MELNIKOV FUNCTIONS OF ARBITRARY ORDER FOR PIECEWISE SMOOTH DIFFERENTIAL SYSTEMS IN $\mathbb{R}^n$ AND APPLICATIONS

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Abstract. In this paper we develop an arbitrary order Melnikov function to study limit cycles bifurcating from a periodic submanifold for autonomous piecewise smooth differential systems in $\mathbb{R}^n$ with two zones separated by a hyperplane. This result not only extends some of the known results on the Melnikov theory in dimension and order but also compensates for some defects of the averaging theory in studying the limit cycle bifurcation of autonomous systems from a periodic submanifold. To demonstrate the application of our theoretical result and its superiority for some systems to the existing averaging theory, we study the maximum number of limit cycles bifurcating from an n-dimensional periodic submanifold caused by non-smooth centers of the fold-fold type, providing an upper bound for any order piecewise polynomial perturbations of degree m. Concerning the planar case of the unperturbed system, a piecewise Hamiltonian system, we obtain a better upper bound for piecewise polynomial Hamiltonian perturbations up to order two. The realizability of these upper bounds is also discussed.

1. Introduction and statement of the main results

Qualitative theory of piecewise smooth (PWS) differential systems has become one of the most booming research objects of ordinary differential equations in recent years. With the help of such systems, we can better model and analyze discontinuous phenomena in nature, such as the switching of circuit systems, the impact of mechanical devices, the activity of neurons in the central nervous system, the vibration of oscillators with dry friction, see [3,12,27,46]. On the contrary, an in-depth understanding of these discontinuous phenomena also has inspired the investigation of PWS systems.

Consider the n-dimensional PWS system

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{f}^+(\mathbf{x};\varepsilon) & \text{if} \quad \mathbf{x} \in \Sigma^+, \\ \mathbf{f}^-(\mathbf{x};\varepsilon) & \text{if} \quad \mathbf{x} \in \Sigma^-, \end{cases} \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $n \geq 2$, $\varepsilon \in \mathbb{R}$ is a perturbation parameter, $\mathbf{f}^\pm : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ are analytic functions,

$$\Sigma^+ = \{\mathbf{x} \in \mathbb{R}^n : \pi\mathbf{x} > 0\}, \qquad \Sigma^- = \{\mathbf{x} \in \mathbb{R}^n : \pi\mathbf{x} < 0\}$$

are two zones separated by the hyperplane $\Sigma = \{\mathbf{x} \in \mathbb{R}^n : \pi\mathbf{x} = 0\}$, usually called *discontinuity boundary* or *switching boundary* [3]. We denote by $\pi : \mathbb{R}^n \to \mathbb{R}$ the projection onto the first coordinate and by $\pi^\perp : \mathbb{R}^n \to \mathbb{R}^{n-1}$ the projection onto the last $n-1$ ones. For system (1)

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we can define its solution with the Filippov convention, refer to [16] or [33]. In this case, all points in Σ are classified into

- *crossing set*: $\Sigma^c = \{\mathbf{x} \in \Sigma : \pi \mathbf{f}^+(\mathbf{x}; \varepsilon) \cdot \pi \mathbf{f}^-(\mathbf{x}; \varepsilon) > 0\}$, and
- *sliding set*: $\Sigma^s = \{\mathbf{x} \in \Sigma : \pi \mathbf{f}^+(\mathbf{x}; \varepsilon) \cdot \pi \mathbf{f}^-(\mathbf{x}; \varepsilon) \leq 0\}$.

A typical periodic orbit that exists only in PWS systems is the so-called *crossing periodic orbit*, i.e. a closed curve formed by concatenating the orbits of subsystems $\dot{\mathbf{x}} = \mathbf{f}^{\pm}(\mathbf{x}; \varepsilon)$ only at some points of Σ^c. An isolated crossing periodic orbit in the set of all crossing periodic orbits of the differential system is called *crossing limit cycle*. In this paper we are interested in the limit cycle bifurcation of system (1), when the unperturbed system has a periodic submanifold, i.e. a submanifold fulfilled by a continuum of crossing periodic orbits.

One of the tools to study this problem is the averaging theory. The classical averaging theory is effective only for smooth systems [10, 22, 47]. With the Brouwer degree theory, the authors of [6] and [42, 43] extended the first three order averaging theory and the arbitrary order one respectively, only requiring that the considered systems are continuous. At present, stimulated by both the development of theory and practical applications, the averaging theory for PWS systems has been studied intensively, see [39] for high-dimensional systems and the first two orders, [29, 40, 50] for one-dimensional systems and any order, [41] for high-dimensional systems and any order. It is widely known that averaging theory is established originally for non-autonomous smooth or PWS systems, and thus it works for system (1) when we change system (1) to a non-autonomous system by a suitable transformation, as usual generalized polar coordinates. Then we can obtain information about the number of crossing limit cycles of system (1) bifurcating from an unperturbed periodic submanifold via studying the zeros of *averaged functions* associated with the non-autonomous system.

An alternative tool is the Melnikov theory, which can act on system (1) directly. Here the number of crossing limit cycles bifurcating an unperturbed periodic submanifold can be determined by the zeros of *Melnikov functions*. The research of Melnikov theory for smooth systems has a long history, see [13, 17, 21, 30, 31, 54] and the references therein. Whereas for PWS systems it is developing boomingly in the recent decade, and contributions mainly focus on deriving the first order Melnikov function. For instance, the authors of [14, 15] studied the perturbations of general planar system (1) with a periodic annulus; the works [37] and [35] dealt with the perturbations of planar PWS Hamiltonian case and integrable non-Hamiltonian one of system (1) with a periodic annulus respectively, providing a new expression of the first order Melnikov function; Xiong [51] generalized the work [37] by introducing an additional parameter in the considered PWS near-Hamiltonian system. All of the aforementioned references [14, 15, 35, 37, 51] aim to the planar case, while for the high-dimensional case, we quote [23] and [48] where the perturbations of general system (1) with a periodic submanifold containing in an invariant hyperplane and integrable system (1) with a general n-dimensional periodic submanifold were considered separately. Regarding the higher order Melnikov function, the result is much fewer. To our knowledge, only a formula of the second order Melnikov function for planar near-Hamiltonian case of system (1) was given in [38] and [52] independently.

In this paper our first goal is to develop an arbitrary order Melnikov function to study crossing limit cycles of arbitrarily dimensional system (1) bifurcating from a periodic submanifold fulfilled by a continuum of crossing periodic orbits. As far as we know, an expression of such function is still lacking for the general system (1) except for some special classes, such as piecewise polynomial perturbations of a linear center [8, 9].

We stress that it is quite necessary to obtain an arbitrary order Melnikov function for system (1), even if we have had an arbitrary order averaged function developed in [40,41] and the two methods are equivalent in most cases [5,25,38]. The main reason is that the averaging method has the following two deficiencies in studying the limit cycle bifurcation of system (1) from a periodic submanifold. Firstly, although the averaged method can be applied in theory with a suitable transformation, it is very difficult to find such a transformation for many systems, in particular, the ones with a continuum of periodic orbits that are not due to linear centers. Secondly, if we use the usual change of generalized polar coordinates, it is possible for some PWS systems of the form (1) that the derived PWS non-autonomous systems are extremely complicated, which increases the difficulty in computing the expression and zeros of the averaged functions, even the subsystems of the derived systems are discontinuous in each of regions associated to $\Sigma^{\pm}$. For example, as we will see at the end of Section 3, if we transform the PWS system (10) with $n = 2$ into a non-autonomous one using the usual change of polar coordinates $x_1 = r\cos\theta, x_2 = r\sin\theta$, then both subsystems of the obtained system are discontinuous and thus the existing averaging method fails. However, applying the Melnikov theory developed in this paper to system (10) with $n = 2$, we obtain that the Melnikov functions are polynomials divided by a monomial, for which its zeros can be studied easily by mature tools. To some extent, this shows the superiority of the Melnikov method to certain systems. Based on these reasons, we believe that it is quite necessary to provide one alternative bifurcation function when it is not easy to compute and analyze the averaged one. In general, there is no universal rule about how to choose between the two methods. This mainly depends on the studied problem itself and personal preference.

1.1. **Arbitrary order Melnikov function for PWS system (1).** To state our first main theorem on the arbitrary order Melnikov theory for system (1), we next set up our problem precisely and introduce some notations. Without loss of generality we can rewrite system (1) as

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{f}_0^+(\mathbf{x}) + \sum_{i=1}^{\infty} \varepsilon^i \mathbf{f}_i^+(\mathbf{x}) & \text{if} \quad \mathbf{x} \in \Sigma^+, \\ \mathbf{f}_0^-(\mathbf{x}) + \sum_{i=1}^{\infty} \varepsilon^i \mathbf{f}_i^-(\mathbf{x}) & \text{if} \quad \mathbf{x} \in \Sigma^-, \end{cases} \tag{2}$$

where $\mathbf{f}_i^{\pm}(\mathbf{x})$, $i = 0, 1, 2, \cdots$, are analytic. We make the following basic hypothesis for system (2) with $\varepsilon = 0$.

(H) There exists an open subset Ω of $\mathbb{R}^{n-1}$ such that for each $\mathbf{h} = (h_2, h_3, \cdots, h_n) \in \Omega$ the orbit of system (2) with $\varepsilon = 0$ starting at $(0, \mathbf{h})^\top \in \Sigma$ is a crossing periodic orbit, denoted by $\Gamma_{\mathbf{h}}$, which crosses Σ in a transversal way and only twice.

Hypothesis **(H)** states that the unperturbed system of (2) has a periodic submanifold $\mathcal{A}$ fulfilled by a continuum of crossing periodic orbits $\Gamma_{\mathbf{h}}$ for $\mathbf{h} \in \Omega$, namely $\mathcal{A} = \{\Gamma_{\mathbf{h}} : \mathbf{h} \in \Omega\}$. Without loss of generality, in this paper we always consider that each $\Gamma_{\mathbf{h}}$ crosses Σ at $(0, \mathbf{h})^\top$ from Σ^- to Σ^+, i.e. $\pi\mathbf{f}_0^{\pm}(0, \mathbf{h}) > 0$. In this case if we denote by $\mathbf{x}_0^{\pm}(t, 0, \mathbf{h})$ the solution of the unperturbed subsystem of $(2)_{\pm}$ starting at $(0, \mathbf{h})^\top \subset \Sigma$ with $\mathbf{h} \in \Omega$, then hypothesis **(H)** implies that there exists $t_0^+(\mathbf{h}) > 0$ and $t_0^-(\mathbf{h}) < 0$ such that $\mathbf{x}_0^{\pm}(t_0^{\pm}(\mathbf{h}), 0, \mathbf{h}) \in \Sigma$, $\mathbf{x}_0^+(t, 0, \mathbf{h}) \in \Sigma^+$ for $0 < t < t_0^+(\mathbf{h})$ and $\mathbf{x}_0^-(t, 0, \mathbf{h}) \in \Sigma^-$ for $t_0^-(\mathbf{h}) < t < 0$. Moreover, by the transversality we have

$$\pi\mathbf{f}_0^{\pm}(\mathbf{x}_0^{\pm}(t_0^{\pm}(\mathbf{h}), 0, \mathbf{h})) < 0. \tag{3}$$

Our goal is to establish some criteria for determining the persistence of crossing periodic orbits in $\mathcal{A}$, when we perturb it inside this class of all PWS systems of the form (2).

For sufficiently smooth function $\mathbf{g}:\mathbb{R}^n\to\mathbb{R}^n$, following notions of [42] we similarly define a symmetric L-multilinear map $\partial^L\mathbf{g}(\mathbf{x})/\partial\mathbf{x}^L$, which acts on a 'product' of l n-dimensional vectors, as

$$\frac{\partial^L\mathbf{g}(\mathbf{x})}{\partial\mathbf{x}^L}\bigodot_{j=1}^{l}\mathbf{u}_j=\sum_{i_1,\cdots,i_L=1}^{n}\frac{\partial^L\mathbf{g}(\mathbf{x})}{\partial x_{i_1}\cdots\partial x_{i_L}}u_{1i_1}\cdots u_{li_L}, \tag{4}$$

where L and l are positive integers, $\mathbf{x}=(x_1,x_2,\cdots,x_n)^\top\in\mathbb{R}^n$ and $\mathbf{u}_j=(u_{j1},u_{j2},\cdots,u_{jn})^\top\in\mathbb{R}^n$. Moreover, we denote $\mathbf{u}^b=\bigodot_{j=1}^b\mathbf{u}\in\mathbb{R}^{nb}$ for positive integer b and n-dimensional vector u. To clarify this notation we consider $n=2$ and $l=2$ as an example. In this case we have

$$\frac{\partial\mathbf{g}(\mathbf{x})}{\partial\mathbf{x}}\bigodot_{j=1}^{2}\mathbf{u}_j=\frac{\partial\mathbf{g}(\mathbf{x})}{\partial x_1}u_{11}u_{21}+\frac{\partial\mathbf{g}(\mathbf{x})}{\partial x_2}u_{12}u_{22}$$

for $L=1$ and

$$\frac{\partial^2\mathbf{g}(\mathbf{x})}{\partial\mathbf{x}^2}\bigodot_{j=1}^{2}\mathbf{u}_j=\frac{\partial^2\mathbf{g}(\mathbf{x})}{\partial x_1^2}u_{11}u_{21}+\frac{\partial^2\mathbf{g}(\mathbf{x})}{\partial x_1\partial x_2}u_{11}u_{22}+\frac{\partial^2\mathbf{g}(\mathbf{x})}{\partial x_2\partial x_1}u_{12}u_{21}+\frac{\partial^2\mathbf{g}(\mathbf{x})}{\partial x_2^2}u_{12}u_{22}$$

for $L=2$.

