9} < 0$, then the number of the sign changes of this RSL is at least two. By the symmetry the vector field (4) has at most four elementary equilibrium points different from $q_{1,2,3}$. On the other hand the vector field (4) has four infinite nodes A_1, B_1 and $O_{1,2}$. The known infinite and finite equilibrium

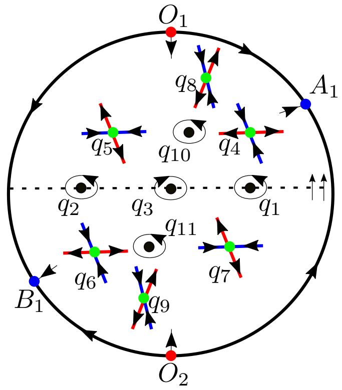


FIGURE 5. The local phase portraits of the vector field (4) with the five centers $q_{1,2,3,10,11}$ and the six saddles $q_{4,5,6,7,8,9}$ when $\Delta_1 < 0$ and $a_{21} < 0$.

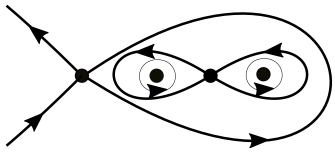


FIGURE 6. An eight-figure loop is inside a center-loop.

points have total index 6. Then total index of the remaining finite equilibrium points must be -4, i.e. the remaining finite equilibrium points are two saddles $q_{4,5}$. Similar to subcase (ii.2.1) the phase portraits are topologically equivalent to the phase portraits 1.10–1.12 of Figure 1.

(ii.2.2.2) If $\Delta_1 < 0$ and $a_{21} < 0$ we have $\widehat{d_1}\widehat{d_2} < 0$, $\widehat{d_2}\widehat{d_3} < 0$ and $\widehat{d_8}\widehat{d_9} < 0$, the number of the sign changes of this RSL is at least zero, then the vector field (4) has at most eight elementary equilibrium points different from $q_{1,2,3}$. From Propositions 2.5 and 2.8 the known equilibrium points of the vector field (4) have total index 10. Hence the total index of the remaining finite equilibrium points must be -8, i.e. they may be either six saddles and two centers, or four saddles.

Assume that the vector field (4) has eight extra equilibrium points, which are six saddles $q_{4,5,6,7,8,9}$ and two centers $q_{10,11}$. The local phase portraits of the all equilibrium points in the Poincaré disc can see Figure 5. The corresponding RSL of $-f(-y)$ must be

$$[-1, -1, -1, -1, -1, -1, -1, -1],$$

i.e. $M_{10,12,13,14} > 0$ and $M_{11} < 0$. With a computer algebra system such as Mathematica, by solving these inequalities we obtain the condition $a_{21} < \text{Root}[M_{14}, 1]$.

If the saddle q_4 is on the boundary of the period annulus of the centers $q_{1,10}$, creating an eight-figure loop, then the saddle q_8 is on the boundary of the period annulus of this eight-figure loop, see Figure 6. By the symmetry the saddle q_6 is on the boundary of the period annulus of the centers $q_{2,11}$, creating an eight-figure loop, the saddle q_9 is on the boundary of the period annulus of this eight-figure loop. Then the saddles $q_{5,7}$ must be on the boundary of the period annulus of the center q_3, creating a heteroclinic loop. The global phase portrait is topologically equivalent to the phase portrait 1.13 of Figure 1, which can be realized when $a_{21} = -1$, $b_{03} = -0.68$ and $a_{02} = -2.1$.

Or these two eight-figure loops are inside the tri-heteroclinic loop, which is created by the saddles $q_{5,7}$ and the center q_3. Then one attracting and one repelling separatrices of the saddle q_8 connect the ones of the saddle q_9. The global phase portrait is topologically equivalent to the phase portrait 1.15 of Figure 1, which can be realized when $a_{21} = -1$, $b_{03} = -0.67$ and $a_{02} = -2.1$.

From the phase portraits 1.13 to 1.15 it follows by the continuity of the phase portraits with respect to the parameters that there must exist the phase portrait 1.14 of Figure 1, which can be realized when $a_{21} = -1$, $b_{03} \approx -0.6765$ and $a_{02} = -2.1$. That is the saddles $q_{5,7,8}$ are on the boundary of the period annulus of one eight-figure loop, and the saddles $q_{5,7,9}$ are on the boundary of the period annulus of an other eight-figure loop.

Assume that the vector field (4) has four extra saddles $q_{4,5,6,7}$, from the above subcase we obtain the condition $a_{21} \geq \text{Root}[M_{14}, 1]$. In this subcase the phase portraits are topologically equivalent to the phase portraits 1.16–1.18 of Figure 1.

(ii.2.2.3) If $\Delta_1 > 0$ the known infinite and finite equilibrium points have total index 10. Hence the total index of the remaining finite equilibrium points must be -8, i.e. they are either four saddles, or six saddles and two centers.

If the vector field (4) has eight extra equilibrium points, then $-f(-y)$ need to have four negative roots. Since $\Delta_1 > 0$ we have $\widehat{d_1}\widehat{d_2} > 0$ and the corresponding RSL must be $[1, 1, 1, 1, 1, 1, 1, 1]$, which cannot be obtained varying the parameters a_{02}, a_{21} and b_{03} by solving the inequalities with $a_{21}^3 - 9b_{03}^2 < 0$, $M_{10,12,14} > 0$ and $M_{11,13} < 0$. Thus this subcase is impossible.

If the vector field (4) has four extra saddles $q_{4,5,6,7}$, similar to the above analysis of the phase portraits 1.10–1.12 we can obtain the phase portraits 1.19–1.21 of Figure 1. On the other hand, if the separatrices of the saddle q_3 connect with the nodes $O_{1,2}$, A_3 and B_3. Then the saddle q_4 is on the boundary of the period annulus of the center q_1, and they create a center-loop. The separatrices of the saddle q_7 connect with the infinite equilibrium points $A_{1,2,3}$ and O_1. By the symmetry the global phase portrait in the Poincaé disc is topologically equivalent to the phase portrait 1.22 of Figure 1, which can be realized $a_{02} = -0.5$, $a_{21} = 0.8$ and $b_{03} = -0.1$.

In summary we have the following result.

Theorem 3.2. *When* $a_{03} = 0$ *the phase portraits of the continuous piecewise* $\mathbb{Z}_2$*-equivariant cubic Hamiltonian vector field* (4) *are topologically equivalent to one of the* 22 *phase portraits showed in Figure* 1*. The corresponding conditions realizing these phase portraits are given in Table 3.*

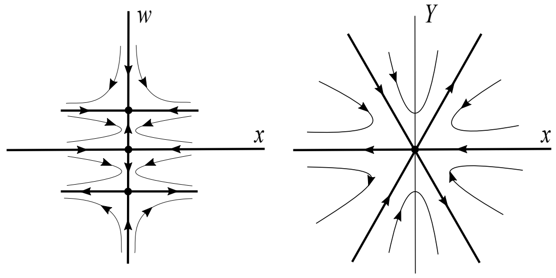
TABLE 3. The conditions for the phase portraits of the vector field (4) with $a_{03} = 0$.

Conditions			Phase portraits
$b_{03} = 0$	$a_{21} = 0$		1.1
	$a_{21} < 0$	$a_{02}^2 + 2a_{21}^3 \geq 0$	1.2
		$a_{02}^2 + 2a_{21}^3 < 0$	1.3
	$a_{21} > 0$	$a_{02}^4 - 12a_{21}^6 \neq 0$	1.4,1.6
		$a_{02}^4 - 12a_{21}^6 = 0$	1.5
$b_{03} \neq 0$	$\Delta_1 = 0$		1.7–1.9
	$\Delta_1 < 0$	$a_{21} \geq 0$	1.10–1.12
		$a_{21} < \text{Root}[M_{14}, 1]$	1.13–1.15
		$\text{Root}[M_{14}, 1] \leq a_{21} < 0$	1.16–1.18
	$\Delta_1 > 0$		1.19–1.22

3.2. **Case** $a_{03} \neq 0$**.** We compute the Gröbner basis for the polynomials $\dot{x}$, $\dot{y}$, $-1 + 3x^2 - 2a_{21}y^2$ and $2a_{21}x - 3b_{03}y$ to analyze the non-elementary points, and obtain the following polynomials

$$b_{03},\ a_{03}^2 + 2a_{02}^2a_{21} + 4a_{03}a_{21}^2 + 4a_{21}^4,\ -a_{03} - 2a_{21}^2 + 2a_{02}a_{21}y,\ 1 + 2a_{21}y^2,\ x,$$
$$a_{02}^2 + a_{03}a_{21} + 2a_{21}^3 + a_{02}a_{03}y,\ a_{02} + a_{03}y + 2a_{21}^2y,\ -a_{21} + a_{02}y + a_{03}y^2.$$

It means that the first vector field of (4) has one possible nilpotent equilibrium point $(0, -\frac{\sqrt{-2a_{21}}}{2a_{21}})$ when the above polynomials are zero and $a_{21} < 0$. We obtain that this possible nilpotent e-quilibrium point is a saddle.



