NILPOTENT BI-CENTERS IN CONTINUOUS PIECEWISE $\mathbb{Z}_2$-EQUIVARIANT CUBIC POLYNOMIAL HAMILTONIAN VECTOR FIELD: CUSP-CUSP TYPE

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Abstract. In this paper we study the global dynamics for a class of continuous piecewise $\mathbb{Z}_2$-equivariant cubic Hamiltonian vector fields with nilpotent bi-centers at $(\pm 1, 0)$. We consider these polynomial vector fields with a challenging case where the bi-centers $(\pm 1, 0)$ come from the combination of two nilpotent cusps separated by $y = 0$. We call it a cusp-cusp type. We use the Poincaré compactification, the blow-up theory, the index theory and the theory of discriminant sequence for determining the number of distinct or negative real roots of a polynomial, to classify the global phase portraits of these vector fields in the Poincaré disc.

1. Introduction and statement of the main results

From the works of Poincaré [34] and Dulac [17] the *center-focus problem*, i.e. the problem of distinguishing between a focus and a center, has been one of the important problems in the qualitative theory of planar differential vector fields. To overcome this classical problem, many methods have been developed, such as Poincaré-Liapunov method, Melnikov function method, Poincaré compactification method, and so on, see [3, 4, 5, 7, 18, 23, 33]. Thus, for instance, in the articles [14, 15, 25, 38, 39, 40] these methods have been used for studying the center-focus problem of the quadratic and cubic polynomial differential vector fields.

In recent years in order to model better some natural phenomena many authors started to analyze the non-smooth vector fields see for instance [1, 2, 6, 19, 32]. In this paper we deal with the following family of piecewise smooth vector fields

$$(1)\quad (\dot{x}, \dot{y}) = \begin{Bmatrix} (-y,\ x) \\ (0,\ x) \\ (0,\ 0) \end{Bmatrix} + \big(F^{\pm}(x,y), G^{\pm}(x,y)\big) = \big(P^{\pm}(x,y), Q^{\pm}(x,y)\big) \quad \text{for} \quad \pm\,\Gamma(x,y) \geq 0,$$

where $\Gamma : \mathbb{R}^2 \to \mathbb{R}$ is a $\mathcal{C}^\infty$ function, $F^{\pm}(x,y)$ and $G^{\pm}(x,y)$ are real polynomials without constant and linear terms. In fact, the vector field of (1) has two different regions $\Gamma^+ = \{(x,y) \in \mathbb{R}^2 : \Gamma(x,y) > 0\}$ and $\Gamma^- = \{(x,y) \in \mathbb{R}^2 : \Gamma(x,y) < 0\}$ separated by the line $S = \Gamma^{-1}(0)$. We say that an equilibrium point q of the piecewise smooth vector field (1) is a *center* if there is a neighborhood U of this equilibrium point, such that $U \setminus \{q\}$ is filled with periodic orbits. When the origin of the piecewise smooth vector field (1) is a center, it is called a *linear type center*, a *nilpotent center*, or a *degenerate center* if in (1) we have $(-y, x)$, $(0, x)$, or $(0, 0)$, respectively. It is well known that the center-focus problem of a piecewise smooth vector field (1) becomes much more difficult than for smooth vector fields. The classical methods have been developed for studying the center-focus problem of a differential vector field (1) with the linear type, see [8, 16, 21, 22, 36]. But this problem for the piecewise smooth quadratic polynomial differential vector fields still remains open.

As far as we know the center-focus problem for a non-elementary equilibrium point of a polynomial vector field is much more challenging compared with the study for an elementary equilibrium point. In order to overcome this type problem the authors of [20, 30, 31, 35] developed some computationally efficient methods for planar smooth vector fields with a nilpotent

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