We define Melnikov functions $\mathcal{M}_k(\mathbf{h}):\Omega\to\mathbb{R}^{n-1}$ for $k=1,2,\cdots$ by

$$\mathcal{M}_k(\mathbf{h})=\pi^\perp\mathcal{M}_k^+(\mathbf{h})-\pi^\perp\mathcal{M}_k^-(\mathbf{h}), \tag{5}$$

where

$$\begin{aligned}\mathcal{M}_k^\pm(\mathbf{h})&=\mathbf{x}_k^\pm(t_0^\pm(\mathbf{h}),0,\mathbf{h})\\&\quad+k!\sum_{l=1}^{k}\frac{1}{(k-l)!}\sum_{S_l}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\frac{\partial^L\mathbf{x}_{k-l}^\pm(t_0^\pm(\mathbf{h}),0,\mathbf{h})}{\partial t^L}\bigodot_{j=1}^{l}t_j^\pm(\mathbf{h})^{b_j},\end{aligned} \tag{6}$$

$\mathbf{x}_k^\pm(t,0,\mathbf{h}):\mathbb{R}\times\{0\}\times\Omega\to\mathbb{R}^n$ and $t_k^\pm(\mathbf{h}):\Omega\to\mathbb{R}$ for $k=1,2,\cdots$ are defined recurrently as

$$\begin{aligned}\mathbf{x}_1^\pm(t,0,\mathbf{h})&=A_0^\pm(t,0,\mathbf{h})\int_0^t A_0^\pm(s,0,\mathbf{h})^{-1}\mathbf{f}_1^\pm(\mathbf{x}_0^\pm(s,0,\mathbf{h}))ds,\\ \mathbf{x}_k^\pm(t,0,\mathbf{h})&=k!A_0^\pm(t,0,\mathbf{h})\int_0^t A_0^\pm(s,0,\mathbf{h})^{-1}\Bigg(\mathbf{f}_k^\pm(\mathbf{x}_0^\pm(s,0,\mathbf{h}))\\&\quad+\sum_{l=1}^{k-1}\sum_{S_l}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\frac{\partial^L\mathbf{f}_{k-l}^\pm(\mathbf{x}_0^\pm(s,0,\mathbf{h}))}{\partial\mathbf{x}^L}\bigodot_{j=1}^{l}\mathbf{x}_j^\pm(s,0,\mathbf{h})^{b_j}\\&\quad+\sum_{S_k\setminus\sigma}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_k!k!^{b_k}}\frac{\partial^L\mathbf{f}_0^\pm(\mathbf{x}_0^\pm(s,0,\mathbf{h}))}{\partial\mathbf{x}^L}\bigodot_{j=1}^{k}\mathbf{x}_j^\pm(s,0,\mathbf{h})^{b_j}\Bigg)ds,\ \ k\geq 2,\end{aligned} \tag{7}$$

and

$$
\begin{aligned}
t_1^{\pm}(\mathbf{h}) &= -\frac{1}{\pi \mathbf{f}_0^{\pm}(\mathbf{x}_0^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h}))}\pi \mathbf{x}_1^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h}),\\
t_k^{\pm}(\mathbf{h}) &= -\frac{1}{\pi \mathbf{f}_0^{\pm}(\mathbf{x}_0^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h}))}\pi \Bigg(\mathbf{x}_k^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h})\\
&\qquad + k! \sum_{l=1}^{k-1} \frac{1}{(k-l)!} \sum_{S_l} \frac{1}{b_1! b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} \frac{\partial^L \mathbf{x}_{k-l}^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h})}{\partial t^L} \bigodot_{j=1}^{l} t_j^{\pm}(\mathbf{h})^{b_j}\\
&\qquad + k! \sum_{S_k \setminus \sigma} \frac{1}{b_1! b_2! 2!^{b_2} \cdots b_k! k!^{b_k}} \frac{\partial^L \mathbf{x}_0^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h})}{\partial t^L} \bigodot_{j=1}^{k} t_j^{\pm}(\mathbf{h})^{b_j} \Bigg), \qquad k \geq 2,
\end{aligned} \tag{8}
$$

$A_0^{\pm}(t,0,\mathbf{h})$ is a fundamental matrix solution of the variational equation of system $\dot{\mathbf{x}} = \mathbf{f}_0^{\pm}(\mathbf{x})$ along the solution $\mathbf{x}_0^{\pm}(t,0,\mathbf{h})$, S_l for $l = 1,2,\cdots,k$ is the set of all l-tuples of non-negative integers $(b_1, b_2, \cdots, b_{l-1}, b_l)$ satisfying $b_1 + 2b_2 + \cdots + lb_l = l$, $\sigma = \{(0,0,\cdots,0,1)\}$ and $L = b_1 + b_2 + \cdots + b_l$. Here $t_k^{\pm}(\mathbf{h})$ for $k = 1,2,\cdots$ are well defined due to (3).

Now we state the first main theorem.

Theorem 1. *For piecewise analytic differential system* (2) *satisfying hypothesis* **(H)** *the functions* $\mathcal{M}_k(\mathbf{h})$ *for* $k = 1,2,\cdots$, *defined in* (5), *are analytic in* $\mathbf{h} \in \Omega$. *Moreover, letting* $k_0 \geq 1$ *be the first positive integer such that* $\mathcal{M}_{k_0}(\mathbf{h}) \not\equiv \mathbf{0}$ *we have the following statements.*

- (i) *If* $\mathcal{M}_{k_0}(\mathbf{h}_*) \neq \mathbf{0}$ *for some* $\mathbf{h}_* \in \Omega$, *then there exist no crossing periodic orbits in a small neighborhood of the crossing periodic orbit* $\Gamma_{\mathbf{h}_*} \subset \mathcal{A}$ *for* $|\varepsilon| > 0$ *sufficiently small.*
- (ii) *If* $\mathbf{h}_* \in \Omega$ *is a simple zero of* $\mathcal{M}_{k_0}(\mathbf{h})$, *i.e.* $\mathcal{M}_{k_0}(\mathbf{h}_*) = \mathbf{0}$ *and the Jacobian matrix of* $\mathcal{M}_{k_0}(\mathbf{h})$ *at* $\mathbf{h} = \mathbf{h}_*$ *has no zero eigenvalues, then there exists a unique crossing periodic orbit in a small neighborhood of the crossing periodic orbit* $\Gamma_{\mathbf{h}_*} \subset \mathcal{A}$ *for* $|\varepsilon| > 0$ *sufficiently small.*
- (iii) *If* $\mathcal{M}_{k_0}(\mathbf{h})$ *has* q *simple zeros on* Ω, *then there exist* q *crossing periodic orbits bifurcating from* $\mathcal{A}$ *for* $|\varepsilon| > 0$ *sufficiently small.*
- (iv) *If* $\mathcal{M}_{k_0}(\mathbf{h})$ *has at most* q *zeros on* Ω, *taking into multiplicities account, then there exist at most* q *crossing periodic orbits bifurcating from* $\mathcal{A}$ *for* $|\varepsilon| > 0$ *sufficiently small.*

1.2. **Application to the perturbations of a PWS system. In smooth differential systems a classical perturbation problem is to determine the maximum number of limit cycles bifurcating from the periodic orbits of a polynomial Hamiltonian system with a center when it is perturbed inside the set of all polynomial differential systems with a given degree $m \geq 1$. For example, the perturbation of the harmonic oscillator $\dot{x}_1 = -x_2, \dot{x}_2 = x_1$ and quasi-homogeneous Hamiltonian systems are considered in [28] and [18] respectively, obtaining an upper bound via the Melnikov theory up to any order. This is essentially the weak Hilbert's 16th problem [1, 34].** In recent years, it has been extended to PWS systems, considering the piecewise polynomial perturbations of $\dot{x}_1 = -x_2, \dot{x}_2 = x_1$ in two zones separated by a straight line, e.g. [8, 9, 36, 44], or considering the piecewise polynomial perturbations of the $(d+2)$-dimensional reversible system $\dot{x}_1 = -x_2, \dot{x}_2 = x_1, \dot{\mathbf{y}} = \mathbf{0}$ in two zones separated by a hyperplane, e.g. [45, 48], where $\mathbf{y} \in \mathbb{R}^d$ with $d \geq 1$. **It is worth mentioning that the**

method used in the above references to compute the higher order Melnikov function is the Françoise algorithm [17] and its generalization [19, 21]. However, in this paper we use the Faá di Bruno's formula [32, 42] to compute the higher order Melnikov function as we will see later. Clearly, the number of zeros of Melnikov functions is totally determined by the considered system and it is independent of the methods we use to compute Melnikov functions. Thus, no matter which method we use, we should get the same result on the number of zeros of Melnikov functions. However, how to choose these two methods for specific perturbation problems and identify their possible link is still unknown.

Motivated by these works and as an application to Theorem 1, the second goal of this paper is to bring the weak Hilbert's 16th problem to piecewise polynomial perturbations of the following n-dimensional PWS system,

$$\dot{\mathbf{x}} = \begin{cases} (2x_2, -1, 0, \cdots, 0)^\top & \text{if} \quad x_1 > 0, \\ (2x_2, \ 1\ , 0, \cdots, 0)^\top & \text{if} \quad x_1 < 0, \end{cases} \tag{9}$$

where $\mathbf{x} = (x_1, x_2, x_3, \cdots, x_n)^\top \in \mathbb{R}^n$ for $n \geq 2$. In the whole paper $n = 2$ means that system (9) is a planar one with the variables x_1 and x_2.

System (9) with $n = 2$ originally comes from the work [7], and it was obtained as a normal form of the planar PWS system with a non-smooth center of the fold-fold type, that is, the center is a fold-fold singularity, see [7, 49]. Here a point in the switching boundary is said to be a *non-smooth center* if all orbits in a small deleted neighborhood of it are crossing periodic orbits, and a fold-fold singularity is a point where both smooth vector fields are quadratically tangent to the switching boundary, see [24, 33] for a detailed definition. For $n > 2$, we note that $(x_1, x_2) \in \mathbb{R}^2$ for any fixed $(x_3, \cdots, x_n) = (c_3, \cdots, c_n) \in \mathbb{R}^{n-2}$ is an invariant plane of system (9), and the dynamics on this invariant plane coincides with the one of system (9) with $n = 2$. This means that $(0, 0, c_3, \cdots, c_n)$ is a non-smooth center of system (9) restricted to the plane (x_1, x_2).

In particular we have the next proposition.

Proposition 2. *For each $\mathbf{h} = (h_2, h_3, \cdots, h_n) \in \mathbb{R}^+ \times \mathbb{R}^{n-2}$ the orbit of system* (9) *starting at $(0, \mathbf{h})^\top$ is a crossing periodic orbit, which intersects the switching boundary $x_1 = 0$ in a transversal way and only twice.*

Proposition 2 implies that system (9) satisfies hypothesis **(H)**, and thus we can apply the Melnikov functions developed in subsection 1.1 to study the limit cycle bifurcation for the piecewise polynomial perturbations of system (9),

$$\dot{\mathbf{x}} = \begin{cases} (2x_2, -1, 0, \cdots, 0)^\top + \displaystyle\sum_{i=1}^{\infty} \varepsilon^i \Big(p^+_{i,1}(\mathbf{x}), p^+_{i,2}(\mathbf{x}), p^+_{i,3}(\mathbf{x}), \cdots, p^+_{i,n}(\mathbf{x})\Big)^\top & \text{if } x_1 > 0, \\ (2x_2, \ 1\ , 0, \cdots, 0)^\top + \displaystyle\sum_{i=1}^{\infty} \varepsilon^i \Big(p^-_{i,1}(\mathbf{x}), p^-_{i,2}(\mathbf{x}), p^-_{i,3}(\mathbf{x}), \cdots, p^-_{i,n}(\mathbf{x})\Big)^\top & \text{if } x_1 < 0, \end{cases} \tag{10}$$

where $p^{\pm}_{i,j} : \mathbb{R}^n \to \mathbb{R}$ for $j = 1, 2, 3, \cdots, n$ are polynomials of degree $m \geq 1$.

Theorem 3. *For $|\varepsilon| > 0$ sufficiently small the maximum number of crossing limit cycles of system* (10) *bifurcating from the unperturbed crossing periodic orbits is at most m^{n-1} (resp. $k^{n-1}(2m+1)^{n-1}$) by using the Melnikov method of order one (resp. $k \geq 2$). Moreover, this upper bound can be reached for order one.*

We must mention that the non-polynomial perturbations of system (9) with $n = 2$ and $n = 3$ also was studied in [7] and [11] respectively. In those papers it was showed that any finitely or infinitely many crossing limit cycles can bifurcate by some non-polynomial piecewise $\mathcal{C}^\infty$-perturbations.

On the other hand, since system (9) with $n = 2$ is a piecewise Hamiltonian system with Hamiltonian functions $H_0^+(x_1, x_2) = -x_1 - x_2^2$ and $H_0^-(x_1, x_2) = x_1 - x_2^2$, we are also interested in the following piecewise polynomial Hamiltonian perturbations of system (9) with $n = 2$,

$$(11)\quad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} 2x_2 \\ -1 \end{pmatrix} + \varepsilon \begin{pmatrix} -H^+_{1,x_2}(x_1, x_2) \\ H^+_{1,x_1}(x_1, x_2) \end{pmatrix} + \varepsilon^2 \begin{pmatrix} -H^+_{2,x_2}(x_1, x_2) \\ H^+_{2,x_1}(x_1, x_2) \end{pmatrix} & \text{if } x_1 > 0, \\ \begin{pmatrix} 2x_2 \\ 1 \end{pmatrix} + \varepsilon \begin{pmatrix} -H^-_{1,x_2}(x_1, x_2) \\ H^-_{1,x_1}(x_1, x_2) \end{pmatrix} + \varepsilon^2 \begin{pmatrix} -H^-_{2,x_2}(x_1, x_2) \\ H^-_{2,x_1}(x_1, x_2) \end{pmatrix} & \text{if } x_1 < 0, \end{cases}$$

where $H_1^\pm, H_2^\pm : \mathbb{R}^2 \to \mathbb{R}$ are polynomials of degree $m + 1 \geq 2$, the subscripts x_1 and x_2 denote derivative. So we have the next theorem.

Theorem 4. *For $|\varepsilon| > 0$ sufficiently small the maximum number of crossing limit cycles of system* (11) *bifurcating from the unperturbed crossing periodic orbits is at most* $[m/2]$ (*resp.* $m - 1$) *by using the first* (*resp. second*) *order Melnikov method, where* $[\cdot]$ *denotes the integer part function. Moreover, these upper bounds can be reached.*

It is worth mentioning that Yang, Han and Huang in [53] studied the Hopf bifurcation of planar piecewise polynomial Hamiltonian systems with two zones separated by a straight line. From Theorem 1.2 and Remark 1.1 of that paper, it follows that $m - 1$ small amplitude crossing limit cycles can bifurcate from the fold-fold singularity O of the unperturbed system of (11). However, we deal with crossing limit cycles bifurcating from the unperturbed crossing periodic orbits, and then our result allows that the bifurcated crossing limit cycles are of large amplitude by resorting to the Melnikov method developed in this paper.

The paper is organized as follows. In Section 2 we provide the proof of Theorem 1 after introducing a displacement function. Section 3 contains the proofs of Proposition 2 and Theorem 3. Section 4 is devoted to proving Theorem 4.

2. Proof of Theorem 1

We start to prove Theorem 1. For each $\mathbf{h}_* \in \Omega$ we construct a displacement function around the crossing periodic orbit $\Gamma_{\mathbf{h}_*}$ as follows. Let $\mathbf{x}^\pm(t, 0, \mathbf{h}; \varepsilon)$ be the solution of subsystem $(2)_\pm$ with the initial value $\mathbf{x}^\pm(0, 0, \mathbf{h}; \varepsilon) = (0, \mathbf{h})^\top \in \Sigma$. By hypothesis **(H)** and the analytic dependency on initial values and parameters, there exists a neighborhood $\Omega_* \subset \Omega$ of $\mathbf{h}_*$ and a constant $\varepsilon_0 > 0$ such that $\mathbf{x}^\pm(t, 0, \mathbf{h}; \varepsilon)$ satisfying $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and $\mathbf{h} \in \Omega_*$ evolves in $\Sigma^\pm$ respectively until it transversally reaches Σ after a travelling time $t^\pm(\mathbf{h}; \varepsilon)$, i.e.

$$(12)\qquad \pi \mathbf{x}^\pm(t^\pm(\mathbf{h}; \varepsilon), 0, \mathbf{h}; \varepsilon) \equiv 0.$$

Note that $\mathbf{x}^\pm(t, 0, \mathbf{h}; 0) = \mathbf{x}_0^\pm(t, 0, \mathbf{h})$ and $t^\pm(\mathbf{h}; 0) = t_0^\pm(\mathbf{h})$ in the settings and notations given below **(H)**. Therefore we can define a displacement function $\mathcal{D}(\mathbf{h}; \varepsilon) : \Omega_* \times (-\varepsilon_0, \varepsilon_0) \to \Sigma$ as

$$(13)\qquad \mathcal{D}(\mathbf{h}; \varepsilon) = \pi^\perp \mathbf{x}^+(t^+(\mathbf{h}; \varepsilon), 0, \mathbf{h}; \varepsilon) - \pi^\perp \mathbf{x}^-(t^-(\mathbf{h}; \varepsilon), 0, \mathbf{h}; \varepsilon).$$

Moreover we have the following lemma.