(a) (b)

FIGURE 7. The local phase portraits at the origin of systems (26) and (27).

Then we calculate the Gröbner basis for the four polynomials $\dot{x}$, $\dot{y}$, $2a_{21}x-3b_{03}y$ and M_8, and obtain twenty-nine polynomials. We can find that one polynomial $a_{21}^2(a_{02}^2+4a_{03}a_{21}-9b_{03}^2)$ must be zero when the first vector field of (4) has a nilpotent equilibrium point. We take $a_{21}=0$ and again compute the Gröbner basic for the above four polynomials and obtain

$$b_{03}y,\ \ a_{02}^2y,\ \ y(2a_{02}+3a_{03}y),\ \ a_{02}y^2,\ \ (-1+x)x(1+x).$$

It means that there is no nilpotent equilibrium points different from $q_{1,2,3}$ when $a_{21}=0$. Otherwise we take $a_{03}=-\frac{a_{02}^2-9b_{03}^2}{4a_{21}}\neq 0$ and compute the Gröbner basic for the above four polynomials, then we obtain the four polynomials

$$b_{03}(3a_{02}^2+8a_{21}^3-27b_{03}^2),\ \ -b_{03}(6a_{02}a_{21}+8a_{21}^3y-27b_{03}^2y),\ \ -2a_{21}+a_{02}y,\ \ \frac{2a_{21}x-3b_{03}y}{a_{21}}.$$

Hence it has one nilpotent equilibrium point $(\frac{3b_{03}}{a_{02}},\frac{2a_{21}}{a_{02}})$ different from $q_{1,2,3}$ when $b_{03}(3a_{02}^2+8a_{21}^3-27b_{03}^2)=0$ and $a_{21}<0$, which is a cusp.

Next we analyze the degenerate equilibrium points of the first vector field of (4). We calculate the Gröbner basis for five polynomials $\dot{x}$, $\dot{y}$, $-1+3x^2-2a_{21}y^2$, $y(2a_{21}x-3b_{03}y)$ and M_8, and we obtain the nine polynomials

$$\begin{aligned}&b_{03},\ a_{02}^4-8a_{03}^3,\ a_{02}^2+4a_{03}a_{21},\ 2a_{03}^2+a_{02}^2a_{21},\\&-a_{03}+2a_{21}^2,\ a_{02}+2a_{03}y,\ -2a_{21}+a_{02}y,\ 1+2a_{21}y^2,\ x.\end{aligned}$$

It means that it has one degenerate equilibrium point $(0,-\frac{a_{02}}{2a_{03}})$ when the above polynomials are zero and $a_{21}<0$. For analyzing the local phase portrait of this degenerate equilibrium point we need to do blow-ups. We do the change $y\to Y-\frac{a_{02}}{2a_{03}}$ and $a_{03}\to 2a_{21}^2$ and the vector field (4) becomes

$$\begin{aligned}\dot{x}=&-\frac{a_{02}}{4a_{21}}x^2-\frac{a_{02}}{2}Y^2+a_{21}x^2Y+2a_{21}^2Y^3,\\ \dot{Y}=&\frac{x^3}{2}+\frac{a_{02}}{2a_{21}}xY-a_{21}xY^2.\end{aligned}\tag{26}$$

Applying the directional blow-up $(x,Y)\to(x,w)$ with $w=\frac{Y}{x}$ and eliminating the common factor x we have

$$\begin{aligned}\dot{x}=&-\frac{a_{02}}{4a_{21}}x-\frac{1}{2}a_{02}xw^2+a_{21}x^2w+2a_{21}^2x^2w^3,\\ \dot{w}=&\frac{x}{2}+\frac{3a_{02}}{4a_{21}}w-2a_{21}xw^2+\frac{a_{02}}{2}w^3-2a_{21}^2xw^4.\end{aligned}\tag{27}$$

For $x=0$ system (27) has three saddles $(0,0)$ and $(0,\pm\frac{\sqrt{-6a_{21}}}{2a_{21}})$, see Figure 7 (a). Going back through the change of variables to system (26) we obtain that the local phase portrait at the origin has six hyperbolic sectors, as it is shown in Figure 7 (b).

Now we consider the elementary finite equilibrium points of the vector field (4). Similar to the previous case we divide the study with two subcases $b_{03}=0$ and $b_{03}\neq 0$.

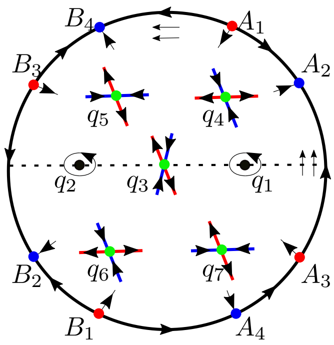


FIGURE 8. The local phase portraits at all equilibrium points of the vector field (4) with $-2a_{21}^2 < a_{03} < 0$, $b_{03} = 0$ and $a_{21} > 0$.

(i) Assume that $b_{03} = 0$ we have $M_2 = a_{03}(a_{03} + 2a_{21}^2)^2$. Then the vector field (4) is also symmetry respect to the x-axis and the y-axis.

(i.1) If $a_{03} = -2a_{21}^2 \neq 0$, then $M_2 = 0$. When $a_{21} > 0$ the vector field (4) has no finite equilibrium points different from $q_{1,2,3}$, the saddle q_3 must be on the boundary of the period annulus of the centers $q_{1,2}$, creating an eight-figure loop. From Proposition 2.7 we obtain that only condition XII of Table 2 satisfies this subcase. And the vector field (4) has four E-H equilibrium points A_i and B_i $(i = 1, 2)$, whose elliptic sectors are outside the Poincaré disc. Then the infinite and finite equilibrium points have total index 2. By the symmetries in this subcase the phase portrait is topologically equivalent to the phase portrait 1.23 of Figure 1.

When $a_{21} < 0$ we obtain that only condition XX of Proposition 2.7 satisfies this subcase. Then the vector field (4) has no infinite equilibrium points and five finite equilibrium points, where $q_{1,2,3}$ are three centers and $q_{4,5} = (0, \pm\frac{a_{02}+\sqrt{a_{02}^2-8a_{21}^3}}{4a_{21}^2})$ are two saddles, and these five equilibrium points create a tri-heteroclinic loop. In this subcase the only possible phase portrait is topologically equivalent to the phase portrait 1.24 of Figure 1.

(i.2) If $a_{03} + 2a_{21}^2 \neq 0$, then $M_2 \neq 0$.

(i.2.1) We consider condition I, i.e. $a_{21} > 0$, $-2a_{21}^2 < a_{03} < 0$, and the vector field (4) has eight infinite equilibrium points A_i and B_i $(i = 1, 2, 3, 4)$, which are nodes. From the previous analysis the known infinite and finite equilibrium points have total index 10. Hence the total index of the remaining finite equilibrium points must be -8. Then we obtain that the vector field (4) has four saddles

$$q_{4,5} = \left(\pm \frac{\sqrt{a_{03}^2 + 2a_{02}^2 a_{21} + 4a_{03}a_{21}^2 + 4a_{21}^4}}{a_{03} + 2a_{21}^2}, -\frac{a_{02}}{a_{03} + 2a_{21}^2} \right),$$
$$q_{6,7} = \left(\mp \frac{\sqrt{a_{03}^2 + 2a_{02}^2 a_{21} + 4a_{03}a_{21}^2 + 4a_{21}^4}}{a_{03} + 2a_{21}^2}, \frac{a_{02}}{a_{03} + 2a_{21}^2} \right),$$

different from $q_{1,2,3}$. The local phase portraits at all equilibrium points in the Poincaré disc are drawn in Figure 8.

Similar to the analysis of the phase portraits 1.4–1.6, we obtain the phase portraits 1.25–1.27 of Figure 1. In fact we can obtain the condition of the corresponding phase portrait 1.26. Since the five saddles $q_{3,4,5,6,7}$ are in the same energy level, we have the Hamiltonian

$$(28)\qquad \begin{aligned} H^+|_{q_{4,5}} = H^-|_{q_{6,7}} &= -\frac{2a_{02}^4 - 3a_{03}^3 - 18a_{03}^2 a_{21}^2 - 36a_{03}a_{21}^4 - 24a_{21}^6}{24(a_{03} + 2a_{21}^2)^3} \\ &= -\frac{M_{15}}{24(a_{03} + 2a_{21}^2)^3} = 0, \end{aligned}$$

i.e. $a_{02} = -\sqrt[4]{\frac{3(a_{03}+2a_{21}^2)^3}{2}}$.

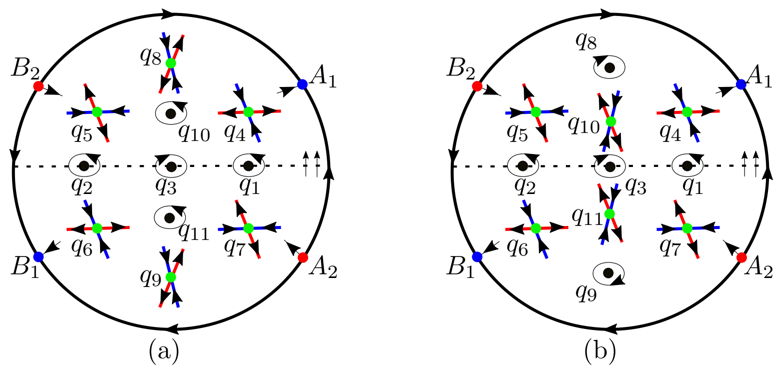


FIGURE 9. The possible local phase portraits at all equilibrium points of the vector field (4) with $a_{21} < 0$, $b_{03} = 0$ and $0 < a_{03} < -\frac{a_{02}^2}{4a_{21}}$.

(i.2.2) Since $M_2 \neq 0$ we consider the subcase when the vector field (4) has four infinite nodes A_i and B_i $(i = 1, 2)$, where one of the conditions V, IX and X satisfies this subcase.

(i.2.2.1) Assume that condition V (i.e. $a_{21} < 0$ and $a_{03} > 0$) holds. From the previous analysis, the known equilibrium points have total index 10. Hence the total index of the remaining finite equilibrium points must be -8.

If $a_{03} > -\frac{a_{02}^2}{4a_{21}}$ the extra finite equilibrium points are four saddles $q_{4,5,6,7}$, which are in the same energy level. Since there are no other finite equilibrium points, these four saddles $q_{4,5,6,7}$ must connect to each other one by one. The only possible global phase portrait in the Poincaé disc is topologically equivalent to the phase portrait 1.28 of Figure 1.

If $a_{03} = -\frac{a_{02}^2}{4a_{21}}$ and $a_{02}^4 + 8a_{21}^3 \neq 0$ the extra finite equilibrium points are four saddles $q_{4,5,6,7}$ and two cusps $\widetilde{q}_{8,9} = (0, \pm\frac{2a_{21}}{a_{02}})$. Similar to the above subcase we obtain that the phase portrait is topologically equivalent to the phase portrait 1.29 of Figure 2.

If $a_{03} = -\frac{a_{02}^2}{4a_{21}}$ and $a_{02}^4 + 8a_{21}^3 = 0$ then the vector field (4) has two extra degenerate equilibrium points $\widetilde{q}_{4,5} = (0, \pm\frac{2a_{21}}{a_{02}})$, whose phase portrait has six hyperbolic sectors. Then these two equilibrium point must be on the boundary of the period annulus of the centers $q_{1,2,3}$. In this subcase the global phase portrait is topologically equivalent to the phase portrait 1.30 of Figure 2.

If $0 < a_{03} < -\frac{a_{02}^2}{4a_{21}}$ there are several subcases. First if $a_{03}^2 + 2a_{02}^2a_{21} + 4a_{03}a_{21}^2 + 4a_{21}^4 < 0$, i.e $a_{02} < -\sqrt{-2a_{21}^3}$ and $0 < a_{03} < -2a_{21}^2 + \sqrt{-2a_{02}^2a_{21}}$ then the vector field (4) has four extra finite equilibrium points $q_{8,9,10,11}$, where

$$q_{8,9} = \left(0, \pm\frac{-a_{02} + \sqrt{a_{02}^2 + 4a_{03}a_{21}}}{2a_{03}}\right),$$
$$q_{10,11} = \left(0, \mp\frac{a_{02} + \sqrt{a_{02}^2 + 4a_{03}a_{21}}}{2a_{03}}\right),$$

which are four saddles. Then we have that the global phase portrait is topologically equivalent to the phase portrait 1.18 of Figure 1.

If $a_{03}^2 + 2a_{02}^2a_{21} + 4a_{03}a_{21}^2 + 4a_{21}^4 > 0$ then the vector field (4) has eight extra equilibrium points q_i for $i = 4, 5, \cdots, 10, 11$. We compute the Jacobian matrices of the vector field (4) at

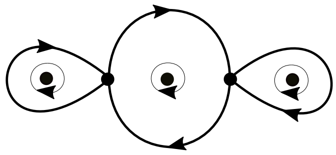


FIGURE 10. A homo-heteroclinic loop.

$q_{8,9,10,11}$ and obtain

$$
(29)\quad
\begin{aligned}
J|_{q_{8,9}} &= \begin{pmatrix} 0 & -\frac{-a_{02}^2-4a_{03}a_{21}+a_{02}\sqrt{a_{02}^2+4a_{03}a_{21}}}{2a_{03}} \\ -\frac{a_{03}^2+a_{02}^2a_{21}+2a_{03}a_{21}^2-a_{02}a_{21}\sqrt{a_{02}^2+4a_{03}a_{21}}}{2a_{03}^2} & 0 \end{pmatrix}, \\
J|_{q_{10,11}} &= \begin{pmatrix} 0 & \frac{-a_{02}^2-4a_{03}a_{21}+a_{02}\sqrt{a_{02}^2+4a_{03}a_{21}}}{2a_{03}} \\ -\frac{a_{03}^2+a_{02}^2a_{21}+2a_{03}a_{21}^2+a_{02}a_{21}\sqrt{a_{02}^2+4a_{03}a_{21}}}{2a_{03}^2} & 0 \end{pmatrix}.
\end{aligned}
$$

By the eigenvalues of $J|_{q_{8,9,10,11}}$ we obtain that $q_{8,9}$ are two saddles and $q_{10,11}$ are two centers when (a): $M_{16} > 0$, where

$$
(30)\quad
\begin{aligned}
M_{16} =& - a_{02}^2a_{03}^3 - 2a_{02}^4a_{03}a_{21} - 4a_{03}^4a_{21} - 10a_{02}^2a_{03}^2a_{21}^2 - 8a_{03}^3a_{21}^3 \\
&+ a_{02}a_{03}^3\sqrt{a_{02}^2 + 4a_{03}a_{21}} + 2a_{02}^3a_{03}a_{21}\sqrt{a_{02}^2 + 4a_{03}a_{21}} \\
&+ 6a_{02}a_{03}^2a_{21}^2\sqrt{a_{02}^2 + 4a_{03}a_{21}}.
\end{aligned}
$$

And $q_{8,9}$ are two centers and $q_{10,11}$ are two saddles when (b): $M_{16} < 0$. Then we have the local phase portraits at all equilibrium points of the vector field (4) in the Poincaré disc, see Figure 9 (a) and (b), respectively. Using the Hamiltonian quantifies we know that $q_{4,5,6,7}$ are in the same energy level. By the symmetries the separatrices of the saddles $q_{4,5,6,7}$ connect with the other one by one.

Subcase (a): That is either $-2\sqrt{-2a_{21}^3} < a_{02} \leq -\sqrt{-2a_{21}^3}$ and $-2a_{21}^2 + \sqrt{-2a_{02}^2a_{21}} < a_{03} < -\frac{a_{02}^2}{4a_{21}}$, or $-\sqrt{-2a_{21}^3} < a_{02} < 0$ and $0 < a_{03} < -\frac{a_{02}^2}{4a_{21}}$. If the saddles $q_{4,7}$ are on the boundary of the period annulus of the center q_1, they create a heteroclinic loop. Then the saddle q_8 is on the boundary of the period annulus of the center q_{10}, creating a center-loop. By the symmetries we obtain the global phase portrait 1.31 of Figure 2, which can be realized when $a_{02} = -1$, $a_{21} = -1$ and $a_{03} = 0.24$.

If the centers $q_{4,5}$ are also on the boundary of the period annulus of the center q_{10}, by the symmetries they create a quadruple-heteroclinic loop with the saddles $q_{4,5,6,7}$ and the centers $q_{1,2,3,10,11}$. Then the one attracting and one repelling separatrices of the saddle q_8 connect with the ones of the saddle q_9. The global phase portrait is topologically equivalent to the phase portrait 1.33 of Figure 2, which can be realized when $a_{02} = -1$, $a_{21} = -1$ and $a_{03} = 0.2$.