Lemma 5. *The displacement function $\mathcal{D}(\mathbf{h};\varepsilon):\Omega_*\times(-\varepsilon_0,\varepsilon_0)\to\Sigma$ defined in* (13) *can be written in power series of ε as*

$$\mathcal{D}(\mathbf{h};\varepsilon)=\sum_{k=1}^{\infty}\frac{\mathcal{M}_k(\mathbf{h})}{k!}\varepsilon^k,$$

where $\mathcal{M}_k(\mathbf{h})$ for $k=1,2,\cdots$ are defined in (5).

Lemma 5 will be proved later on. The fact that $\mathcal{M}_k(\mathbf{h})$ for $k=1,2,\cdots$ are analytic is obtained directly from their definitions (5). Clearly $\mathcal{D}(\mathbf{h}_\varepsilon;\varepsilon)=\mathbf{0}$ for some $\mathbf{h}_\varepsilon\in\Omega_*$ if and only if the solution of system (2) starting at $(0,\mathbf{h}_\varepsilon)^\top$ is a crossing periodic orbit. Thus we consider the system of equations $\mathcal{D}(\mathbf{h};\varepsilon)=\mathbf{0}$ in order to study the persistence of the crossing periodic orbit $\Gamma_{\mathbf{h}_*}$. In the assumption that $\mathcal{M}_k(\mathbf{h})\equiv\mathbf{0}$ for $k=1,2,\cdots,k_0-1$, the system of equations $\mathcal{D}(\mathbf{h};\varepsilon)=\mathbf{0}$ is equivalent to

$$\widetilde{\mathcal{D}}(\mathbf{h};\varepsilon)=\mathbf{0},$$

where

$$\widetilde{\mathcal{D}}(\mathbf{h};\varepsilon)=\frac{\mathcal{M}_{k_0}(\mathbf{h})}{k_0!}+\sum_{k=k_0+1}^{\infty}\frac{\mathcal{M}_k(\mathbf{h})}{k!}\varepsilon^{k-k_0}.$$

If $\mathcal{M}_{k_0}(\mathbf{h}_*)\neq\mathbf{0}$, then $\widetilde{\mathcal{D}}(\mathbf{h};\varepsilon)\neq\mathbf{0}$ for $\|\mathbf{h}-h_*\|$ and $|\varepsilon|$ sufficiently small. This means that there exist no crossing periodic orbits in a small neighborhood of $\Gamma_{\mathbf{h}_*}$ for $|\varepsilon|$ sufficiently small, i.e. statement (i) holds.

If $\mathcal{M}_{k_0}(\mathbf{h}_*)=\mathbf{0}$, then $\widetilde{\mathcal{D}}(\mathbf{h}_*;0)=\mathbf{0}$. Moreover, since we are assuming that the Jacobian matrix of $\mathcal{M}_{k_0}(\mathbf{h})$ at $\mathbf{h}=\mathbf{h}_*$ has no zero eigenvalues, a direct application of the Implicit Function Theorem yields that there exists a unique function $\mathbf{h}=\mathbf{h}(\varepsilon)$, defined in a small neighborhood of $0\in\mathbb{R}$, such that $\mathbf{h}(0)=\mathbf{h}_*$ and $\widetilde{\mathcal{D}}(\mathbf{h}(\varepsilon);\varepsilon)\equiv\mathbf{0}$. Hence, we obtain a unique crossing periodic orbit in a small neighborhood of $\Gamma_{\mathbf{h}_*}$ for $|\varepsilon|$ sufficiently small, i.e. statement (ii) holds.

If $\mathcal{M}_{k_0}(\mathbf{h})$ has q simple zeros on Ω, denoted by $\mathbf{h}_i$, $i=1,2,\cdots,q$, then for each $\mathbf{h}_i$ there exists $\varepsilon_i>0$ such that for $|\varepsilon|<\varepsilon_i$ there is a unique crossing periodic orbit in a small neighborhood of $\Gamma_{\mathbf{h}_i}$ by statement (ii). Choosing $\varepsilon_0:=\min\{\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_q\}$, we get q crossing periodic orbits bifurcating from the periodic submanifold $\mathcal{A}$ for $|\varepsilon|<\varepsilon_0$, i.e. statement (iii) holds.

Since $\mathcal{M}_k(\mathbf{h})$ for $k=k_0,k_0+1,\cdots$ are analytic and $\widetilde{\mathcal{D}}(\mathbf{h};0)=\mathcal{M}_{k_0}(\mathbf{h})/k_0!$, we verify that the function $\widetilde{\mathcal{D}}(\mathbf{h};\varepsilon)$ satisfies all the conditions of [26, Theorem 3.1] if $\mathcal{M}_{k_0}(\mathbf{h})$ has at most q zeros on Ω, taking into multiplicities account. Thus by [26, Theorem 3.1] we get statement (iv) and then complete the proof of Theorem 1. □

Now we shall prove Lemma 5.

Proof of Lemma 5. First we prove the following claim.

Claim: *We can write the solution $\mathbf{x}^\pm(t,0,\mathbf{h};\varepsilon)$ around $\varepsilon=0$ as*

$$\mathbf{x}^\pm(t,0,\mathbf{h};\varepsilon)=\mathbf{x}_0^\pm(t,0,\mathbf{h})+\sum_{k=1}^{\infty}\frac{\mathbf{x}_k^\pm(t,0,\mathbf{h})}{k!}\varepsilon^k, \tag{14}$$

where $\mathbf{x}_0^\pm(t,0,\mathbf{h})$ is the solution of subsystem $(2)_\pm$ with $\varepsilon=0$ starting at $(0,\mathbf{h})^\top$, and $\mathbf{x}_k^\pm(t,0,\mathbf{h})$ for $k=1,2,\cdots$ are defined in (7).

We prove this claim following the idea of the proof of Lemma 1 in [42, Appendix A] where a non-autonomous version was obtained. By the analytic dependency on parameters we can write the solution $\mathbf{x}^{\pm}(t,0,\mathbf{h};\varepsilon)$ in the form of (14). Thus next all we need to do is to derive the expression of $\mathbf{x}_k^{\pm}(t,0,\mathbf{h})$ for $k=1,2,\cdots$.

Clearly $\mathbf{x}^{\pm}(t,0,\mathbf{h};\varepsilon)$ satisfies

$$\mathbf{x}^{\pm}(t,0,\mathbf{h};\varepsilon)=(0,\mathbf{h})^{\top}+\sum_{i=0}^{\infty}\varepsilon^i\int_0^t \mathbf{f}_i^{\pm}(\mathbf{x}^{\pm}(s,0,\mathbf{h};\varepsilon))ds. \tag{15}$$

Taking the k-th derivative of (15) with respect to ε for $k=1,2,\cdots$, we get

$$\mathbf{x}_k^{\pm}(t,0,\mathbf{h})=\left.\frac{\partial^k\mathbf{x}^{\pm}(t,0,\mathbf{h};\varepsilon)}{\partial\varepsilon^k}\right|_{\varepsilon=0}=\sum_{l=0}^{k}\frac{k!}{l!}\int_0^t\left.\frac{\partial^l\mathbf{f}_{k-l}^{\pm}(\mathbf{x}^{\pm}(s,0,\mathbf{h};\varepsilon))}{\partial\varepsilon^l}\right|_{\varepsilon=0}ds. \tag{16}$$

Then applying the Faá di Bruno's formula (see [32,42] or the Appendix) to compute the l-th derivative of $\mathbf{f}_{k-l}^{\pm}(\mathbf{x}^{\pm}(s,0,\mathbf{h};\varepsilon))$ with respect to ε for $l=1,2,\cdots$, we get

$$\begin{aligned}&\left.\frac{\partial^l\mathbf{f}_{k-l}^{\pm}(\mathbf{x}^{\pm}(s,0,\mathbf{h};\varepsilon))}{\partial\varepsilon^l}\right|_{\varepsilon=0}\\ &=\sum_{S_l}\frac{l!}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\left.\frac{\partial^L\mathbf{f}_{k-l}^{\pm}(\mathbf{x}^{\pm}(s,0,\mathbf{h};\varepsilon))}{\partial\mathbf{x}^L}\right|_{\varepsilon=0}\bigodot_{j=1}^{l}\left(\left.\frac{\partial^j\mathbf{x}^{\pm}(s,0,\mathbf{h};\varepsilon)}{\partial\varepsilon^j}\right|_{\varepsilon=0}\right)^{b_j}\\ &=\sum_{S_l}\frac{l!}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\frac{\partial^L\mathbf{f}_{k-l}^{\pm}(\mathbf{x}_0^{\pm}(s,0,\mathbf{h}))}{\partial\mathbf{x}^L}\bigodot_{j=1}^{l}\mathbf{x}_j^{\pm}(s,0,\mathbf{h})^{b_j},\end{aligned} \tag{17}$$

where S_l and L are defined below (8). Thus substituting (17) into (16) we have

$$\mathbf{x}_1^{\pm}(t,0,\mathbf{h})=\int_0^t\left(\frac{\partial\mathbf{f}_0^{\pm}(\mathbf{x}_0^{\pm}(s,0,\mathbf{h}))}{\partial\mathbf{x}}\mathbf{x}_1^{\pm}(s,0,\mathbf{h})+\mathbf{f}_1^{\pm}(\mathbf{x}_0^{\pm}(s,0,\mathbf{h}))\right)ds,$$

and

$$\begin{aligned}\mathbf{x}_k^{\pm}(t,0,\mathbf{h})&=k!\int_0^t\Bigg(\mathbf{f}_k^{\pm}(\mathbf{x}_0^{\pm}(s,0,\mathbf{h}))\\ &\qquad+\sum_{l=1}^{k}\sum_{S_l}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\frac{\partial^L\mathbf{f}_{k-l}^{\pm}(\mathbf{x}_0^{\pm}(s,0,\mathbf{h}))}{\partial\mathbf{x}^L}\bigodot_{j=1}^{l}\mathbf{x}_j^{\pm}(s,0,\mathbf{h})^{b_j}\Bigg)ds\\ &=\int_0^t\Bigg(\frac{\partial\mathbf{f}_0^{\pm}(\mathbf{x}_0^{\pm}(s,0,\mathbf{h}))}{\partial\mathbf{x}}\mathbf{x}_k^{\pm}(s,0,\mathbf{h})+k!\mathbf{f}_k^{\pm}(\mathbf{x}_0^{\pm}(s,0,\mathbf{h}))\\ &\qquad+k!\sum_{l=1}^{k-1}\sum_{S_l}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\frac{\partial^L\mathbf{f}_{k-l}^{\pm}(\mathbf{x}_0^{\pm}(s,0,\mathbf{h}))}{\partial\mathbf{x}^L}\bigodot_{j=1}^{l}\mathbf{x}_j^{\pm}(s,0,\mathbf{h})^{b_j}\\ &\qquad+k!\sum_{S_k\setminus\sigma}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_k!k!^{b_k}}\frac{\partial^L\mathbf{f}_0^{\pm}(\mathbf{x}_0^{\pm}(s,0,\mathbf{h}))}{\partial\mathbf{x}^L}\bigodot_{j=1}^{k}\mathbf{x}_j^{\pm}(s,0,\mathbf{h})^{b_j}\Bigg)ds\end{aligned}$$

for $k = 2, 3, \cdots$, where $\sigma = \{(0, 0, \cdots, 0, 1)\}$, also see below (8). This means that $\mathbf{x}_1^{\pm}(t, 0, \mathbf{h})$ and $\mathbf{x}_k^{\pm}(t, 0, \mathbf{h})$ for $k = 2, 3, \cdots$ obey the differential equations

$$\begin{aligned}
\frac{\partial \mathbf{x}_1^{\pm}(t,0,\mathbf{h})}{\partial t} &= \frac{\partial \mathbf{f}_0^{\pm}(\mathbf{x}_0^{\pm}(t,0,\mathbf{h}))}{\partial \mathbf{x}} \mathbf{x}_1^{\pm}(t,0,\mathbf{h}) + \mathbf{f}_1^{\pm}(\mathbf{x}_0^{\pm}(t,0,\mathbf{h})), \quad \text{and} \\
\frac{\partial \mathbf{x}_k^{\pm}(t,0,\mathbf{h})}{\partial t} &= \frac{\partial \mathbf{f}_0^{\pm}(\mathbf{x}_0^{\pm}(t,0,\mathbf{h}))}{\partial \mathbf{x}} \mathbf{x}_k^{\pm}(t,0,\mathbf{h}) + k!\Bigg(\mathbf{f}_k^{\pm}(\mathbf{x}_0^{\pm}(t,0,\mathbf{h})) \\
&\quad + \sum_{l=1}^{k-1} \sum_{S_l} \frac{1}{b_1! b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} \frac{\partial^L \mathbf{f}_{k-l}^{\pm}(\mathbf{x}_0^{\pm}(t,0,\mathbf{h}))}{\partial \mathbf{x}^L} \bigodot_{j=1}^{l} \mathbf{x}_j^{\pm}(t,0,\mathbf{h})^{b_j} \\
&\quad + \sum_{S_k \setminus \sigma} \frac{1}{b_1! b_2! 2!^{b_2} \cdots b_k! k!^{b_k}} \frac{\partial^L \mathbf{f}_0^{\pm}(\mathbf{x}_0^{\pm}(t,0,\mathbf{h}))}{\partial \mathbf{x}^L} \bigodot_{j=1}^{k} \mathbf{x}_j^{\pm}(t,0,\mathbf{h})^{b_j} \Bigg)
\end{aligned}$$

satisfying the initial value $\mathbf{x}_1^{\pm}(0, 0, \mathbf{h}) \equiv \mathbf{0}$ and $\mathbf{x}_k^{\pm}(0, 0, \mathbf{h}) \equiv \mathbf{0}$ respectively. Solving these linear differential equations we obtain (7) and thus this claim holds.