From the phase portraits 1.31 and 1.33 it follows by the continuity of the phase portraits with respect to the parameters the existence of the phase portrait 1.32 of Figure 2, which can be realized when $a_{02} = -1$, $a_{21} = -1$ and $a_{03} \approx 0.228747$. In this subcase the values of the Hamiltonian at these six saddles $q_{4,5,6,7,8,9}$ are $H^+(x,y)|_{q_{4,5,8}} = H^-(x,y)|_{q_{6,7,9}}$, i.e.

$$
-\frac{M_{15}}{24(a_{03} + 2a_{21}^2)^3} = \frac{(-a_{02} + \sqrt{a_{02}^2 + 4a_{03}a_{21}})^2(-a_{02}^2 - 6a_{03}a_{21} + a_{02}\sqrt{a_{02}^2 + 4a_{03}a_{21}})}{96a_{03}^3}.
$$

By solving the above equation we obtain $-2\sqrt{-2a_{21}^3} < a_{02} < 0$ and $a_{03} = \mathrm{R}[M_{17}, 1]$, where

$$
\begin{aligned}
M_{17} =& - 8a_{02}^4a_{21}^4 + 32a_{02}^2a_{21}^7 + (-12a_{02}^4a_{21}^2 + 32a_{02}^2a_{21}^5 + 144a_{21}^8)a_{03} \\
&+ (-6a_{02}^4 + 8a_{02}^2a_{21}^3 + 288a_{21}^6)a_{03}^2 + 216a_{21}^4a_{03}^3 + 72a_{21}^2a_{03}^4 + 9a_{03}^5.
\end{aligned}
$$

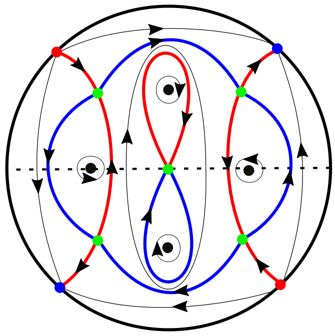


FIGURE 11. With condition IX or X, one possible phase portrait of vector field (4) when $a_{03} \neq 0$ and $b_{03} = 0$.

Subcase (b): That is $a_{02} < -2\sqrt{-2a_{21}^3}$ and $-2a_{21}^2 + \sqrt{-2a_{02}^2 a_{21}} < a_{03} < -\frac{a_{02}^2}{4a_{21}}$. If the saddles $q_{4,5}$ are on the boundary of the period annulus of the center q_8, then it creates a heteroclinic loop. By the symmetry the saddles $q_{6,7}$ are on the boundary of the period annulus of the center q_9. Then the saddles $q_{10,11}$ are on the boundary of the period annulus of the centers $q_{1,2,3}$, creating a tri-heteroclinic loop. Hence we have the global phase portrait 1.34 of Figure 2, which can be realized when $a_{02} = -1$, $a_{21} = -0.4$ and $a_{03} = 0.6$.

If the saddles $q_{4,7}$ are on the boundary of the period annulus of the center q_1, the saddles $q_{5,6}$ are on the boundary of the period annulus of the center q_2, then they create two heteroclinic loops. By the symmetries the saddles $q_{10,11}$ must be on the boundary of the period annulus of the center q_3, then the saddles $q_{10,11}$ are also on the boundary of the period annulus of the centers $q_{8,9}$, respectively. Thus they create a homo-heteroclinic loop, see Figure 10. The global phase portrait is topologically equivalent to the phase portrait 1.36 of Figure 2, which can be realized when $a_{02} = -1$, $a_{21} = -0.4$ and $a_{03} = 0.61$.

From the phase portraits 1.34 and 1.36 it follows by the continuity of the phase portraits with respect to the parameters the existence of the phase portrait 1.35 of Figure 2, which can be realized when $a_{02} = -1$, $a_{21} = -0.4$ and $a_{03} \approx 0.606045$. Similar to the phase portrait 1.32, we obtain that the values of Hamiltonian at these six saddles $q_{4,5,6,7,10,11}$ are equal. We obtain the conditions $a_{02} < -2\sqrt{-2a_{21}^3}$ and $a_{03} = \mathrm{R}[M_{17}, 1]$.

(i.2.2.2) Assume that condition IX or X holds. From the previous analysis the known infinite and finite equilibrium points have total index 6. Hence the total index of the remaining finite equilibrium points must be -4. Since $a_{21} \geq 0$ these remaining equilibrium points can not be cusps. And we obtain that the extra equilibrium points are four saddles $q_{4,5,6,7}$ and two centers $q_{8,9}$. If the saddles $q_{4,5}$ are on the boundary of the period annulus of the center q_8, they create a heteroclinic loop. By the symmetries the saddles $q_{6,7}$ are on the boundary of the period annulus of the center q_9. Then the saddle q_3 is on the boundary of the period annulus of the centers $q_{1,2}$, and they create an eight-figure loop. Then we have the global phase portrait 1.37 of Figure 2, which can be realized when $a_{02} = -1$, $a_{21} = -0.4$ and $a_{03} = 0.6$.

If the saddles $q_{4,7}$ are on the boundary of the period annulus of the center q_1, they create a heteroclinic loop. By the symmetries the saddles $q_{5,6}$ are on the boundary of the period annulus of the center q_2. Then the saddles $q_{8,9}$ are on the boundary of the period annulus of the center q_3, they create an eight-figure loop. We obtain the phase portrait of Figure 11, but this phase portrait is topologically equivalent to the phase portrait 1.37 of Figure 2.

From the phase portrait 1.37 to the phase portrait of Figure 11 it follows by the continuity of the phase portraits with respect to the parameters the existence of the phase portrait 1.38 of Figure 2, which can be realized when $a_{02} \approx -1.10668$, $a_{21} = 0$ and $a_{03} = 1$. In this subcase the five saddles $q_{3,4,5,6,7}$ are in the same energy level of the Hamiltonian (28) and we have $a_{02} = -\sqrt[4]{\frac{3(a_{03}+2a_{21}^2)^3}{2}}$.

(i.2.3) We consider the subcase when the vector field (4) has no infinite equilibrium points, then one of the conditions XIV, XVII and XIX must be satisfied in this subcase.

(i.2.3.1) Assume that the condition XIV holds. Then the vector field (4) has no extra finite equilibrium points different from $q_{1,2,3}$, where q_3 is a saddle. Then the global phase portrait is topologically equivalent to the phase portrait 1.39 of Figure 2.

(i.2.3.2) Assume that one of the conditions XVII and XIX holds. Then the vector field (4) has two extra finite equilibrium points $q_{8,9}$, where q_3 is a center and $q_{8,9}$ are two saddles. The global phase portrait in the Poincaré disc is topologically equivalent to the phase portrait 1.24 of Figure 1.

Table 4. The conditions for the phase portraits of the vector field (4) with $a_{03} \neq 0$ and $b_{03} = 0$.

Conditions			Phase portraits
$a_{03} = -2a_{21}^2$	$a_{21} > 0$		1.23
	$a_{21} < 0$		1.24
$a_{03} + 2a_{21}^2 \neq 0$	$a_{21} > 0$	$-2a_{21}^2 < a_{03} < 0$, $a_{02} \neq -\sqrt[4]{\frac{3(a_{03}+2a_{21}^2)^3}{2}}$	1.25,1.27
		$-2a_{21}^2 < a_{03} < 0$, $a_{02} = -\sqrt[4]{\frac{3(a_{03}+2a_{21}^2)^3}{2}}$	1.26
		$a_{03} > 0$, $a_{02} \neq -\sqrt[4]{\frac{3(a_{03}+2a_{21}^2)^3}{2}}$	1.37
		$a_{03} > 0$, $a_{02} = -\sqrt[4]{\frac{3(a_{03}+2a_{21}^2)^3}{2}}$	1.38
		$a_{03} < -2a_{21}^2$	1.39
	$a_{21} = 0$	$a_{03} > 0$, $a_{02} \neq -\sqrt[4]{\frac{3a_{03}^3}{2}}$	1.37
		$a_{03} > 0$, $a_{02} = -\sqrt[4]{\frac{3a_{03}^3}{2}}$	1.38
		$a_{03} < 0$	1.24
	$a_{21} < 0$	$a_{03} > -\frac{a_{02}^2}{4a_{21}}$	1.28
		$a_{03} = -\frac{a_{02}^2}{4a_{21}}$, $a_{02}^2 + 8a_{21}^3 \neq 0$	1.29
		$a_{03} = -\frac{a_{02}^2}{4a_{21}}$, $a_{02}^2 + 8a_{21}^3 = 0$	1.30
		$a_{02} < -\sqrt{-2a_{21}^3}$, $0 < a_{03} < -2a_{21}^2 + \sqrt{-2a_{02}^2 a_{21}}$	1.18
		$-2\sqrt{-2a_{21}^3} < a_{02} \leq -\sqrt{-2a_{21}^3}$, $-2a_{21}^2 + \sqrt{-2a_{02}^2 a_{21}} < a_{03} < -\frac{a_{02}^2}{4a_{21}}$; $-\sqrt{-2a_{21}^3} < a_{02} < 0$, $0 < a_{03} < -\frac{a_{02}^2}{4a_{21}}$	1.31,1.33
		$-2\sqrt{-2a_{21}^3} < a_{02} < 0$, $a_{03} = \mathrm{R}[M_{17}, 1]$	1.32
		$a_{02} < -2\sqrt{-2a_{21}^3}$, $a_{03} \neq \mathrm{R}[M_{17}, 1]$, $-2a_{21}^2 + \sqrt{-2a_{02}^2 a_{21}} < a_{03} < -\frac{a_{02}^2}{4a_{21}}$	1.34,1.36
		$a_{02} < -2\sqrt{-2a_{21}^3}$, $a_{03} = \mathrm{R}[M_{17}, 1]$	1.35
		$a_{03} < 0$	1.24

In summary we have the following result.

Theorem 3.3. *When $a_{03} \neq 0$ and $b_{03} = 0$ the phase portraits of the continuous piecewise $\mathbb{Z}_2$-equivariant cubic Hamiltonian vector field* (4) *are topologically equivalent to one of the* 18 *phase portraits showed in Figures* 1 *and* 2*. The corresponding conditions realizing these phase portraits are given in Table 4.*

(ii) We assume that $b_{03} \neq 0$. The explicit expressions of the finite equilibrium points different from q_k for $k = 1, 2, 3$, and their eigenvalues in terms of the four parameters a_{02}, a_{03}, a_{21} and b_{03} are more complicated. From the first vector field of (4) we compute the Gröbner basis for

$\dot{x}$ and $\dot{y}$ and we obtain sixteen polynomials, the following two polynomials

$$y^2\Big[x(a_{02}a_{21}+a_{03}a_{21}y+2a_{21}^3y-9b_{03}^2y)$$
$$-b_{03}(3a_{21}-3a_{02}y-3a_{03}y^2+2a_{21}^2y^2)\Big]$$

and

$$-y^2\Big[-a_{21}(a_{02}^2-9b_{03}^2)+a_{02}(a_{02}^2-2a_{03}a_{21}-4a_{21}^3-9b_{03}^2)y$$
$$+(3a_{02}^2a_{03}-a_{03}^2a_{21}+4a_{02}^2a_{21}^2-4a_{03}a_{21}^3-4a_{21}^5-9a_{03}b_{03}^2+30a_{21}^2b_{03}^2)y^2 \tag{31}$$
$$+a_{02}M_{18}y^3+M_2y^4\Big]=-y^2f(y),$$

are enough for our analysis, where

$$M_{18}=(a_{03}+2a_{21}^2)(3a_{03}+2a_{21}^2)-36a_{21}b_{03}^2.$$

Thus to find the number of finite equilibrium points of the first vector field of (4) for $y>0$ is equivalent to find the number of positive real roots of $f(y)$.

Subcase (ii.1): Assume that $M_2=0$, i.e. the coefficient of the quartic term of $f(y)$ in (31) is zero.

(ii.1.1) If $M_{18}=0$, then $f(y)$ has at most two positive real roots. Correspondingly the vector field (4) has at most four finite elementary equilibrium points for $y>0$. By solving $M_2=M_{18}=0$, from $a_{03}\neq 0$ we have $8a_{21}^3-27b_{03}^2=0$ and $a_{03}=\frac{2}{3}a_{21}^2$. Hence only condition XI of Proposition 2.7 satisfies this subcase. Then the vector field (4) has four infinite equilibrium points A_i and B_i $(i=1,2)$, which are nodes. The origin q_3 is a saddle. The total index of the known infinite and finite equilibrium points is 6. Hence the total index of the remaining finite equilibrium points must be -4. Since the vector field (4) has no cusps for $y>0$ when $a_{21}\geq 0$, these remaining equilibrium points are two saddles. By the symmetry in this subcase the phase portraits are topologically equivalent to the phase portraits 1.10–1.12 of Figure 1.

(ii.1.2) Otherwise $M_{18}\neq 0$, then the coefficient of the cubic term of $f(y)$ in (31) is nonzero. From $M_2=0$ we find that only condition II or XIII of Proposition 2.7 satisfies this subcase.

(ii.1.2.1) When condition II holds the vector field (4) has six infinite equilibrium points $A_{1,2,3}$ and $B_{1,2,3}$, where $A_{1,2}$ and $B_{1,2}$ are four nodes, A_3 and B_3 are two E-H points. If the hyperbolic sectors of A_3 and B_3 are outside the Poincaré disc, then they do not appear in the phase portrait of our piecewise differential vector field. The known infinite and finite equilibrium points have total index 10. Hence the total index of the remaining finite equilibrium points must be -8, i.e. they are four saddles. In fact from $a_{03}=\mathrm{R}[M_2,3]$ we have that the cubic polynomial $f(y)$ has two positive real roots, and its constant term is positive and the coefficient of the cubic term is also positive. In this subcase the phase portraits are topologically equivalent to the phase portraits 1.40–1.41 of Figure 2.

If the elliptic sectors of A_3 and B_3 are outside the Poincaré disc, then the known infinite and finite equilibrium points have total index 6. Hence the total index of the remaining finite equilibrium point must be -4. Similarly from $a_{03}=\mathrm{R}[M_2,2]$ the remaining equilibrium points are two saddles. By the symmetry the phase portraits are topologically equivalent to the phase portraits 1.7–1.9 of Figure 1.

(ii.1.2.2) When condition XIII holds the vector field (4) has two infinite equilibrium points A_1 and B_1, which are two E-H equilibrium points.

(ii.1.2.2.1) We assume that $a_{21}\geq 0$ the vector field (4) has no extra equilibrium points, where the origin q_3 is a saddle. In this subcase the phase portrait is topologically equivalent to the phase portrait 1.1 of Figure 1.

(ii.1.2.2.2) We assume that $a_{21}<0$. There is no parameters a_{02}, a_{21}, a_{03} and b_{03} by solving $M_7>0$, $M_2=0$ and $M_{18}<0$, we obtain $M_{18}>0$. Further the coefficient of the cubic term of $f(y)$ in (31) is negative and the constant one is positive, then the polynomial $f(y)$ cannot have two positive real roots.

If the elliptic sectors of A_1 and B_1 are inside the Poincaré disc, the known infinite and finite equilibrium points have total index 10. Hence the total index of the remaining finite equilibrium points must be -8, i.e. they are four saddles and two cusps. From the previous analysis we have $a_{03} = -\frac{a_{02}^2-9b_{03}^2}{4a_{21}}$ when one finite equilibrium point of the vector field (4) is a cusp, then we obtain

$$\begin{aligned} M_2 = & \frac{1}{64a_{21}^2}(-a_{02}^3 + 8a_{02}a_{21}^3 - 9a_{02}^2b_{03} + 8a_{21}^3b_{03} - 27a_{02}b_{03}^2 - 27b_{03}^3) \\ & \times (a_{02}^3 - 8a_{02}a_{21}^3 - 9a_{02}^2b_{03} + 8a_{21}^3b_{03} + 27a_{02}b_{03}^2 - 27b_{03}^3). \end{aligned}$$

But there is no values of the parameters a_{02}, a_{21} and b_{03} such that $M_2 \leq 0$, so we obtain that $a_{03} \neq \frac{a_{02}^2-9b_{03}^2}{4a_{21}}$ when $M_2 \leq 0$, then the vector field (4) has no cusps when $M_2 \leq 0$. Thus this subcase cannot hold.

If the hyperbolic sectors of A_1 and B_1 are inside the Poincaré disc, the known infinite and finite equilibrium points have total index 6. Hence the total index of the remaining finite equilibrium points must be -4, i.e. they are either two saddles, or four saddles and two centers.

When the remaining equilibrium points are two saddles $q_{4,5}$, then they create a tri-heteroclinic loop with the centers $q_{1,2,3}$. We obtain that the phase portrait is topologically equivalent to the phase portrait 1.2 of Figure 1.

Otherwise the vector field (4) has four centers $q_{1,2,8,9}$ and five saddles $q_{3,4,5,6,7}$. Since the number of saddles is more than the number of centers, there is at least one saddle on the boundary of the period annulus of two centers. Assume that the saddle q_4 is on the boundary of the period annulus of the centers $q_{1,8}$, by the symmetry, the saddle q_6 is on the boundary of the period annulus of the centers $q_{2,9}$. Then they create two eight-figure loops, which are inside one tri-heteroclinic loop creating by the center q_3 and the saddles $q_{5,7}$. Then we obtain the phase portrait 1.42 of Figure 2, which can be realized when $a_{02} = -1$, $a_{21} = -1$, $a_{03} \approx -0.01995$ and $b_{03} = -0.1$.

TABLE 5. The conditions for the phase portraits of the vector field (4) with $a_{03}b_{03} \neq 0$ and $M_2 = 0$.

Conditions			Phase portraits
$8a_{21}^3 - 27b_{03}^2 = 0$, $a_{03} = \frac{2}{3}a_{21}^2$			1.10–1.12
$8a_{21}^3 - 27b_{03}^2 > 0$	$a_{21} > 0$	$a_{03} = \mathrm{R}[M_2, 2]$	1.7–1.9
		$a_{03} = \mathrm{R}[M_2, 3]$	1.40–1.41
$8a_{21}^3 - 27b_{03}^2 \neq 0$	$a_{21} \geq 0$	$a_{03} = \mathrm{R}[M_2, 1]$	1.1
	$a_{21} < 0$	$a_{03} = \mathrm{R}[M_2, 1]$	1.2,1.42

In summary we have the following result.