From (14) it follows that

$$\mathbf{x}^{\pm}(t^{\pm}(\mathbf{h};\varepsilon), 0, \mathbf{h}; \varepsilon) = \sum_{k=0}^{\infty} \frac{\mathbf{x}_k^{\pm}(t^{\pm}(\mathbf{h};\varepsilon), 0, \mathbf{h})}{k!} \varepsilon^k. \tag{18}$$

Taking the k-th derivative of (18) with respect to ε for $k = 1, 2, \cdots$, we get

$$\left. \frac{\partial^k \mathbf{x}^{\pm}(t^{\pm}(\mathbf{h};\varepsilon), 0, \mathbf{h}; \varepsilon)}{\partial \varepsilon^k} \right|_{\varepsilon=0} = k! \sum_{l=0}^{k} \frac{1}{l!(k-l)!} \left. \frac{\partial^l \mathbf{x}_{k-l}^{\pm}(t^{\pm}(\mathbf{h};\varepsilon), 0, \mathbf{h})}{\partial \varepsilon^l} \right|_{\varepsilon=0}. \tag{19}$$

Again, applying the Faá di Bruno's formula to compute the l-th derivative of $\mathbf{x}_{k-l}^{\pm}(t^{\pm}(\mathbf{h};\varepsilon), 0, \mathbf{h})$ with respect to ε for $l = 1, 2, \cdots$, we get

$$\begin{aligned}
&\left. \frac{\partial^l \mathbf{x}_{k-l}^{\pm}(t^{\pm}(\mathbf{h};\varepsilon), 0, \mathbf{h})}{\partial \varepsilon^l} \right|_{\varepsilon=0} \\
&= \sum_{S_l} \frac{l!}{b_1! b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} \left. \frac{\partial^L \mathbf{x}_{k-l}^{\pm}(t^{\pm}(\mathbf{h};\varepsilon), 0, \mathbf{h})}{\partial t^L} \right|_{\varepsilon=0} \bigodot_{j=1}^{l} \left(\left. \frac{\partial^j t^{\pm}(\mathbf{h};\varepsilon)}{\partial \varepsilon^j} \right|_{\varepsilon=0} \right)^{b_j} \\
&= \sum_{S_l} \frac{l!}{b_1! b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} \frac{\partial^L \mathbf{x}_{k-l}^{\pm}(t_0^{\pm}(\mathbf{h}), 0, \mathbf{h})}{\partial t^L} \bigodot_{j=1}^{l} \left(\frac{\partial^j t^{\pm}(\mathbf{h};0)}{\partial \varepsilon^j} \right)^{b_j}.
\end{aligned} \tag{20}$$

Thus substituting (20) into (19) we can write $\mathbf{x}^{\pm}(t^{\pm}(\mathbf{h};\varepsilon),0,\mathbf{h};\varepsilon)$ in the power series of ε as

$$\begin{aligned}
&\mathbf{x}^{\pm}(t^{\pm}(\mathbf{h};\varepsilon),0,\mathbf{h};\varepsilon)\\
&= \mathbf{x}_0^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h}) + \sum_{k=1}^{\infty}\frac{1}{k!}\Bigg(\mathbf{x}_k^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h})\\
&\quad + k!\sum_{l=1}^{k}\frac{1}{(k-l)!}\sum_{S_l}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\frac{\partial^L\mathbf{x}_{k-l}^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h})}{\partial t^L}\bigodot_{j=1}^{l}\left(\frac{\partial^j t^{\pm}(\mathbf{h};0)}{\partial\varepsilon^j}\right)^{b_j}\Bigg)\varepsilon^k\\
&= \mathbf{x}_0^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h}) + \left(\mathbf{x}_1^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h}) + \frac{\partial\mathbf{x}_0^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h})}{\partial t}\frac{\partial t^{\pm}(\mathbf{h};0)}{\partial\varepsilon}\right)\varepsilon\\
&\quad + \sum_{k=2}^{\infty}\frac{1}{k!}\Bigg(\mathbf{x}_k^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h}) + \frac{\partial\mathbf{x}_0^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h})}{\partial t}\frac{\partial^k t^{\pm}(\mathbf{h};0)}{\partial\varepsilon^k}\\
&\quad + k!\sum_{l=1}^{k-1}\frac{1}{(k-l)!}\sum_{S_l}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\frac{\partial^L\mathbf{x}_{k-l}^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h})}{\partial t^L}\bigodot_{j=1}^{l}\left(\frac{\partial^j t^{\pm}(\mathbf{h};0)}{\partial\varepsilon^j}\right)^{b_j}\\
&\quad + k!\sum_{S_k\setminus\sigma}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_k!k!^{b_k}}\frac{\partial^L\mathbf{x}_0^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h})}{\partial t^L}\bigodot_{j=1}^{k}\left(\frac{\partial^j t^{\pm}(\mathbf{h};0)}{\partial\varepsilon^j}\right)^{b_j}\Bigg)\varepsilon^k.
\end{aligned}\tag{21}$$

Since $t^{\pm}(\mathbf{h};\varepsilon)$ satisfies equation (12), it follows from (21) that

$$\pi\left(\mathbf{x}_1^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h}) + \frac{\partial\mathbf{x}_0^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h})}{\partial t}\frac{\partial t^{\pm}(\mathbf{h};0)}{\partial\varepsilon}\right)\equiv 0,$$

and

$$\begin{aligned}
&\pi\Bigg(\mathbf{x}_k^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h}) + \frac{\partial\mathbf{x}_0^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h})}{\partial t}\frac{\partial^k t^{\pm}(\mathbf{h};0)}{\partial\varepsilon^k}\\
&\quad + k!\sum_{l=1}^{k-1}\frac{1}{(k-l)!}\sum_{S_l}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\frac{\partial^L\mathbf{x}_{k-l}^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h})}{\partial t^L}\bigodot_{j=1}^{l}\left(\frac{\partial^j t^{\pm}(\mathbf{h};0)}{\partial\varepsilon^j}\right)^{b_j}\\
&\quad + k!\sum_{S_k\setminus\sigma}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_k!k!^{b_k}}\frac{\partial^L\mathbf{x}_0^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h})}{\partial t^L}\bigodot_{j=1}^{k}\left(\frac{\partial^j t^{\pm}(\mathbf{h};0)}{\partial\varepsilon^j}\right)^{b_j}\Bigg)\equiv 0,
\end{aligned}$$

from which we easily obtain

$$\frac{\partial t^{\pm}(\mathbf{h};0)}{\partial\varepsilon} = t_1^{\pm}(\mathbf{h}),\qquad \frac{\partial^k t^{\pm}(\mathbf{h};0)}{\partial\varepsilon^k} = t_k^{\pm}(\mathbf{h}),\quad k\geq 2,$$

because $\partial\mathbf{x}_0^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h})/\partial t = \mathbf{f}_0^{\pm}(\mathbf{x}_0^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h}))$, where $t_1^{\pm}(\mathbf{h})$ and $t_k^{\pm}(\mathbf{h})$ are defined recurrently in (8). Then from (6) and the first equality of (21) we have

$$\mathbf{x}^{\pm}(t^{\pm}(\mathbf{h};\varepsilon),0,\mathbf{h};\varepsilon) = \mathbf{x}_0^{\pm}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h}) + \sum_{k=1}^{\infty}\frac{\mathcal{M}_k^{\pm}(\mathbf{h})}{k!}\varepsilon^k.$$

This, together with (5) and (13), means that the displacement function $\mathcal{D}(h;\varepsilon)$ can be written as

$$\mathcal{D}(\mathbf{h};\varepsilon) = \pi^{\perp}\mathbf{x}_0^{+}(t_0^{\pm}(\mathbf{h}),0,\mathbf{h}) - \pi^{\perp}\mathbf{x}_0^{-}(t_0^{-}(\mathbf{h}),0,\mathbf{h}) + \sum_{k=1}^{\infty}\frac{\mathcal{M}_k(\mathbf{h})}{k!}\varepsilon^k = \sum_{k=1}^{\infty}\frac{\mathcal{M}_k(\mathbf{h})}{k!}\varepsilon^k,$$

where the second equality is due that each orbit of system (2) with $\varepsilon = 0$ starting at $(0, \mathbf{h}) \in \Omega_* \subset \Omega$ is a crossing periodic orbit. This ends the proof of Lemma 5. □

3. Proofs of Proposition 2 and Theorem 3

The purpose of this section is to prove Proposition 2 and Theorem 3. We start with the proof of Proposition 2.

Proof of Proposition 2. Let $\mathbf{x}_0^{\pm}(t, \mathbf{z}) = (x_{0,1}^{\pm}(t, \mathbf{z}), x_{0,2}^{\pm}(t, \mathbf{z}), \cdots, x_{0,n}^{\pm}(t, \mathbf{z}))^{\top}$ be the solution of subsystem $(9)_{\pm}$ with the initial value $\mathbf{z} = (z_1, z_2, \cdots, z_n)^{\top} \in \mathbb{R}^n$. Then

$$x_{0,1}^{\pm}(t, \mathbf{z}) = \mp t^2 + 2z_2 t + z_1, \qquad x_{0,2}^{\pm}(t, \mathbf{z}) = \mp t + z_2,$$
$$x_{0,j}^{\pm}(t, \mathbf{z}) = z_j, \quad j = 3, \cdots, n.$$

Hence for $\mathbf{h} = (h_2, h_3, \cdots, h_n) \in \mathbb{R}^+ \times \mathbb{R}^{n-2}$ the forward orbit of subsystem $(9)_+$ starting at $(0, \mathbf{h})^{\top}$ evolves in the half plane $x_1 > 0$ for $0 < t < 2h_2$ and maps $(0, \mathbf{h})^{\top}$ to the point $(0, -h_2, h_3, \cdots, h_n)^{\top}$. Furthermore the forward orbit of subsystem $(9)_-$ starting at $(0, -h_2, h_3, \cdots, h_n)^{\top}$ evolves in the half plane $x_1 < 0$ for $0 < t < 2h_2$ and maps $(0, -h_2, h_3, \cdots, h_n)^{\top}$ to the point $(0, \mathbf{h})^{\top}$. In conclusion, for each $\mathbf{h} \in \mathbb{R}^+ \times \mathbb{R}^{n-2}$ the orbit of system (9) with the initial value $(0, \mathbf{h})^{\top}$ is a crossing periodic orbit which crosses the switching line $x_1 = 0$ only twice. Moreover, since $\dot{x}_1 = 2x_2 \neq 0$ for $x_2 \neq 0$, all the crossing periodic orbits of system (9) intersect $x_1 = 0$ in a transversal way. This concludes the proof of Proposition 2. □

From Proposition 2 we know that system (10) satisfies hypothesis **(H)**. Hence the Melnikov function of order k associated to system (10) are just the one $\mathcal{M}_k(\mathbf{h})$ defined in (5) by taking $\Omega = \mathbb{R}^+ \times \mathbb{R}^{n-2}$ and

$$\begin{aligned} &\mathbf{f}_0^{\pm}(\mathbf{x}) = (2x_2, \mp 1, 0, \cdots, 0)^{\top}, \\ &\mathbf{f}_i^{\pm}(\mathbf{x}) = (p_{i,1}^{\pm}(\mathbf{x}), p_{i,2}^{\pm}(\mathbf{x}), \cdots, p_{i,n}^{\pm}(\mathbf{x}))^{\top}, \quad i = 1, 2, \cdots. \end{aligned} \tag{22}$$

Moreover, from the proof of Proposition 2 we see that for $\mathbf{h} = (h_2, h_3, \cdots, h_n) \in \mathbb{R}^+ \times \mathbb{R}^{n-2}$,

$$\mathbf{x}_0^{\pm}(t, 0, \mathbf{h}) = (\mp t^2 + 2h_2 t, \mp t + h_2, h_3, \cdots, h_n)^{\top}, \qquad t_0^{\pm}(\mathbf{h}) = \pm 2h_2, \tag{23}$$

and

$$A_0^{\pm}(t, 0, \mathbf{h}) = \begin{pmatrix} 1 & 2t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I \end{pmatrix}, \qquad A_0^{\pm}(t, 0, \mathbf{h})^{-1} = \begin{pmatrix} 1 & -2t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I \end{pmatrix},$$

where $A^{\pm}(t, 0, \mathbf{h})$ is introduced below (8) and I is an $(n-2) \times (n-2)$ identity matrix.

Consequently, restricted to system (10), we can simplify $\mathcal{M}_k(\mathbf{h})$ defined in (5) to

$$\mathcal{M}_k(\mathbf{h}) = \left(\mathcal{M}_{k,2}^{+}(\mathbf{h}) - \mathcal{M}_{k,2}^{-}(\mathbf{h}), \mathcal{M}_{k,3}^{+}(\mathbf{h}) - \mathcal{M}_{k,3}^{-}(\mathbf{h}), \cdots, \mathcal{M}_{k,n}^{+}(\mathbf{h}) - \mathcal{M}_{k,n}^{-}(\mathbf{h})\right)^{\top} \tag{24}$$

for $\mathbf{h} \in \mathbb{R}^+ \times \mathbb{R}^{n-2}$ and $k = 1, 2, \cdots$, where

$$\begin{aligned} \mathcal{M}_{k,j}^{\pm}(\mathbf{h}) = {}& x_{k,j}^{\pm}(\pm 2h_2, 0, \mathbf{h}) \\ & + k! \sum_{l=1}^{k} \frac{1}{(k-l)!} \sum_{S_l} \frac{1}{b_1! b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} \frac{\partial^L x_{k-l,j}^{\pm}(\pm 2h_2, 0, \mathbf{h})}{\partial t^L} \bigodot_{r=1}^{l} t_r^{\pm}(\mathbf{h})^{b_r} \end{aligned} \tag{25}$$

for $j = 2, 3, \cdots, n$, $x^{\pm}_{k,j}(t, 0, \mathbf{h})$ is the j-th component of $\mathbf{x}^{\pm}_k(t, 0, \mathbf{h}) : \mathbb{R}\times\{0\}\times\mathbb{R}^+\times\mathbb{R}^{n-2} \to \mathbb{R}^n$ and they are defined recurrently as

$$
\begin{aligned}
&x^{\pm}_{1,1}(t, 0, \mathbf{h}) = \int_0^t p^{\pm}_{1,1}(\mathbf{x}^{\pm}_0(s, 0, \mathbf{h})) - 2sp^{\pm}_{1,2}(\mathbf{x}^{\pm}_0(s, 0, \mathbf{h}))ds + 2t\int_0^t p^{\pm}_{1,2}(\mathbf{x}^{\pm}_0(s, 0, \mathbf{h}))ds,\\
&x^{\pm}_{1,j}(t, 0, \mathbf{h}) = \int_0^t p^{\pm}_{1,j}(\mathbf{x}^{\pm}_0(s, 0, \mathbf{h}))ds, \qquad j \geq 2,\\
&x^{\pm}_{k,1}(t, 0, \mathbf{h}) = k!\int_0^t g^{\pm}_{k,1}(s, 0, \mathbf{h}) - 2sg^{\pm}_{k,2}(s, 0, \mathbf{h})ds + 2k!t\int_0^t g^{\pm}_{k,2}(s, 0, \mathbf{h})ds, \quad k \geq 2,\\
&x^{\pm}_{k,j}(t, 0, \mathbf{h}) = k!\int_0^t g^{\pm}_{k,j}(s, 0, \mathbf{h})ds, \qquad k \geq 2,\ j \geq 2,
\end{aligned}
\tag{26}
$$

with

$$
\begin{aligned}
g^{\pm}_{k,j}(s, 0, \mathbf{h}) = {} & p^{\pm}_{k,j}(\mathbf{x}^{\pm}_0(s, 0, \mathbf{h}))\\
& + \sum_{l=1}^{k-1}\sum_{S_l}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\frac{\partial^L p^{\pm}_{k-l,j}(\mathbf{x}^{\pm}_0(s, 0, \mathbf{h}))}{\partial \mathbf{x}^L}\bigodot_{r=1}^{l}\mathbf{x}^{\pm}_r(s, 0, \mathbf{h})^{b_r}
\end{aligned}
\tag{27}
$$

for $j = 1, 2\cdots, n$, and $t^{\pm}_k(\mathbf{h}) : \mathbb{R}^+\times\mathbb{R}^{n-2} \to \mathbb{R}$ are defined recurrently as

$$
\begin{aligned}
t^{\pm}_1(\mathbf{h}) = {} & \frac{x^{\pm}_{1,1}(\pm 2h_2, 0, \mathbf{h})}{2h_2},\\
t^{\pm}_k(\mathbf{h}) = {} & \frac{1}{2h_2}\Bigg(x^{\pm}_{k,1}(\pm 2h_2, 0, \mathbf{h})\\
& + k!\sum_{l=1}^{k-1}\frac{1}{(k-l)!}\sum_{S_l}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}}\frac{\partial^L x^{\pm}_{k-l,1}(\pm 2h_2, 0, \mathbf{h})}{\partial t^L}\bigodot_{r=1}^{l}t^{\pm}_r(\mathbf{h})^{b_r}\\
& + k!\sum_{S_k\setminus\sigma}\frac{1}{b_1!b_2!2!^{b_2}\cdots b_k!k!^{b_k}}\frac{\partial^L x^{\pm}_{0,1}(\pm 2h_2, 0, \mathbf{h})}{\partial t^L}\bigodot_{r=1}^{k}t^{\pm}_r(\mathbf{h})^{b_r}\Bigg), \qquad k \geq 2.
\end{aligned}
\tag{28}
$$

In the computation of $x^{\pm}_{k,j}(t, 0, \mathbf{h})$ for $k \geq 2, j \geq 2$ we used the facts that $L = b_1+b_2+\cdots+b_k \geq 2$ for $(b_1, b_2, \cdots, b_k) \in S_k \setminus \sigma$, and

$$
\frac{\partial^L f^{\pm}_{0,j}(\mathbf{x})}{\partial x_{i_1}\cdots\partial x_{i_L}} \equiv 0 \qquad \text{for } L \geq 2 \text{ and any } i_1, \cdots, i_L,
$$

since the vector fields $\mathbf{f}^{\pm}_0(\mathbf{x})$ associated to system (10) are linear, see (22).

To study the number of zeros of the k-th order Melnikov function $\mathcal{M}_k(\mathbf{h})$ in (24), we need the following two technical lemmas.

Lemma 6. *Consider all the functions given in* (26). *Then* $x^{\pm}_{k,1}(t, 0, \mathbf{h})$ (*resp.* $x^{\pm}_{k,j}(t, 0, \mathbf{h})$, $j = 2, 3, \cdots, n$) *for* $k = 1, 2, \cdots$ *are polynomials of degree* $2km + 2$ (*resp.* $2km + 1$) *in the variables* t *and* $\mathbf{h}$.

Proof. We prove this lemma by the induction method. Since $p^{\pm}_{1,j}(\mathbf{x})$ for $j = 1, 2, \cdots, n$ are polynomials of degree m, $p^{\pm}_{1,j}(\mathbf{x}^{\pm}_0(t, 0, \mathbf{h}))$ are polynomials of degree $2m$ in the variables t and $\mathbf{h}$ by (23). Then $x^{\pm}_{1,1}(t, 0, \mathbf{h})$ are polynomials of degree $2m+2$ and $x^{\pm}_{1,j}(t, 0, \mathbf{h})$ for $j = 2, \cdots, n$ are polynomials of degree $2m + 1$ in the variables t and $\mathbf{h}$, i.e. Lemma 6 holds for $k = 1$.

It remains to prove this lemma for $k = k_0 \geq 2$, provided that it holds for $k = 2, 3, ..., k_0 - 1$. In fact, from (23) again $p^{\pm}_{k_0,j}(\mathbf{x}^{\pm}_0(t, 0, \mathbf{h}))$ are polynomials of degree $2m$ in the variables t and $\mathbf{h}$. In addition, by the meaning of the symbol $\bigodot$ given in (4),

$$\frac{\partial^L p^{\pm}_{k_0-l,j}(\mathbf{x}^{\pm}_0(t,0,\mathbf{h}))}{\partial \mathbf{x}^L} \bigodot_{r=1}^{l} \mathbf{x}^{\pm}_r(t,0,\mathbf{h})^{b_r} \qquad \text{for} \quad l \leq k_0 - 1$$

are polynomials of degree $2(m-L) + \sum_{r=1}^{l}(2rm+2)b_r = 2lm + 2m$ in the variables t and $\mathbf{h}$, as we are assuming that this lemma holds for $k = 1, 2, 3, \cdots, k_0 - 1$. This means that $g^{\pm}_{k_0,j}(t, 0, \mathbf{h})$ in (27) are polynomials of degree $2(k_0 - 1)m + 2m = 2k_0 m$ in the variables t and $\mathbf{h}$. Therefore Lemma 6 for $k = k_0 \geq 2$ follows directly from the definitions of $x^{\pm}_{k_0,1}(t, 0, \mathbf{h})$ and $x^{\pm}_{k_0,j}(t, 0, \mathbf{h})$ for $j = 2, \cdots, n$ given in (26). This ends the proof of Lemma 6. □

Lemma 7. *Consider the functions $t^{\pm}_k(\mathbf{h})$ for $k = 1, 2, \cdots$ given in (28). Then we can write them into the form*

$$t^{\pm}_k(\mathbf{h}) = \frac{\tau^{\pm}_k(\mathbf{h})}{h_2^{k-1}}$$

with $\tau^{\pm}_k(\mathbf{h})$ polynomials of degree $k(2m+1)$.

Proof. We prove this lemma by the induction method. From Lemma 6 we know that $x^{\pm}_{1,1}(t, 0, \mathbf{h})$ are polynomials of degree $2m+2$ in the variables t and $\mathbf{h}$, so that $x^{\pm}_{1,1}(\pm 2h_2, 0, \mathbf{h})$ are polynomials of degree $2m + 2$ in the variables $\mathbf{h}$. Moreover, it follows from (26) that $x^{\pm}_{1,1}(0, 0, \mathbf{h}) \equiv 0$, which implies that $x^{\pm}_{1,1}(\pm 2h_2, 0, \mathbf{h})$ has a factor h_2. Using the definition of $t^{\pm}_1(\mathbf{h})$ we get $t^{\pm}_1(\mathbf{h}) = x^{\pm}_{1,1}(\pm 2h_2, 0, \mathbf{h})/(2h_2)$ is of degree $2m+1$. This provides Lemma 7 for $k = 1$ by taking $\tau^{\pm}_1(\mathbf{h}) = t^{\pm}_1(\mathbf{h})$.

Assuming that Lemma 7 holds for $k = 2, 3, \cdots, k_0 - 1$, we only need to prove it for $k = k_0 \geq 2$ in order to complete the proof of Lemma 7. In this case recalling (28) we obtain

$$t^{\pm}_{k_0}(\mathbf{h}) = \frac{1}{2h_2}\Big(x^{\pm}_{k_0,1}(\pm 2h_2, 0, \mathbf{h}) + k_0! u^{\pm}_{k_0}(\mathbf{h}) + k_0! v^{\pm}_{k_0}(\mathbf{h})\Big), \tag{29}$$

where

$$u^{\pm}_{k_0}(\mathbf{h}) = \sum_{l=1}^{k_0-1} \frac{1}{(k_0-l)!} \sum_{S_l} \frac{1}{b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}} \frac{\partial^L x^{\pm}_{k_0-l,1}(\pm 2h_2, 0, \mathbf{h})}{\partial t^L} \bigodot_{r=1}^{l} \left(\frac{\tau^{\pm}_r(\mathbf{h})}{h_2^{r-1}}\right)^{b_r},$$

$$v^{\pm}_{k_0}(\mathbf{h}) = \sum_{S_{k_0}\backslash\sigma} \frac{1}{b_1!b_2!2!^{b_2}\cdots b_{k_0}!k_0!^{b_{k_0}}} \frac{\partial^L x^{\pm}_{0,1}(\pm 2h_2, 0, \mathbf{h})}{\partial t^L} \bigodot_{r=1}^{k_0} \left(\frac{\tau^{\pm}_r(\mathbf{h})}{h_2^{r-1}}\right)^{b_r}.$$

By Lemma 6 $\partial^L x^{\pm}_{k_0-l,1}(\pm 2h_2, 0, \mathbf{h})/\partial t^L$ are polynomials of degree $2(k_0 - l)m + 2 - L$. Moreover we are assuming that $\tau^{\pm}_j(\mathbf{h})$ are polynomials of degree $j(2m+1)$ for $j = 1, 2, \cdots, k_0 - 1$. Thus, using $b_1 + 2b_2 + \cdots + lb_l = l$ and $b_1 + b_2 + \cdots + b_l = L$, we obtain that

$$h_2^{l-L} \frac{\partial^L x^{\pm}_{k_0-l,1}(\pm 2h_2, 0, \mathbf{h})}{\partial t^L} \bigodot_{r=1}^{l} \left(\frac{\tau^{\pm}_r(\mathbf{h})}{h_2^{r-1}}\right)^{b_r} = \frac{\partial^L x^{\pm}_{k_0-l,1}(\pm 2h_2, 0, \mathbf{h})}{\partial t^L} \bigodot_{r=1}^{l} \tau^{\pm}_r(\mathbf{h})^{b_r}$$

are polynomials of degree $2k_0m+2+l-L$. So, together with the definition of $u^{\pm}_{k_0}(\mathbf{h})$, we can write $u^{\pm}_{k_0}(\mathbf{h})$ into the form

$$u^{\pm}_{k_0}(\mathbf{h})=\frac{\mu^{\pm}_{k_0}(\mathbf{h})}{h_2^{k_0-2}} \tag{30}$$

with $\mu^{\pm}_{k_0}(\mathbf{h})$ polynomials of degree $k_0(2m+1)$.

On the other hand, we find $L=b_1+b_2+\cdots+b_{k_0-1}+b_{k_0}\geq 2$ for $(b_1,b_2,\cdots,b_{k_0})\in S_{k_0}\setminus\sigma$, i.e. $b_{k_0}=0$. Moreover from (23) we have $\partial^L x^{\pm}_{0,1}(\pm 2h_2,0,\mathbf{h})/\partial t^L=\mp 2$ (resp. 0) if $L=2$ (resp. >2). Thus, using again $b_1+2b_2+\cdots+k_0b_{k_0}=k_0$ and $b_1+b_2+\cdots+b_{k_0}=L$, we get that

$$h_2^{k_0-L}\frac{\partial^L x^{\pm}_{0,1}(\pm 2h_2,0,\mathbf{h})}{\partial t^L}\bigodot_{r=1}^{k_0}\left(\frac{\tau^{\pm}_r(\mathbf{h})}{h_2^{r-1}}\right)^{b_r}=\frac{\partial^L x^{\pm}_{0,1}(\pm 2h_2,0,\mathbf{h})}{\partial t^L}\bigodot_{r=1}^{k_0}\tau^{\pm}_r(\mathbf{h})^{b_r}$$

are polynomials of degree $k_0(2m+1)$ if $L=2$, while if $L>2$ they are identically zero. So, together with the definition of $v^{\pm}_{k_0}(\mathbf{h})$, we can write $v^{\pm}_{k_0}(\mathbf{h})$ into the form

$$v^{\pm}_{k_0}(\mathbf{h})=\frac{\nu^{\pm}_{k_0}(\mathbf{h})}{h_2^{k_0-2}} \tag{31}$$

with $\nu^{\pm}_{k_0}(\mathbf{h})$ polynomials of degree $k_0(2m+1)$.

Joining (29), (30), (31) and the fact that $x^{\pm}_{k_0,1}(\pm 2h_2,0,\mathbf{h})$ are polynomials of degree $2k_0m+2$, which is obtained in Lemma 6, we finally get Lemma 7 for $k=k_0\geq 2$, provided that it holds for $k=1,2,\cdots,k_0-1$. That is, the proof of Lemma 7 is finished. □

With Lemmas 6 and 7 we can obtain an upper bound for the maximum number of isolated zeros of the k-th order Melnikov function $\mathcal{M}_k(\mathbf{h})$ defined in (24) associated to system (10) as it is stated in the following proposition.

Proposition 8. *The k-th order Melnikov function $\mathcal{M}_k(\mathbf{h})$ defined in* (24) *associated to system* (10) *has at most $k^{n-1}(2m+1)^{n-1}$ isolated zeros in $\mathbb{R}^+\times\mathbb{R}^{n-2}$.*

Proof. Considering the second summand in (25), we can write it as $M^{\pm}_{k,j}(\mathbf{h})/h_2^{k-1}$ for $k=1,2\cdots$ and $j=2,\cdots,n$ with $M^{\pm}_{k,j}(\mathbf{h})$ polynomials of degree $k(2m+1)$, because

$$h_2^{l-L}\frac{\partial^L x^{\pm}_{k-l,j}(\pm 2h_2,0,\mathbf{h})}{\partial t^L}\bigodot_{r=1}^{l}t^{\pm}_j(\mathbf{h})^{b_r}=\frac{\partial^L x^{\pm}_{k-l,j}(\pm 2h_2,0,\mathbf{h})}{\partial t^L}\bigodot_{r=1}^{l}\tau^{\pm}_r(\mathbf{h})^{b_r}$$

are polynomials of degree $2(k-l)m+1-L+\sum_{r=1}^{l}r(2m+1)b_r=2km+1+l-L$ by Lemmas 6, 7 and the facts that $b_1+2b_2+\cdots+lb_l=l$ and $b_1+b_2+\cdots+b_l=L$. Let

$$\widetilde{\mathcal{M}}^{\pm}_{k,j}(\mathbf{h})=h_2^{k-1}x^{\pm}_{k,j}(\pm 2h_2,0,\mathbf{h})+M^{\pm}_{k,j}(\mathbf{h})$$

for $j=2,3,\cdots,n$. Then $\widetilde{\mathcal{M}}^{\pm}_{k,j}(\mathbf{h})$ are polynomials of degree $k(2m+1)$, since $x^{\pm}_{k,j}(\pm 2h_2,0,\mathbf{h})$ for $j=2,3,\cdots,n$ are polynomials of degree $2km+1$ by Lemma 6. Moreover, it follows from (25) that

$$\mathcal{M}^{\pm}_{k,j}(\mathbf{h})=\frac{\widetilde{\mathcal{M}}^{\pm}_{k,j}(\mathbf{h})}{h_2^{k-1}} \tag{32}$$

for $k=1,2\cdots$ and $j=2,\cdots,n$.