Theorem 3.4. *When $a_{03}b_{03} \neq 0$ and $M_2 = 0$ the phase portraits of the continuous piecewise $\mathbb{Z}_2$-equivariant cubic Hamiltonian vector field* (4) *are topologically equivalent to one of the* 11 *phase portraits showed in Figures* 1 *and* 2. *The corresponding conditions realizing these phase portraits are given in Table 5.*

Subcase (ii.2): Assume that $M_2 \neq 0$, i.e. the coefficient of the quartic term of $f(y)$ in (31) is nonzero.

(ii.2.1) Assume that $a_{21} = 0$. Then the polynomial $f(y) = y\overline{f}(y)$, where

$$\overline{f}(y) = a_{02}(a_{02}^2 - 9b_{03}^2) + 3a_{03}(a_{02}^2 - 3b_{03}^2)y + 3a_{02}a_{03}^2y^2 + (a_{03}^3 + 54b_{03}^4)y^3.$$

Now we compute the number of negative real roots of $-\bar{f}(-y)$ in order to analyze the positive real roots of $\bar{f}(y)$. So we consider the discriminant sequence

$$\{\bar{d}_1\bar{d}_2, \bar{d}_2\bar{d}_3, \bar{d}_3\bar{d}_4, \bar{d}_4\bar{d}_5, \bar{d}_5\bar{d}_6, \bar{d}_6\bar{d}_7\} \tag{32}$$

associated to $-\bar{f}(-y)$, and we have

$$\begin{aligned} &\bar{d}_1 = a_{03}^3 + 54b_{03}^4 = \overline{M}_2, \qquad \bar{d}_2 = 3\overline{M}_2^2, \qquad \bar{d}_3 = -3a_{02}a_{03}^2\overline{M}_2^2,\\ &\bar{d}_4 = 54a_{03}b_{03}^2\overline{M}_2^2\overline{M}_{10}, \qquad \bar{d}_5 = -54a_{03}^2b_{03}^2\overline{M}_2^2\overline{M}_{11}, \qquad \bar{d}_6 = 2916b_{03}^6\overline{M}_2^2\overline{M}_{12},\\ &\bar{d}_7 = -2916a_{02}b_{03}^6(a_{02}^2 - 9b_{03}^2)\overline{M}_2^2\overline{M}_{12}, \end{aligned} \tag{33}$$

where

$$\begin{aligned} \overline{M}_{10} =& a_{03}^3 - 18a_{02}^2b_{03}^2 + 54b_{03}^4,\\ \overline{M}_{11} =& -a_{02}^2a_{03}^3 + 27a_{02}^4b_{03}^2 + 6a_{03}^3b_{03}^2 - 135a_{02}^2b_{03}^4 + 324b_{03}^6,\\ \overline{M}_{12} =& a_{03}^6 - 27a_{02}^6b_{03}^2 - 54a_{02}^2a_{03}^3b_{03}^2 + 486a_{02}^4b_{03}^4 + 54a_{03}^3b_{03}^4 - 2187a_{02}^2b_{03}^6. \end{aligned}$$

Since the coefficient of the cubic term of $\bar{f}(y)$ is nonzero, $\bar{f}(y)$ has at most three positive real roots. Hence the vector field (4) has at most six finite equilibrium points different from $q_{1,2,3}$.

(ii.2.1.1) We consider condition IV or VII, i.e. $a_{03} > 3b_{03}\sqrt[3]{2b_{03}}$, where the vector field (4) has four infinite nodes $A_{1,2}$ and $B_{1,2}$. Since the origin q_3 is a saddle the total index of the known infinite and finite equilibrium points is 6. Hence the total index of the remaining finite equilibrium points must be -4, i.e. they are either two saddles, or two saddles and two cusp, or four saddles and two centers. But the vector field (4) has no cusps for $y > 0$ when $a_{21} \geq 0$.

When the remaining finite equilibrium points are two saddles $q_{4,5}$, we obtain that the global phase portraits in the Poincaré disc are topologically equivalent to the phase portraits 1.10–1.12 of Figure 1.

When the remaining finite equilibrium points are four saddles $q_{4,5,6,7}$ and two centers $q_{8,9}$, we assume that the saddles $q_{4,6}$ are on the boundary of the period annulus of the centers $q_{1,2}$, respectively. If the saddles $p_{5,7}$ are on the boundary of the period annulus of the centers $q_{8,9}$, respectively. Then the vector field (4) has four center-loops. The global phase portrait in the Poincaré disc is topologically equivalent to the phase portrait 1.43 of Figure 2, which can be realized when $a_{03} = 1$, $b_{03} = -0.1$ and $a_{02} = -1$.

If the saddle q_3 is on the boundary of the period annulus of the centers $q_{8,9}$, creating one eight-figure loop. Then in this case the phase portrait is topologically equivalent to the phase portrait 1.45 of Figure 2, which can be realized when $a_{03} = 1$, $b_{03} = -0.1$ and $a_{02} = -0.8$.

From the phase portraits 1.43 and 1.45 it follows by the continuity of the phase portraits with respect to the parameters the existence of the phase portrait 1.44 of Figure 2, which can be realized when $a_{03} = 1$, $b_{03} = -0.1$ and $a_{02} \approx -0.83$.

(ii.2.1.2) We consider that the condition XV holds, i.e. $a_{03} < 3b_{03}\sqrt[3]{2b_{03}}$, where the vector field (4) has no infinite equilibrium points and $\overline{M}_2 < 0$. Further we have $\overline{d}_1\overline{d}_2 < 0$, $\overline{d}_2\overline{d}_3 > 0$, $\overline{d}_3\overline{d}_4 > 0$, $\overline{d}_4\overline{d}_5 < 0$ and $\overline{d}_6\overline{d}_7 > 0$ so the number of the sign changes is at least three, i.e. $-\overline{f}(-y)$ has no negative real roots. Hence the vector field (4) has no finite equilibrium points different from $q_{1,2,3}$. We obtain that the global phase portrait in the Poincaré disc is topologically equivalent to the phase portrait 1.39 of Figure 2.

(ii.2.2) Assume that $a_{21} \neq 0$. We consider the discriminant sequence

$$\{\widetilde{d}_1\widetilde{d}_2, \widetilde{d}_2\widetilde{d}_3, \widetilde{d}_3\widetilde{d}_4, \widetilde{d}_4\widetilde{d}_5, \widetilde{d}_5\widetilde{d}_6, \widetilde{d}_6\widetilde{d}_7, \widetilde{d}_7\widetilde{d}_8, \widetilde{d}_8\widetilde{d}_9\} \tag{34}$$

associated to

$$\begin{aligned} &-f(-y) = a_{21}(a_{02}^2 - 9b_{03}^2) + a_{02}(a_{02}^2 - 2a_{03}a_{21} - 4a_{21}^3 - 9b_{03}^2)y\\ &-(3a_{02}^2a_{03} - a_{03}^2a_{21} + 4a_{02}^2a_{21}^2 - 4a_{03}a_{21}^3 - 4a_{21}^5 - 9a_{03}b_{03}^2 + 30a_{21}^2b_{03}^2)y^2\\ &+a_{02}M_{18}y^3 - M_2y^4, \end{aligned}$$

from (31), and we have

$$
(35)\qquad
\begin{aligned}
&\widetilde{d_1} = -M_2, \qquad \widetilde{d_2} = 4M_2^2, \qquad \widetilde{d_3} = a_{02}M_7M_2^2, \qquad \widetilde{d_4} = M_2^2\widetilde{M}_{10},\\
&\widetilde{d_5} = M_2^2\widetilde{M}_{11}, \qquad \widetilde{d_6} = 2M_2^2\widetilde{M}_{12}, \qquad \widetilde{d_7} = -2a_{02}M_2^2\widetilde{M}_{13}, \qquad \widetilde{d_8} = 4b_{03}^2M_2^2\widetilde{M}_{14},\\
&\widetilde{d_9} = 4a_{21}(a_{02}^2 - 9b_{03}^2)b_{03}^2M_2^2\widetilde{M}_{14},
\end{aligned}
$$