Finally combining (24) and (32) we obtain

$$\mathcal{M}_k(\mathbf{h}) = \frac{1}{h_2^{k-1}} \widetilde{\mathcal{M}}_k(\mathbf{h}), \tag{33}$$

where

$$\widetilde{\mathcal{M}}_k(\mathbf{h}) = \left(\widetilde{\mathcal{M}}_{k,2}^+(\mathbf{h}) - \widetilde{\mathcal{M}}_{k,2}^-(\mathbf{h}), \widetilde{\mathcal{M}}_{k,3}^+(\mathbf{h}) - \widetilde{\mathcal{M}}_{k,3}^-(\mathbf{h}), \cdots, \widetilde{\mathcal{M}}_{k,n}^+(\mathbf{h}) - \widetilde{\mathcal{M}}_{k,n}^-(\mathbf{h})\right)^\top$$

is a map consisting of $n-1$ polynomials of degree $k(2m+1)$. Obviously $\mathcal{M}_k(\mathbf{h})$ and $\widetilde{\mathcal{M}}_k(\mathbf{h})$ have the same zeros in $\mathbb{R}^+ \times \mathbb{R}^{n-2}$. In conclusion, $\mathcal{M}_k(\mathbf{h})$ has at most $k^{n-1}(2m+1)^{n-1}$ isolated zeros in $\mathbb{R}^+ \times \mathbb{R}^{n-2}$ by the Bézout Theorem [4], because each component of $\widetilde{\mathcal{M}}_k(\mathbf{h})$ is a polynomial of degree $k(2m+1)$. That is Proposition 8 follows. □

For the maximum number of isolated zeros of $\mathcal{M}_1(\mathbf{h})$ we have a more precise result.

Proposition 9. *The first order Melnikov function $\mathcal{M}_1(\mathbf{h})$ defined in* (24) *associated to system* (10) *has at most m^{n-1} isolated zeros in $\mathbb{R}^+ \times \mathbb{R}^{n-2}$. Moreover, there exists a system of the form* (10) *such that $\mathcal{M}_1(\mathbf{h})$ has exactly m^{n-1} simple zeros in $\mathbb{R}^+ \times \mathbb{R}^{n-2}$.*

Proof. Consider functions $x_{1,j}^{\pm}(t, 0, \mathbf{h})$ for $j = 1, \cdots, n$ and $\mathbf{h} = (h_2, h_3, \cdots, h_n) \in \mathbb{R}^+ \times \mathbb{R}^{n-2}$ given in (26), and take

$$p_{1,j}^{\pm}(\mathbf{x}) = \sum_{i_1+i_2+\cdots+i_n=0}^{m} a_{i_1 i_2 \cdots i_n, j}^{\pm} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}.$$

We have

$$\begin{aligned}
& x_{1,1}^{\pm}(\pm 2h_2, 0, \mathbf{h}) \\
&= \int_0^{\pm 2h_2} \Bigg(\sum_{i_1+i_2+\cdots+i_n=0}^{m} a_{i_1 i_2 \cdots i_n, 1}^{\pm} (\mp s^2 + 2h_2 s)^{i_1} (\mp s + h_2)^{i_2} h_3^{i_3} \cdots h_n^{i_n} \\
& \quad -2s \sum_{i_1+i_2+\cdots+i_n=0}^{m} a_{i_1 i_2 \cdots i_n, 2}^{\pm} (\mp s^2 + 2h_2 s)^{i_1} (\mp s + h_2)^{i_2} h_3^{i_3} \cdots h_n^{i_n} \Bigg) ds \\
& \quad \pm 4h_2 \int_0^{\pm 2h_2} \Bigg(\sum_{i_1+i_2+\cdots+i_n=0}^{m} a_{i_1 i_2 \cdots i_n, 2}^{\pm} (\mp s^2 + 2h_2 s)^{i_1} (\mp s + h_2)^{i_2} h_3^{i_3} \cdots h_n^{i_n} \Bigg) ds \\
&= \sum_{i_1+i_2+\cdots+i_n=0}^{m} (a_{i_1 i_2 \cdots i_n, 1}^{\pm} \pm 2h_2 a_{i_1 i_2 \cdots i_n, 2}^{\pm}) I_{i_1, i_2}^{\pm}(h_2) h_3^{i_3} \cdots h_n^{i_n} \\
& \quad \pm 2 \sum_{i_1+i_2+\cdots+i_n=0}^{m} a_{i_1 i_2 \cdots i_n, 2}^{\pm} I_{i_1, i_2+1}^{\pm}(h_2) h_3^{i_3} \cdots h_n^{i_n},
\end{aligned} \tag{34}$$

and

$$\begin{aligned}
& x_{1,j}^{\pm}(\pm 2h_2, 0, \mathbf{h}) \\
&= \int_0^{\pm 2h_2} \Bigg(\sum_{i_1+i_2+\cdots+i_n=0}^{m} a_{i_1 i_2 \cdots i_n, j}^{\pm} (\mp s^2 + 2h_2 s)^{i_1} (\mp s + h_2)^{i_2} h_3^{i_3} \cdots h_n^{i_n} \Bigg) ds \\
&= \sum_{i_1+i_2+\cdots+i_n=0}^{m} a_{i_1 i_2 \cdots i_n, j}^{\pm} I_{i_1, i_2}^{\pm}(h_2) h_3^{i_3} \cdots h_n^{i_n}
\end{aligned} \tag{35}$$

for $j = 2, \cdots, n$, where

$$I^{\pm}_{k,l}(h_2) = \int_0^{\pm 2h_2} (\mp s^2 + 2h_2 s)^k (\mp s + h_2)^l ds, \qquad k, l \geq 0.$$

We compute

$$\begin{aligned} I^{\pm}_{k,l}(h_2) &= \int_0^{\pm 2h_2} \sum_{p=0}^{k} \frac{k!}{p!(k-p)!} (\mp s^2)^p (2h_2 s)^{k-p} \sum_{q=0}^{l} \frac{l!}{q!(l-q)!} (\mp s)^q h_2^{l-q} ds \\ &= \int_0^{\pm 2h_2} \sum_{p=0}^{k} \frac{k!}{p!(k-p)!} (\mp 1)^p s^{2p} 2^{k-p} h_2^{k-p} s^{k-p} \sum_{q=0}^{l} \frac{l!}{q!(l-q)!} (\mp 1)^q s^q h_2^{l-q} ds \\ &= \omega^{\pm}_{k,l} h_2^{2k+l+1}, \end{aligned} \tag{36}$$

and

$$\omega^{\pm}_{k,l} = \sum_{p=0}^{k} \sum_{q=0}^{l} \frac{(\pm 1)^{k+1} (-1)^{p+q} 2^{2k+q+1} k! l!}{(k+p+q+1) p!(k-p)! q!(l-q)!}.$$

That is, $I^{\pm}_{k,l}(h_2)$ is a monomial of degree $2k + l + 1$.

On the other hand, we prove

$$I^{\pm}_{k,l}(h_2) \equiv 0 \text{ (resp. } \not\equiv 0) \qquad \text{for } l \geq 0 \text{ odd (resp. even)}. \tag{37}$$

In fact, due to $-s^2 + 2h_2 s > 0$ for $0 < s < 2h_2$ and $s^2 + 2h_2 s < 0$ for $-2h_2 < s < 0$, we have

$$I^{\pm}_{k,0}(h_2) = \int_0^{\pm 2h_2} (\mp s^2 + 2h_2 s)^k ds \not\equiv 0. \tag{38}$$

Moreover,

$$I^{\pm}_{k,1}(h_2) = \frac{1}{2(k+1)} \int_0^{\pm 2h_2} d(\mp s^2 + 2h_2 s)^{k+1} \equiv 0. \tag{39}$$

For $l \geq 2$, using the integration by parts method we get

$$\begin{aligned} I^{\pm}_{k,l}(h_2) &= \frac{1}{2(k+1)} \int_0^{\pm 2h_2} (\mp s + h_2)^{l-1} d(\mp s^2 + 2h_2 s)^{k+1} \\ &= \pm \frac{l-1}{2(k+1)} \int_0^{\pm 2h_2} (\mp s^2 + 2h_2 s)^{k+1} (\mp s + h_2)^{l-2} ds \\ &= \pm \frac{l-1}{2(k+1)} I^{\pm}_{k+1,l-2}(h_2). \end{aligned} \tag{40}$$

Hence (37) follows from (38), (39) and (40). This, together with (36), means that $\omega^{\pm}_{k,l} = 0$ for $l \geq 0$ odd and $\omega^{\pm}_{k,l} \neq 0$ for $l \geq 0$ even.

From (23), (25) and (28) it follows that

$$\begin{aligned} \mathcal{M}^{\pm}_{1,2}(\mathbf{h}) &= x^{\pm}_{1,2}(\pm 2h_2, 0, \mathbf{h}) \mp \frac{x^{\pm}_{1,1}(\pm 2h_2, 0, \mathbf{h})}{2h_2}, \quad \text{and} \\ \mathcal{M}^{\pm}_{1,j}(\mathbf{h}) &= x^{\pm}_{1,j}(\pm 2h_2, 0, \mathbf{h}), \qquad j = 3, \cdots, n. \end{aligned} \tag{41}$$

Therefore, joining (34), (35), (36) and (41), we get

$$
\begin{aligned}
&\mathcal{M}^+_{1,2}(\mathbf{h}) - \mathcal{M}^-_{1,2}(\mathbf{h})\\
=&\sum_{i_1+i_2+\cdots+i_n=0}^{m} a^+_{i_1 i_2\cdots i_n,2}\omega^+_{i_1,i_2} h_2^{2i_1+i_2+1} h_3^{i_3}\cdots h_n^{i_n}\\
&-\sum_{i_1+i_2+\cdots+i_n=0}^{m}\left(\frac{a^+_{i_1 i_2\cdots i_n,1}}{2} + h_2 a^+_{i_1 i_2\cdots i_n,2}\right)\omega^+_{i_1,i_2} h_2^{2i_1+i_2} h_3^{i_3}\cdots h_n^{i_n}\\
&-\sum_{i_1+i_2+\cdots+i_n=0}^{m} a^+_{i_1 i_2\cdots i_n,2}\omega^+_{i_1,i_2+1} h_2^{2i_1+i_2+1} h_3^{i_3}\cdots h_n^{i_n}\\
&-\sum_{i_1+i_2+\cdots+i_n=0}^{m} a^-_{i_1 i_2\cdots i_n,2}\omega^-_{i_1,i_2} h_2^{2i_1+i_2+1} h_3^{i_3}\cdots h_n^{i_n}\\
&-\sum_{i_1+i_2+\cdots+i_n=0}^{m}\left(\frac{a^-_{i_1 i_2\cdots i_n,1}}{2} - h_2 a^-_{i_1 i_2\cdots i_n,2}\right)\omega^-_{i_1,i_2} h_2^{2i_1+i_2} h_3^{i_3}\cdots h_n^{i_n}\\
&+\sum_{i_1+i_2+\cdots+i_n=0}^{m} a^-_{i_1 i_2\cdots i_n,2}\omega^-_{i_1,i_2+1} h_2^{2i_1+i_2+1} h_3^{i_3}\cdots h_n^{i_n}\\
=&-\sum_{i_1+i_2+\cdots+i_n=0}^{m}\left(\frac{a^+_{i_1 i_2\cdots i_n,1}\omega^+_{i_1,i_2}}{2} + \frac{a^-_{i_1 i_2\cdots i_n,1}\omega^-_{i_1,i_2}}{2}\right) h_2^{2i_1+i_2} h_3^{i_3}\cdots h_n^{i_n}\\
&+\sum_{i_1+i_2+\cdots+i_n=0}^{m}\left(a^-_{i_1 i_2\cdots i_n,2}\omega^-_{i_1,i_2+1} - a^+_{i_1 i_2\cdots i_n,2}\omega^+_{i_1,i_2+1}\right) h_2^{2i_1+i_2+1} h_3^{i_3}\cdots h_n^{i_n}
\end{aligned}
\tag{42}
$$

and

$$
\begin{aligned}
&\mathcal{M}^+_{1,j}(\mathbf{h}) - \mathcal{M}^-_{1,j}(\mathbf{h})\\
&= h_2\sum_{i_1+i_2+\cdots+i_n=0}^{m}\left(a^+_{i_1 i_2\cdots i_n,j}\omega^+_{i_1,i_2} - a^-_{i_1 i_2\cdots i_n,j}\omega^-_{i_1,i_2}\right) h_2^{2i_1+i_2} h_3^{i_3}\cdots h_n^{i_n}
\end{aligned}
\tag{43}
$$

for $j = 3,\cdots,n$.

Since $\omega^{\pm}_{k,l} = 0$ for $l \geq 0$ odd and $\omega^{\pm}_{k,l} \neq 0$ for $l \geq 0$ even as proved below (40), all the terms of $\mathcal{M}^+_{1,2}(\mathbf{h}) - \mathcal{M}^-_{1,2}(\mathbf{h})$ and $(\mathcal{M}^+_{1,j}(\mathbf{h}) - \mathcal{M}^-_{1,j}(\mathbf{h}))/h_2$ for $j = 3,\cdots,n$ that contain an odd power of h_2 disappear. Hence if we regard h_2^2 as a new variable, then $\mathcal{M}^+_{1,2}(\mathbf{h}) - \mathcal{M}^-_{1,2}(\mathbf{h})$ and $(\mathcal{M}^+_{1,j}(\mathbf{h}) - \mathcal{M}^-_{1,j}(\mathbf{h}))/h_2$ for $j = 3,\cdots,n$ are polynomials of degree m in the variables $h_2^2, h_3,\cdots,h_n$, so that

$$
\left(\mathcal{M}^+_{1,2}(\mathbf{h}) - \mathcal{M}^+_{1,2}(\mathbf{h}), \frac{1}{h_2}\big(\mathcal{M}^+_{1,3}(\mathbf{h}) - \mathcal{M}^-_{1,3}(\mathbf{h})\big),\cdots,\frac{1}{h_2}\big(\mathcal{M}^+_{1,n}(\mathbf{h}) - \mathcal{M}^-_{1,n}(\mathbf{h})\big)\right)^{\top}
\tag{44}
$$

has at most m^{n-1} isolated zeros in $\mathbb{R}^+ \times \mathbb{R}^{n-2}$ by the Bézout Theorem. By the definition in (24), $\mathcal{M}_1(\mathbf{h})$ and (44) have the same zeros in $\mathbb{R}^+ \times \mathbb{R}^{n-2}$, so that $\mathcal{M}_1(\mathbf{h})$ also has at most m^{n-1} isolated zeros in $\mathbb{R}^+ \times \mathbb{R}^{n-2}$.

Regarding the reachability of this number, we choose $p^+_{1,2}(\mathbf{x}) = p^-_{1,1}(\mathbf{x}) = \cdots = p^-_{1,n}(\mathbf{x}) \equiv 0$ and

$$p^+_{1,1}(\mathbf{x}) = \sum_{i=0}^{m} a^+_{i,1} x_1^i, \qquad p^+_{1,j}(\mathbf{x}) = \sum_{i=0}^{m} a^+_{i,j} x_j^i$$

for $j = 3, \cdots, n$. Then from (42) and (43) it follows that

$$\mathcal{M}^+_{1,2}(\mathbf{h}) - \mathcal{M}^-_{1,2}(\mathbf{h}) = -\sum_{i=0}^{m} \frac{a^+_{i,1}\omega^+_{i,0}}{2} h_2^{2i}, \qquad \frac{1}{h_2}\big(\mathcal{M}^+_{1,j}(\mathbf{h}) - \mathcal{M}^-_{1,j}(\mathbf{h})\big) = \sum_{i=0}^{m} a^+_{i,j}\omega^+_{0,0} h_j^i$$

for $j = 3, \cdots, n$, which are complete polynomials in the variables h_2^2 and h_j respectively. Consequently we can obtain m^{n-1} simple zeros of $\mathcal{M}_1(\mathbf{h})$ in $\mathbb{R}^+ \times \mathbb{R}^{n-2}$ by a suitable choice for parameters $a^+_{i,1}$ and $a^+_{i,j}$. This ends the proof of Proposition 9. □

Having these preliminaries we now prove Theorem 3 as follows.