where

$$
\begin{aligned}
\widetilde{M}_{10} =& 3a_{02}^2a_{03}^4 + 8a_{03}^5a_{21} + 16a_{02}^2a_{03}^3a_{21}^2 + 64a_{03}^4a_{21}^3 + 40a_{02}^2a_{03}^2a_{21}^4 + 192a_{03}^3a_{21}^5 + 64a_{02}^2a_{03}a_{21}^6\\
&+ 256a_{03}^2a_{21}^7 + 48a_{02}^2a_{21}^8 + 128a_{03}a_{21}^9 + 72a_{03}^4b_{03}^2 + 216a_{02}^2a_{03}^2a_{21}b_{03}^2 - 240a_{03}^3a_{21}^2b_{03}^2\\
&- 384a_{02}^2a_{03}a_{21}^3b_{03}^2 - 1888a_{03}^2a_{21}^4b_{03}^2 - 608a_{02}^2a_{21}^5b_{03}^2 - 2368a_{03}a_{21}^6b_{03}^2 - 256a_{21}^8b_{03}^2\\
&- 1296a_{02}^2a_{03}b_{03}^4 - 2160a_{03}^2a_{21}b_{03}^4 + 2160a_{02}^2a_{21}^2b_{03}^4 + 792a_{03}a_{21}^3b_{03}^4 + 3648a_{21}^5b_{03}^4\\
&+ 3888a_{03}b_{03}^6 - 12960a_{21}^2b_{03}^6,\\
\widetilde{M}_{11} =& 3a_{02}^2a_{03}^6a_{21} + 4a_{03}^7a_{21}^2 + 16a_{02}^2a_{03}^5a_{21}^3 - 8a_{02}^4a_{03}^3a_{21}^4 + 48a_{03}^6a_{21}^4 + 12a_{02}^2a_{03}^4a_{21}^5\\
&- 48a_{02}^4a_{03}^2a_{21}^6 + 240a_{03}^5a_{21}^6 - 64a_{02}^2a_{03}^3a_{21}^7 - 96a_{02}^4a_{03}a_{21}^8 + 640a_{03}^4a_{21}^8 - 112a_{02}^2a_{03}^2a_{21}^9\\
&- 64a_{02}^4a_{21}^{10} + 960a_{03}^3a_{21}^{10} + 768a_{03}^2a_{21}^{12} + 64a_{02}^2a_{21}^{13} + 256a_{03}a_{21}^{14} - 54a_{02}^2a_{03}^5b_{03}^2\\
&- 216a_{02}^4a_{03}^3a_{21}b_{03}^2 + 72a_{03}^6a_{21}b_{03}^2 + 54a_{02}^2a_{03}^4a_{21}^2b_{03}^2 + 216a_{02}^4a_{03}^2a_{21}^3b_{03}^2 + 92a_{03}^5a_{21}^3b_{03}^2\\
&+ 1368a_{02}^2a_{03}^3a_{21}^4b_{03}^2 + 1152a_{02}^4a_{03}a_{21}^5b_{03}^2 - 1376a_{03}^4a_{21}^5b_{03}^2 + 2848a_{02}^2a_{03}^2a_{21}^6b_{03}^2\\
&+ 736a_{02}^4a_{21}^7b_{03}^2 - 7168a_{03}^3a_{21}^7b_{03}^2 + 1024a_{02}^2a_{03}a_{21}^8b_{03}^2 - 11904a_{03}^2a_{21}^9b_{03}^2 - 992a_{02}^2a_{21}^{10}b_{03}^2\\
&- 7168a_{03}a_{21}^{11}b_{03}^2 - 512a_{21}^{13}b_{03}^2 + 1458a_{02}^4a_{03}^2b_{03}^4 + 324a_{03}^5b_{03}^4 + 1620a_{02}^2a_{03}^3a_{21}b_{03}^4\\
&- 3888a_{02}^4a_{03}a_{21}^2b_{03}^4 - 3240a_{03}^4a_{21}^2b_{03}^4 - 18144a_{02}^2a_{03}^2a_{21}^3b_{03}^4 - 3240a_{02}^4a_{21}^4b_{03}^4\\
&- 4320a_{03}^3a_{21}^4b_{03}^4 - 15984a_{02}^2a_{03}a_{21}^5b_{03}^4 + 34752a_{03}^2a_{21}^6b_{03}^4 + 4416a_{02}^2a_{21}^7b_{03}^4\\
&+ 61248a_{03}a_{21}^8b_{03}^4 + 11136a_{21}^{10}b_{03}^4 - 7290a_{02}^2a_{03}^2b_{03}^6 + 5832a_{02}^4a_{21}b_{03}^6 - 7776a_{03}^3a_{21}b_{03}^6\\
&+ 69984a_{02}^2a_{03}a_{21}^2b_{03}^6 + 77760a_{03}^2a_{21}^3b_{03}^6 + 3240a_{02}^2a_{21}^4b_{03}^6 - 148608a_{03}a_{21}^5b_{03}^6\\
&- 80640a_{21}^7b_{03}^6 + 17496a_{03}^2b_{03}^8 - 52488a_{02}^2a_{21}b_{03}^8 - 116640a_{03}a_{21}^2b_{03}^8 + 194400a_{21}^4b_{03}^8,\\
\widetilde{M}_{12} =& a_{02}^2a_{03}^8a_{21}^2 + 4a_{02}^4a_{03}^6a_{21}^3 + 4a_{03}^9a_{21}^3 + 4a_{02}^6a_{03}^4a_{21}^4 + 32a_{02}^2a_{03}^7a_{21}^4 + 64a_{02}^4a_{03}^5a_{21}^5\\
&+ 64a_{03}^8a_{21}^5 + 32a_{02}^6a_{03}^3a_{21}^6 + 304a_{02}^2a_{03}^6a_{21}^6 + 368a_{02}^4a_{03}^4a_{21}^7 + \cdots,\\
\widetilde{M}_{13} =& a_{02}^2a_{03}^9a_{21}^3 + 4a_{02}^4a_{03}^7a_{21}^4 + 4a_{03}^{10}a_{21}^4 + 4a_{02}^6a_{03}^5a_{21}^5 + 34a_{02}^2a_{03}^8a_{21}^5 + 72a_{02}^4a_{03}^6a_{21}^6\\
&+ 72a_{03}^9a_{21}^6 + 40a_{02}^6a_{03}^4a_{21}^7 + 368a_{02}^2a_{03}^7a_{21}^7 + 496a_{02}^4a_{03}^5a_{21}^8 + \cdots,\\
\widetilde{M}_{14} =& a_{02}^2a_{03}^6 + 6a_{02}^4a_{03}^4a_{21} + 4a_{03}^7a_{21} + 12a_{02}^6a_{03}^2a_{21}^2 + 36a_{02}^2a_{03}^5a_{21}^2 + 8a_{02}^8a_{21}^3 + 96a_{02}^4a_{03}^3a_{21}^3\\
&+ 48a_{03}^6a_{21}^3 + 80a_{02}^6a_{03}a_{21}^4 + 252a_{02}^2a_{03}^4a_{21}^4 + \cdots.
\end{aligned}
$$

Since the sequence associated to $-f(-y)$ is very completed, it is not easy to analyze all RSL. We consider the following two subcases $a_{21} > 0$ and $a_{21} < 0$ in order to study the phase portraits of vector field (4).

(ii.2.2.1) Assume that $a_{21} > 0$. From the previous analysis we know that the vector field (4) has no cusps.

(ii.2.2.1.1) We consider the vector field (4) has eight infinite nodes A_i and B_i ($i = 1, 2, 3, 4$), i.e. the vector field (4) satisfies condition I. Since $M_2 < 0$ we obtain that the polynomial $-f(-y)$ has four distinct negative real roots if and only if the RSL of (32) is $[1, 1, 1, 1, 1, 1, 1, 1]$, which cannot be obtained varying the parameters a_{02}, a_{21}, a_{03} and b_{03} by solving the inequality $d_i d_{i+1} > 0$, i.e. $M_7 < 0$ and $\widetilde{M}_{10,11,12,13,14} > 0$. And the polynomial $-f(-y)$ has three distinct negative real roots if and only if the RSL of (32) is $[1, 1, 1, 1, 1, 1, 0, 0]$, which also cannot be obtained. Therefore the polynomial $-f(-y)$ has at most two negative real roots. By the symmetry the vector field (4) has at most four additional finite equilibrium points different from $q_{1,2,3}$.

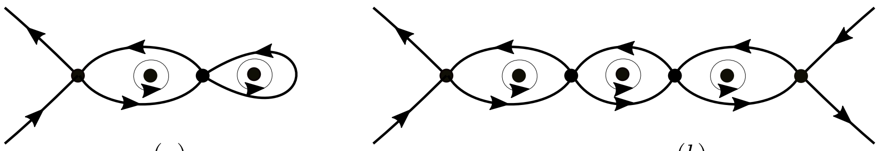


FIGURE 12. (a) A center-heteroclinic loop; (b) A chain-heteroclinic loop.

From the previous analysis we obtain that the known infinite and finite equilibrium points have total index 10. Hence the total index of the remaining finite equilibrium points must be -8, i.e. they are four saddles $q_{4,5,6,7}$. Then we obtain that the global phase portraits in the Poincaré disc are topologically equivalent to the phase portraits 1.19–1.21 of Figure 1.

(ii.2.2.1.2) We consider the vector field (4) has four infinite nodes, i.e. the vector field (4) satisfies one of the conditions III, VI and IX. Since $M_2 > 0$ we have $\widetilde{d}_1\widetilde{d}_2 < 0$ and $\widetilde{d}_8\widetilde{d}_9 > 0$, so the RSL of (32) can not be $[-1, -1, -1, -1, -1, -1, -1, -1]$. Hence the polynomial $-f(-y)$ has at most three negative real roots. By the symmetry the vector field (4) has at most six finite equilibrium points different from $q_{1,2,3}$.

From the previous analysis the known infinite and finite equilibrium points have total index 6. Hence the total index of the remaining finite equilibrium points must be -4, i.e. they are either two saddles, or four saddles and two centers. Similar to the subcase (ii.2.1.1) the global phase portraits in the Poincaré disc are topologically equivalent to the phase portraits 1.10–1.12 of Figure 1 and 1.43–1.45 of Figure 2.

(ii.2.2.1.3) We consider the vector field (4) has no infinite equilibrium points, i.e. it satisfies condition XIV. We find that $-f(-y)$ has no negative real roots, by the symmetry the global phase portrait in the Poincaré disc is topologically equivalent to the phase portrait 1.39 of Figure 2.

(ii.2.2.2) Assume that $a_{21} < 0$.

(ii.2.2.2.1) We consider the vector field (4) has four infinite nodes, i.e. it satisfies condition V or VIII. The polynomial $-f(-y)$ has at most four negative real roots, then the vector field (4) has at most eight finite equilibrium points different from $q_{1,2,3}$. From the previous analysis the known infinite and finite equilibrium points have total index 10. Hence the total index of the remaining finite equilibrium points must be -8, i.e. they are either four saddles, or four saddles and two cusps, or six saddles and two centers.

When the remaining finite equilibrium points are four saddles $q_{4,5,6,7}$ we can obtain the phase portraits 1.16–1.18 of Figure 1.

When the remaining finite equilibrium points are four saddles $q_{4,5,6,7}$ and two cusps $q_{8,9} = (\frac{3b_{03}}{a_{02}}, \frac{2a_{21}}{a_{02}})$, the global phase portraits in the Poincaé disc are topologically equivalent to the phase portraits 1.46 of Figure 2, which can be realized when $a_{21} = -1$, $a_{03} = 2/3$, $b_{03} = -1$ and $a_{02} \approx -3.415651$.

When the remaining finite equilibrium points are six saddles $q_{4,5,6,7,8,9}$ and two centers $q_{10,11}$, the local phase portraits at all equilibrium points of the vector field (4) are topologically equivalent to the one of Figure 9(a). We can obtain the global phase portraits 1.13–1.15 of Figure 1.

On the other hand, if the saddles $q_{4,6,8,9}$ are on the boundary of the period annulus of the centers $q_{1,2,10,11}$, respectively, creating four center-loops. Then the saddles $q_{5,7}$ must be on the boundary of the period annulus of the center q_3, creating a heteroclinic loop. The global phase portrait is topologically equivalent to 1.47 of Figure 2.

From the phase portrait 1.13, which can be realized when $a_{21} = -1$, $a_{03} = 2/3$, $b_{03} = -1$ and $a_{02} = -3.43$, to the phase portrait 1.47, which can be realized when $a_{21} = -1$, $a_{03} = 2/3$, $b_{03} = -1$ and $a_{02} = -3.42$, it follows by the continuity of the phase portraits with respect to the parameters the existence of the phase portrait 1.48 of Figure 2, which can be realized when $a_{21} = -1$, $a_{03} = 1$, $b_{03} = -0.1$ and $a_{02} \approx -3.4217$. In fact the saddle q_4 is on the boundary of the period annulus of the center q_1, and the saddles $q_{4,8}$ are on the boundary of the period annulus of the center q_{10}, they create a center-heteroclinic loop, see Figure 12 (a).

If the saddles $q_{4,6,8,9}$ are on the boundary of the period annulus of the centers $q_{1,2,10,11}$, respectively, they create four center-loops. Then the saddles $q_{5,7}$ must be on the boundary of the period annulus of the center q_3, and they create a heterclinic loop. The global phase portrait is topologically equivalent to the phase portrait 1.49 of Figure 2, which can be realized when $a_{21} = -0.4$, $a_{03} = 0.61$, $b_{03} = -0.01$ and $a_{02} = -1$.

If the saddles $q_{4,6}$ are on the boundary of the period annulus of the centers $q_{1,2}$, respectively, they create two center-loops. The saddles $q_{5,7}$ can be on the boundary of the period annulus of the centers $q_{3,10,11}$, they create a homo-heteroclinic loop. Then one attracting and one repelling separatrices of the saddle q_8 connect with the ones of the saddle q_9. This global phase portrait is topologically equivalent to the phase portrait 1.51 of Figure 2, which can be realized when $a_{21} = -0.41$, $a_{03} = 0.61$, $b_{03} = -0.01$ and $a_{02} = -1$.

From the phase portraits 1.49 and 1.51 it follows by the continuity of the phase portraits with respect to the parameters the existence of the phase portrait 1.50 of Figure 2, which can be realized when $a_{21} \approx -0.406$, $a_{03} = 0.61$, $b_{03} = -0.01$ and $a_{02} = -1$. In fact the saddles $q_{5,7}$ are on the boundary of the period annulus of the center q_3, simultaneously, the saddles $q_{5,8}$ are on the boundary of the period annulus of the center q_{10}, and the saddles $q_{7,9}$ are on the boundary of the period annulus of the center q_{11}, they create a chain-heteroclinic loop, see Figure 12 (b).

(ii.2.2.2.2) We consider vector field (4) has no infinite equilibrium points, i.e. it satisfies one of the conditions XVI, XVII and XIX. Since $M_2 < 0$ and $a_{21} < 0$ we have $\widetilde{d}_1\widetilde{d}_2 > 0$ and $\widetilde{d}_8\widetilde{d}_9 < 0$, the number of the sign changes of this RSL is at least one. Then $-f(-y)$ has at most three negative real roots. By the symmetry the vector field (4) has at most six finite equilibrium points different from $q_{1,2,3}$. Similar to the subcase (ii.1.2.2.3), the total index of the known infinite and finite equilibrium points is 6, we have that the remaining finite equilibrium points are either two saddles, or four saddles and two centers.

When the remaining finite equilibrium points are two saddles $q_{4,5}$ the global phase portrait in the Poincaé disc is topologically equivalent to 1.24 of Figure 1.

When the remaining finite equilibrium points are four saddles $q_{4,5,6,7}$ and two centers $q_{8,9}$ the global phase portrait in the Poincaré disc is topologically equivalent to the phase portrait 1.52 of Figure 2.

Table 6. The conditions for the phase portraits of the vector field (4) with $a_{03}b_{03}M_2 \neq 0$.

Conditions		Phase portraits
$a_{21} > 0$	$-\frac{2\sqrt{6a_{21}^3}}{9} < b_{03} < \frac{2\sqrt{6a_{21}^3}}{9}$, $\mathrm{R}[M_2, 2] < a_{03} < \mathrm{R}[M_2, 3]$	1.19–1.21
	$b_{03} \geq \frac{2\sqrt{6a_{21}^3}}{9}$ or $b_{03} \leq -\frac{2\sqrt{6a_{21}^3}}{9}$, $a_{03} > \mathrm{R}[M_7, 1]$; $-\frac{2\sqrt{6a_{21}^3}}{9} < b_{03} < \frac{2\sqrt{6a_{21}^3}}{9}$, $\mathrm{R}[M_7, 1] < a_{03} < \mathrm{R}[M_2, 2]$ or $a_{03} > \mathrm{R}[M_2, 3]$; $\mathrm{R}[M_2, 1] < a_{03} < \mathrm{R}[M_7, 1]$	1.10–1.12, 1.43–1.45
	$a_{03} < \mathrm{R}[M_2, 1]$	1.39
$a_{21} = 0$	$a_{03} > 3b_{03}\sqrt[3]{2b_{03}}$	1.10–1.12, 1.43–1.45
	$a_{03} < 3b_{03}\sqrt[3]{2b_{03}}$	1.39
$a_{21} < 0$	$\mathrm{R}[M_2, 1] < a_{03}$	1.13–1.18, 1.47–1.51
	$a_{03} = -\frac{a_{02}^2 - 9b_{03}^2}{4a_{21}}$, $3a_{02}^2 + 8a_{21}^3 - 27b_{03}^2 = 0$	1.46
	$a_{03} < \mathrm{R}[M_2, 1]$	1.24,1.52

In summary we have the following result.

Theorem 3.5. *When $a_{03}b_{03}M_2 \neq 0$ the phase portraits of the continuous piecewise $\mathbb{Z}_2$-equivariant cubic Hamiltonian vector field* (4) *are topologically equivalent to one of the* 24 *phase portraits showed in Figures* 1 *and* 2*. The corresponding conditions realizing these phase portraits are given in Table 6.*

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[1] School of Statistics and Mathematics, Guangdong University of Finance and Economics, Guangzhou, 510320, PR China

[2] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: chenting0715@126.com (T.Chen),

E-mail address: jllibre@mat.uab.cat (J.Llibre).

*Corresponding author