Proof of Theorem 3. By statement (iv) of Theorem 1, the upper bounds for order one and order $k \geq 2$ obtained in Theorem 3 are direct conclusions of Propositions 9 and 8 respectively. Regarding the realizability of the upper bound for order one, we can resort to Proposition 9 and statement (iii) of Theorem 1. □

We indicate that the averaging method is not a good choice for studying the limit cycles of system (10). In fact, considering system (10) with $n = 2$ as an example, by the usual change to polar coordinates $x_1 = r\cos\theta, x_2 = r\sin\theta$ we can transform it into the following non-autonomous piecewise form

$$\frac{dr}{d\theta} = \begin{cases} F_0^+(r,\theta) + F_1^+(r,\theta)\varepsilon + \mathcal{O}(\varepsilon^2) & \text{if} \quad \theta \in (0,\pi), \\ F_0^-(r,\theta) + F_1^-(r,\theta)\varepsilon + \mathcal{O}(\varepsilon^2) & \text{if} \quad \theta \in (\pi, 2\pi), \end{cases}$$

where

$$F_0^{\pm}(r,\theta) = \frac{2r^2\sin\theta\cos\theta \mp r\sin\theta}{\mp\cos\theta - 2r\sin^2\theta},$$
$$F_1^{\pm}(r,\theta) = \mp\frac{rp^{\pm}_{1,1}(r\cos\theta, r\sin\theta) \pm 2r^2\sin\theta p^{\pm}_{1,2}(r\cos\theta, r\sin\theta)}{(\cos\theta \pm 2r\sin^2\theta)^2}.$$

Note that $F_0^+(r,\theta)$ and $F_1^+(r,\theta)$ are discontinuous in $(r,\theta) \in \mathbb{R}^+ \times (0,\pi)$, and $F_0^-(r,\theta)$ and $F_1^-(r,\theta)$ are discontinuous in $(r,\theta) \in \mathbb{R}^+ \times (\pi, 2\pi)$. Thus, according to [40], the averaging method cannot be applied to the above non-autonomous piecewise system. On the other hand, our results show that Melnikov functions associated to system (10) are polynomials divided by a monomial, see (33), for which an upper bound for the maximum number of zeros can be obtained by the Bézout Theorem. Therefore, by contrast, the Melnikov method developed in this paper is more successful than the averaging method for system (10).

4. Proof of Theorem 4

In this section we will complete the proof of Theorem 4. First we observe the following important proposition that helps us to simplify system (11).

Proposition 10. *Consider piecewise polynomial Hamiltonian systems* (11) *and*

$$(45)\qquad \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{cases} \begin{pmatrix} 2x_2 \\ -1 \end{pmatrix} + \varepsilon \begin{pmatrix} -H^+_{1,x_2}(0,x_2) \\ 0 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} -H^+_{2,x_2}(0,x_2) \\ 0 \end{pmatrix} & \text{if } x_1 > 0, \\ \begin{pmatrix} 2x_2 \\ 1 \end{pmatrix} + \varepsilon \begin{pmatrix} -H^-_{1,x_2}(0,x_2) \\ 0 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} -H^-_{2,x_2}(0,x_2) \\ 0 \end{pmatrix} & \text{if } x_1 < 0. \end{cases}$$

For $|\varepsilon| > 0$ *sufficiently small the number of crossing limit cycles of the two systems bifurcating from the unperturbed crossing periodic orbits are the same.*

Proof. Assume that system (11) has a crossing limit cycle Γ_ε which tends to a crossing periodic orbit Γ_0 in the unperturbed periodic annulus as $\varepsilon \to 0$. Let $(0, p_1(\varepsilon))$ and $(0, p_2(\varepsilon))$ with $p_1(\varepsilon) > p_2(\varepsilon)$ be the two intersections between Γ_ε and the switching line $x_1 = 0$. Then $p_1(\varepsilon)$ tends to a value $\hat{p} > 0$ and $p_2(\varepsilon)$ tends to $-\hat{p}$ as $\varepsilon \to 0$, where the points $(0, \hat{p})$ and $(0, -\hat{p})$ are the intersections between Γ_0 and the switching line $x_1 = 0$.

We next prove that system (45) also admits a crossing limit cycle that intersects the switching line $x_1 = 0$ at $(0, p_1(\varepsilon))$ and $(0, p_2(\varepsilon))$ for $|\varepsilon| > 0$ enough small. In fact, we observe that the Hamiltonian functions of subsystems in (11) are

$$H^+(x_1, x_2) = -x_1 - x_2^2 + \varepsilon H_1^+(x_1, x_2) + \varepsilon^2 H_2^+(x_1, x_2),$$
$$H^-(x_1, x_2) = x_1 - x_2^2 + \varepsilon H_1^-(x_1, x_2) + \varepsilon^2 H_2^-(x_1, x_2),$$

respectively. Thus $(0, p_1(\varepsilon))$ and $(0, p_2(\varepsilon))$ obey the system of equations

$$H^+(0, p_1(\varepsilon)) = H^+(0, p_2(\varepsilon)), \qquad H^-(0, p_1(\varepsilon)) = H^-(0, p_2(\varepsilon)),$$

or equivalently, the system

$$(46)\qquad \begin{aligned} &-p_1(\varepsilon)^2 + \varepsilon H_1^+(0, p_1(\varepsilon)) + \varepsilon^2 H_2^+(0, p_1(\varepsilon)) = -p_2(\varepsilon)^2 + \varepsilon H_1^+(0, p_2(\varepsilon)) + \varepsilon^2 H_2^+(0, p_2(\varepsilon)), \\ &-p_1(\varepsilon)^2 + \varepsilon H_1^-(0, p_1(\varepsilon)) + \varepsilon^2 H_2^-(0, p_1(\varepsilon)) = -p_2(\varepsilon)^2 + \varepsilon H_1^-(0, p_2(\varepsilon)) + \varepsilon^2 H_2^-(0, p_2(\varepsilon)). \end{aligned}$$

On the other hand, the Hamiltonian functions of subsystems in (45) are given by

$$\widetilde{H}^+(x_1, x_2) = -x_1 - x_2^2 + \varepsilon H_1^+(0, x_2) + \varepsilon^2 H_2^+(0, x_2),$$
$$\widetilde{H}^-(x_1, x_2) = x_1 - x_2^2 + \varepsilon H_1^-(0, x_2) + \varepsilon^2 H_2^-(0, x_2),$$

respectively. Using (46) we obviously get

$$\widetilde{H}^+(0, p_1(\varepsilon)) = \widetilde{H}^+(0, p_2(\varepsilon)), \qquad \widetilde{H}^-(0, p_1(\varepsilon)) = \widetilde{H}^-(0, p_2(\varepsilon)).$$

This means that the points $(0, p_1(\varepsilon))$ and $(0, p_2(\varepsilon))$ lie in the same orbit for both subsystems of (45) and for $|\varepsilon| > 0$ enough small. Moreover, due to $p_1(\varepsilon) \to \hat{p}$ and $p_2(\varepsilon) \to -\hat{p}$ as $\varepsilon \to 0$, $(0, p_1(\varepsilon))$ and $(0, p_2(\varepsilon))$ are crossing points of system (45). Therefore for $|\varepsilon| > 0$ enough small system (45) admits a crossing periodic orbit that passes through $(0, p_1(\varepsilon))$ and $(0, p_2(\varepsilon))$ and that tends to Γ_0 as $\varepsilon \to 0$. In particular this crossing periodic orbit is a crossing limit cycle. Otherwise, by reversing the above analysis process, we can obtain that Γ_ε is no longer a crossing limit cycle, which contradicts the assumption at the beginning of this proof.

On the contrary, if we assume that system (45) has a crossing limit cycle that tends to a crossing periodic orbit Γ_0 in the unperturbed periodic annulus as $\varepsilon \to 0$, we can similarly prove that system (11) also has a crossing limit cycle that tends to Γ_0 as $\varepsilon \to 0$. So Proposition 10 is proved. ∎

Proposition 10 tells that we can equivalently consider the piecewise polynomial Hamiltonian system (45) to study the number of crossing limit cycles of the piecewise polynomial Hamiltonian system (11) bifurcating from the unperturbed crossing periodic orbits. Thus the rest of this section is devoted to computing the first two Melnikov functions associated to system (45) by using the formula (24) with (25)–(28), and then obtaining the number of zeros of the two Melnikov functions. With these results we will prove Theorem 4. In what follows we always take

$$H_1^{\pm}(0,x_2)=\sum_{i=0}^{m+1}a_i^{\pm}x_2^i,\qquad H_2^{\pm}(0,x_2)=\sum_{i=0}^{m+1}b_i^{\pm}x_2^i. \tag{47}$$

Proposition 11. *The first order Melnikov function associated to system* (45) *with* (47) *is*

$$\mathcal{M}_1(h)=\sum_{j=0}^{[m/2]}(a_{2j+1}^{+}-a_{2j+1}^{-})h^{2j} \tag{48}$$

for $h\in\mathbb{R}^+$*, and thus it has at most* $[m/2]$ *isolated zeros in* $\mathbb{R}^+$*, where* $[\cdot]$ *denotes the integer part function. Moreover, there exists a choice of parameters* $a_{2j+1}^{\pm}$ *such that it has exactly* $[m/2]$ *simple zeros in* $\mathbb{R}^+$*.*

Proof. Since system (45) is a particular case of system (10) with $n=2$ in which all terms vanish except for

$$p_{1,1}^{\pm}(x_1,x_2)=-H_{1,x_2}^{\pm}(0,x_2),\qquad p_{2,1}^{\pm}(x_1,x_2)=-H_{2,x_2}^{\pm}(0,x_2), \tag{49}$$

the first order Melnikov function associated to system (45) can be computed from (24) for $h=h_2\in\mathbb{R}^+$ with (25)–(28). In fact, from (23), (26) and (49) we compute

$$\begin{aligned}x_{1,1}^{\pm}(\pm2h,0,h)&=-\int_0^{\pm2h}H_{1,x_2}^{\pm}(0,\mp s+h)ds\\&=-\sum_{i=0}^{m+1}ia_i^{\pm}\int_0^{\pm2h}(\mp s+h)^{i-1}ds\\&=\pm\sum_{i=0}^{m+1}a_i^{\pm}((-1)^i-1)h^i\\&=\mp2\sum_{j=0}^{[m/2]}a_{2j+1}^{\pm}h^{2j+1},\end{aligned}$$

and $x_{1,2}^{\pm}(\pm2h,0,h)\equiv0$. Then

$$t_1^{\pm}(h)=\mp\sum_{j=0}^{[m/2]}a_{2j+1}^{\pm}h^{2j} \tag{50}$$

from (28). Therefore, recalling (24) and (25), the first order Melnikov function associated to system (45) is

$$\begin{aligned}\mathcal{M}_1(h) &= \frac{\partial x_{0,2}^{+}(2h,0,h)}{\partial t}t_1^{+}(h) - \frac{\partial x_{0,2}^{-}(2h,0,h)}{\partial t}t_1^{-}(h)\\ &= -t_1^{+}(h) - t_1^{-}(h)\\ &= \sum_{j=0}^{[m/2]}(a_{2j+1}^{+} - a_{2j+1}^{-})h^{2j},\end{aligned}$$

where the second equality is due to (23). Since $a_{2j+1}^{\pm}$ for $j = 0, 1, \cdots, [m/2]$ can be chosen arbitrarily, a straightforward application of the Descartes Theorem (see the Appendix) implies Proposition 11. □

Proposition 12. *The second order Melnikov function associated to system* (45) *with* (47) *is*

$$\mathcal{M}_2(h) = 2\sum_{j=0}^{[m/2]}(b_{2j+1}^{+} - b_{2j+1}^{-})h^{2j} - \sum_{j=0}^{[(m+1)/2]}2j(a_{2j}^{+} - a_{2j}^{-})h^{2j-2}\sum_{j=0}^{[m/2]}a_{2j+1}^{+}h^{2j} \tag{51}$$

for $h \in \mathbb{R}^+$, and thus it has at most $m-1$ isolated zeros in $\mathbb{R}^+$. Moreover, there exists a choice of parameters $a_{2j}^{\pm}, a_{2j+1}^{+}$ and $b_{2j+1}^{\pm}$ such that it has exactly $m-1$ simple zeros in $\mathbb{R}^+$.

Proof. Again, since system (45) is a particular case of system (10) with $n = 2$ in which all terms vanish except for the ones in (49), we can compute the second order Melnikov function for system (45) by (24) with (25)–(28) for $h = h_2 \in \mathbb{R}^+$. To do this, from (23), (26) and (27) it follows that $x_{1,2}^{\pm}(t,0,h) = x_{2,2}^{\pm}(t,0,h) \equiv 0$,

$$\frac{\partial x_{1,1}^{\pm}(\pm 2h,0,h)}{\partial t} = -H_{1,x_2}^{\pm}(0,-h) = -\sum_{i=0}^{m+1} i a_i^{\pm}(-h)^{i-1},$$

and

$$\begin{aligned}x_{2,1}^{\pm}(\pm 2h,0,h) &= -2\int_0^{\pm 2h} H_{2,x_2}^{\pm}(0,\mp s + h)ds\\ &= -2\sum_{i=0}^{m+1} i b_i^{\pm}\int_0^{\pm 2h}(\mp s + h)^{i-1}ds\\ &= \pm 2\sum_{i=0}^{m+1} b_i^{\pm}((-1)^i - 1)h^i\\ &= \mp 4\sum_{j=0}^{[m/2]} b_{2j+1}^{\pm}h^{2j+1},\end{aligned}$$

because $p_{1,1}^{\pm}(x_1,x_2)$ and $p_{2,1}^{\pm}(x_1,x_2)$ satisfy (49), and $p_{1,2}^{\pm}(x_1,x_2) = p_{2,2}^{\pm}(x_1,x_2) \equiv 0$ for system (45). Then, together with (23), (28) and (50),

$$\begin{aligned}t_2^{\pm}(h) &= \frac{1}{2h}\left(x_{2,1}^{\pm}(\pm 2h,0,h) + 2\frac{\partial x_{1,1}^{\pm}(\pm 2h,0,h)}{\partial t}t_1^{\pm}(h) + \frac{\partial^2 x_{0,1}^{\pm}(\pm 2h,0,h)}{\partial t^2}t_1^{\pm}(h)^2\right)\\ &= \frac{1}{2h}\left(\mp 4\sum_{j=0}^{[m/2]} b_{2j+1}^{\pm}h^{2j+1} \pm 2\sum_{i=0}^{m+1} i a_i^{\pm}(-h)^{i-1}\sum_{j=0}^{[m/2]} a_{2j+1}^{\pm}h^{2j} \mp 2\Big(\sum_{j=0}^{[m/2]} a_{2j+1}^{\pm}h^{2j}\Big)^2\right).\end{aligned}$$

Finally, according to (24) and (25), the second Melnikov function associated to system (45) is

$$\begin{aligned}
\mathcal{M}_2(h) &= \frac{\partial^2 x_{0,2}^+(2h,0,h)}{\partial t^2} t_1^+(h)^2 + \frac{\partial x_{0,2}^+(2h,0,h)}{\partial t} t_2^+(h) \\
&\quad - \frac{\partial^2 x_{0,2}^-(-2h,0,h)}{\partial t^2} t_1^-(h)^2 - \frac{\partial x_{0,2}^-(-2h,0,h)}{\partial t} t_2^-(h) \\
&= -t_2^+(h) - t_2^-(h) \\
&= -\frac{1}{2h}\left(-4\sum_{j=0}^{[m/2]} b_{2j+1}^+ h^{2j+1} + 2\sum_{i=0}^{m+1} i a_i^+ (-h)^{i-1} \sum_{j=0}^{[m/2]} a_{2j+1}^+ h^{2j} - 2\Big(\sum_{j=0}^{[m/2]} a_{2j+1}^+ h^{2j}\Big)^2\right) \\
&\quad - \frac{1}{2h}\left(4\sum_{j=0}^{[m/2]} b_{2j+1}^- h^{2j+1} - 2\sum_{i=0}^{m+1} i a_i^- (-h)^{i-1} \sum_{j=0}^{[m/2]} a_{2j+1}^- h^{2j} + 2\Big(\sum_{j=0}^{[m/2]} a_{2j+1}^- h^{2j}\Big)^2\right),
\end{aligned}$$

where we used that $x_{1,2}^\pm(t,0,h) = x_{2,2}^\pm(t,0,h) \equiv 0$ in the computation of the first equality, and the second equality is due to (23).

Note that $\mathcal{M}_2(h)$ makes sense only if $\mathcal{M}_1(h) \equiv 0$, which leads to $a_{2j+1}^+ = a_{2j+1}^-$ for $j = 0, 1, \cdots, [m/2]$ from (48). Hence we can reduce $\mathcal{M}_2(h)$ obtained above to (51). It is easy to observe that $\mathcal{M}_2(h)$ is a polynomial of degree $2m-2$ satisfying that all the odd terms vanish. By the Descartes Theorem $\mathcal{M}_2(h)$ has at most $m-1$ isolated zeros in $\mathbb{R}^+$.

Regarding the reachability, we take $b_{2j+1}^+ = b_{2j+1}^-$ for $j = 0, 1, \cdots, [m/2]$ and $a_{2j}^- = 0$ for $j = 0, 1, \cdots, [(m+1)/2]$. In this case we have $\mathcal{M}_2(h) = -\mathcal{M}_{21}(h)\mathcal{M}_{22}(h)$, where

$$\mathcal{M}_{21}(h) = \sum_{j=0}^{[(m+1)/2]} 2j a_{2j}^+ h^{2j-2}, \qquad \mathcal{M}_{22}(h) = \sum_{j=0}^{[m/2]} a_{2j+1}^+ h^{2j}.$$

If we regard h^2 as a new variable, then $\mathcal{M}_{21}(h)$ and $\mathcal{M}_{22}(h)$ are complete polynomials of degrees $[(m+1)/2]-1$ and $[m/2]$ in the variable h^2 respectively. Thus we can choose a_{2j}^+ and a_{2j+1}^+ in such a way that $\mathcal{M}_{21}(h)$ has $[(m+1)/2]-1$ simple zeros in $\mathbb{R}^+$, and $\mathcal{M}_{22}(h)$ has $[m/2]$ simple zeros in $\mathbb{R}^+$ which are different from the ones of $\mathcal{M}_{21}(h)$. This concludes that $\mathcal{M}_2(h)$ has $[(m+1)/2]-1+[m/2] = m-1$ simple zeros in $\mathbb{R}^+$ for a suitable choice of parameters a_{2j}^+ and a_{2j+1}^+. □

Having these preliminaries we now prove Theorem 4 as follows.

Proof of Theorem 4. By statements (iii) and (iv) of Theorem 1, the upper bounds and their realizability in Theorem 4 can be obtained directly from Propositions 10, 11 and 12. □

Appendix

In this appendix we recall the Faá di Bruno's formula [32,42] on the higher order derivative of a composite function and the Descartes Theorem [2,20].

Faá di Bruno's formula. *Let $g(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{f}(t) : \mathbb{R} \to \mathbb{R}^n$ be two functions with a sufficient number of derivatives. Then*

$$\frac{d^l g(\mathbf{f}(t))}{dt^l} = \sum_{s_l} \frac{l!}{b_1! b_2! \cdots b_l! l!^{b_l}} \frac{\partial^L g(\mathbf{f}(t))}{\partial \mathbf{x}^L} \bigodot_{j=1}^{l} \left(\frac{d^j \mathbf{f}(t)}{dt^j} \right)^{b_j},$$

where S_l is the set of all l-tuples of non-negative integers $(b_1, b_2, \cdots, b_l)$ satisfying $b_1 + 2b_2 + \cdots + lb_l = l$, and $L = b_1 + b_2 + \cdots + b_l$.

Descartes Theorem. *Let $q(x) = a_{i_1 1} x^{i_1} + a_{i_2} x^{i_2} + ... + a_{i_r} x^{i_r}$ be the real polynomial with $0 = i_1 < i_2 < ... < i_r$ with $r > 1$. If $a_{i_j} a_{i_{j+1}} < 0$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of sign. If the number of variations of signs is $r_0 \in \{0, 1, 2, ..., r-1\}$, then the polynomial $q(x)$ has at most r_0 positive real roots. Furthermore, we can choose the coefficients of the polynomial $q(x)$ in such a way that $q(x)$ has exactly $r-1$ simple positive real zeros.*

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References

[1] V.I. Arnold, Ten problems, *Adv. Soviet Math.* **1** (1990), 1–8.
[2] I.S. Berezin, N.P. Zhidkov, *Computing Methods*, Reading, Mass. London, 1965.
[3] M. di Bernardo, C.J. Budd, A.R. Champneys, P. Kowalczyk, *Piecewise-Smooth Dynamical systems: Theory and Applications*, Applied Mathematical Sciences, Vol.163 (Springer Verlag, London), 2008.
[4] É. Bézout, *Théorie générale des équations algébriques*, Ph. D. Pierres, **Paris**, 1779.
[5] A. Buică, On the equivalence of the Melnikov functions method and the averaging method, *Qual. Theory Dyn. Syst.* **16** (2017), 547–560.
[6] A. Buică, J. Llibre, Averaging methods for finding periodic orbits via Brouwer degree, *Bull. Sci. Math.* **128** (2004), 7–22.
[7] C.A. Buzzi, T. de Carvalho, M.A. Teixeira, Birth of limit cycles bifurcating from a nonsmooth center, *J. Math. Pures Appl.* **102** (2014), 36–47.
[8] C.A. Buzzi, M.F.S. Lima, J. Torregrosa, Limit cycles via higher order perturbations for some piecewise differential systems, *Physica D* **371** (2018), 28–47.
[9] C.A. Buzzi, C. Pessoa, J. Torregrosa, Piecewise linear perturbations of a linear center, *Discrete Contin. Dyn. Syst.* **9** (2013), 3915–3936.
[10] M.R. Cândido, J. Llibre, D.D. Novaes, Persistence of periodic solutions for higher order perturbed differential systems via Lyapunov-Schmidt reduction, *Nonlinearity* **30** (2017), 3560–3586.
[11] T. Carvalho, B.R. de Freitas, Birth of isolated nested cylinders and limit cycles in 3D piecewise smooth vector fields with symmetry, *Int. J. Bifur. Chaos* **30** (2020), 2050098.
[12] H. Chen, J. Llibre, Y. Tang, Global dynamics of a SD oscillator, *Nonlin. Dyn.* **91** (2018) 1755–1777.
[13] C. Christopher, C. Li, *Limit cycles of Differential Equations*, Adv. Courses Math. CRM Barcelona, Birkhäauser Verlag, Basel, 2007.
[14] Z. Du, Y. Li, Bifurcation of periodic orbits with multiple crossings in a class of planar Filippov systems, *Math. Comput. Modelling* **55** (2012), 1072–1082.
[15] Z. Du, Y. Li, W. Zhang, Bifurcation of periodic orbits in a class of planar Filippov systems, *Nonlin. Anal.* **69** (2008), 3610–3628.
[16] A.F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Kluwer Academic Publishers, Dordrecht, 1988.

[17] J.P. Françoise, Successive derivatives of a first return map, application to the study of quadratic vector fields, *Ergod. Theory Dyn. Syst.* **16** (1996), 87–96.

[18] J.P. Françoise, H. He, D. Xiao, The number of limit cycles bifurcating from the period annulus of quasi-homogeneous Hamiltonian systems at any order, *J. Differ. Equ.* **276 (2021), 318–341.**

[19] J. P. Françoise, M. Pelletier, Iterated integrals, Gelfand-Leray residue, and first return mapping, *J. Dyn. Control Syst.* **12 (2006), 357–369.**

[20] W. Fulton, *Algebraic Curves*, Mathematics Lecture Note Series, W.A. Benjamin, 1974.

[21] L. Gavrilov, Higher order Poincare-Pontryagin functions and iterated path integrals, *Ann. Fac. Sci. Toulouse* XIV (2005), 663–682.

[22] J. Giné, M. Grau, J. Llibre, Averaging theory at any order for computing periodic orbits, *Physica D* **250** (2013), 58–65.

[23] M.R.A. Gouveia, J. Llibre, D.D. Novaes, C. Pessoa, Piecewise smooth dynamical systems: Persistence of periodic solutions and normal forms, *J. Differ. Equ.* **260** (2016), 6108–6129.

[24] M. Guardia, T.M. Seara, M.A. Teixeira, Generic bifurcations of low codimension of planar Filippov systems, *J. Differ. Equ.* **250** (2011), 1967–2023.

[25] M. Han, V.G. Romanovski, X. Zhang, Equivalence of the Melnikov function method and the averaging method, *Qual. Theory Dyn. Syst.* **15** (2016), 471–479.

[26] M. Han, H. Sun, Z. Balanov, Upper estimates for the number of periodic solutions to multi-dimensional systems *J. Differ. Equ.* **266** (2019), 8281–8293.

[27] J. Harris, B. Ermentrout, Bifurcations in the Wilson-Cowan equations with nonsmooth firing rate, *SIAM J. Appl. Dyn. Syst.* **14** (2015), 43–72.

[28] I.D. Iliev, The number of limit cycles due to polynomial perturbations of the harmonic oscillator, *Math. Proc. Cambridge Philos. Soc.* **127** (1999), 317–322.

[29] J. Itikawa, J. Llibre, D.D. Novaes, A new result on averaging theory for a class of discontinuous planar differential systems with applications, *Rev. Mat. Iberoam.* **33** (2017), 1247–1265.

[30] A. Jebrane, P. Mardešić, M. **Pelletier**, A generalization of Françoise's algorithm for calculating higher order Melnikov functions, *Bull. Sci. Math.* **126** (2002), 705–732.

[31] A. Jebrane, H. Żołądek, A note on higher order Melnikov functions, *Qual. Theory Dyn. Syst.* **6** (2005), 273–287.

[32] W.P. Johnson, The curious history of Faá di Bruno's formula, *Am. Math. Mon.* **109** (2002), 217–234.

[33] Yu.A. Kuznetsov, S. Rinaldi, A. Gragnani, One parameter bifurcations in planar Filippov systems, *Int. J. Bifur. Chaos* **13**(2003), 2157–2188.

[34] J. Li, Hilbert's 16th problem and bifurcations of planar polynomial vector fields, *Int. J. Bifur. Chaos* **13** (2003), 47–106.

[35] S. Li, X. Cen, Y. Zhao, Bifurcation of limit cycles by perturbing piecewise smooth integrable non-Hamiltonian systems, *Nonlin. Anal.: Real World Appl.* **34** (2017), 140–148.

[36] T. Li, J. Llibre, On the 16-th Hilbert problem for discontinuous piecewise polynomial Hamiltonian systems, to appear, 2021.

[37] X. Liu, M. Han, Bifurcation of limit cycles by perturbing piecewise Hamiltonian systems, *Int. J. Bifur. Chaos* **20** (2010), 1379–1390.

[38] S. Liu, M. Han, J. Li, Bifurcation methods of periodic orbits for piecewise smooth systems, *J. Differ. Equ.* **275** (2021), 204–233.

[39] J. Llibre, A.C. Mereu, D.D. Novaes, Averaging theory for discontinuous piecewise differential systems, *J. Differ. Equ.* **258** (2015), 4007–4032.

[40] J. Llibre, D.D. Novaes, C.A.B. Rodrigues, Averaging theory at any order for computing limit cycles of discontinuous piecewise differential systems with many zones, *Physica D* **353-354** (2017), 1–10.

[41] J. Llibre, D.D. Novaes, C.A.B. Rodrigues, Bifurcations from families of periodic solutions in piecewise differential systems, *Physica D* **404** (2020), 132342.

[42] J. Llibre, D.D. Novaes, M.A. Teixeira, Higher order averaging theorem for finding periodic solutions via Brouwer degree, *Nonlinearity* **27** (2014), 563–583.

[43] J. Llibre, D.D. Novaes, M.A. Teixeira, Corrigendum: higher order averaging theory for finding periodic solutions via Brouwer degree, *Nonlinearity* **27** (2014), 2417.

[44] J. Llibre, Y. Tang, Limit cycles of discontinuous piecewise quadratic and cubic polynomial perturbations of a linear center, *Discrete Contin. Dyn. Syst. Ser. B* **24** (2019), 1769–1784.

[45] J. Llibre, M.A. Teixeira, I.O. Zeli, Birth of limit cycles for a class of continuous and discontinuous differential systems in (d+2)-dimension, *Dyn. Syst.* **31** (2016), 237–250.

[46] O. Makarenkov, J.S.W. Lamb, Dynamics and bifurcations of nonsmooth systems: a survey, *Physica D* **241** (2012), 1826–1844.
[47] J.A. Sanders, F. Verhulst, J. Murdock, *Averaging Methods in Nonlinear Dynamical Systems*, Second edition, Applied Mathematical Sciences 59, Springer, New York, 2007.
[48] H. Tian, M. Han, Bifurcation of periodic orbits by perturbing high-dimensional piecewise smooth integrable systems, *J. Differ. Equ.* **263** (2017), 7448–7474.
[49] L. Wei, X. Zhang, Normal form and limit cycle bifurcation of piecewise smooth differential systems with a center, *J. Differ. Equ.* **261** (2016), 1399–1428.
[50] L. Wei, X. Zhang, Averaging theory of arbitrary order for piecewise smooth differential systems and its application, *J. Dyn. Diff. Equat.* **30** (2018), 55–79.
[51] Y. Xiong, Limit cycle bifurcations by perturbing piecewise smooth Hamiltonian systems with multiple parameters, *J. Math. Anal. Appl.* **421** (2015), 260–275.
[52] P. Yang, J.P. Françoise, J. Yu, Second order Melnikov functions of piecewise Hamiltonian systems, *Int. J. Bifur. Chaos* **30** (2020), 2050016.
[53] J. Yang, M. Han, W. Huang, On Hopf bifurcations of piecewise Hamiltonian systems, *J. Differ. Equ.* **250** (2011), 1026–1051.
[54] Z. Zhang, B. Li, High order Melnikov functions and the problem of uniformity in global bifurcation, *Ann. Mat. Pura Appl.* CLXI (1992) 181–212.

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