NILPOTENT BI-CENTER IN CONTINUOUS PIECEWISE $\mathbb{Z}_2$-EQUIVARIANT CUBIC POLYNOMIAL HAMILTONIAN SYSTEMS (I)

TING CHEN[1], SHIMIN LI[1] AND JAUME LLIBRE[2]

Abstract. One of the classical and difficult problems in the theory of planar differential systems is to classify their centers. Here we classify the global phase portraits in the Poincaré disc of the class continuous piecewise differential systems separated by one straight line and formed by two $\mathbb{Z}_2$-equivariant cubic Hamiltonian systems with nilpotent bi-centers at $(\pm 1, 0)$.

1. Introduction and statement of the main results

The problem in the qualitative theory of planar differential equations of distinguishing between a focus and a center is known as the *center-focus problem.* This classical problem started with Poincaré [36] in 1881 and Dulac [15] in 1908, and nowadays the center-focus problem remains as one of main subjects in the qualitative theory of planar polynomial differential systems.

We say that a singular point p of a planar differential system is a *center* if it has a neighborhood U filled with periodic orbits with the unique exception of this singular point.

If a planar polynomial differential system has a *linear type center*, or a *nilpotent center*, or a *degenerate center* at the origin of coordinates, after making a time rescaling and a linear change of variables, this differential system can be written as

$$(1) \qquad (\dot{x}, \dot{y}) = \left\{ \begin{array}{l} (-y,\ x) \\ (y,\ 0) \\ (0,\ 0) \end{array} \right\} + (f(x,y), g(x,y)),$$

respectively. Here the dot denotes derivative with respect to the time t, and $f(x,y)$ and $g(x,y)$ are real polynomials without constant and linear terms.

The focus-center problem for the quadratic polynomial differential systems has been solved see [3, 7, 15, 23, 24, 37, 40, 43]. There are partial results in the classification of the centers for the cubic polynomial differential systems, see for instance [9, 11, 32, 41, 44, 45], but the focus-center problem for the cubic polynomial differential systems still remains open.

Recently Colak *el at.* [12, 13] studied the phase portraits of some cubic Hamiltonian differential systems with a linear type center and a nilpotent center at the origin, respectively. Liu and Li [30] investigated the linear type bi-center problem for $\mathbb{Z}_2$-equivariant differential systems. Here we shall study the $\mathbb{Z}_2$-equivariant polynomial systems having two centers at the singular points.

The study of $\mathbb{Z}_q$-equivariant polynomial systems, whose phase portraits are unchanged by a rotation of $2q$ ($q \in Z^+$) radians around one point, is closely related to the well-known Hilbert 16th problem, for more details see [22, 27, 28]. Chen *el at.* [10] provided all possible phase portraits of $\mathbb{Z}_2$-equivariant cubic polynomial Hamiltonian vector fields with a linear type bi-center. Li *el at.* [25, 26] studied the bi-center and isochronous bi-center problems in some $\mathbb{Z}_2$-equivariant cubic systems.

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However many natural phenomena have been modeled more accurately by dynamical systems whose differential systems are non-smooth (see for instance [1, 6, 31]), increasing contributions have been attracted to the qualitative analysis of non-smooth systems, see [2, 17]. In this paper we deal with the following family of piecewise smooth systems,

$$(\dot{x},\dot{y}) = \begin{cases} \big(f^{+}(x,y), g^{+}(x,y)\big) & \text{if} \quad S(x,y) > 0, \\ \big(f^{-}(x,y), g^{-}(x,y)\big) & \text{if} \quad S(x,y) < 0, \end{cases} \tag{2}$$

where $S : \mathbb{R}^2 \to \mathbb{R}$ is a $\mathcal{C}^\infty$ function and $(f^{\pm}(x,y), g^{\pm}(x,y))$ are smooth vector fields. In fact, systems (2) have two different regions $\Gamma^{\pm} = \{(x,y) \in \mathbb{R}^2 : \pm S(x,y) > 0\}$ separated by the discontinuity line $\Gamma = S^{-1}(0)$.

The focus-center problem in piecewise smooth systems becomes much more difficult and complicated than for the smooth systems. For example, a singular point of systems (2) on the discontinuous curve $S(x,y) = 0$ can be a center even it is neither a center for the first system, nor a center for the second system of (2).

Some methods have been developed for studying the linear type focus-center problem of the piecewise smooth systems (2). Thus Gasull and Torregrosa [20] developed an efficient method for computing the Lyapunov constants of switching polynomial systems, which can be used to determine the center conditions for a linear type singular point. By computing the Lyapunov constants, the authors of [8, 39] gave a complete classification on the linear type center conditions of the origin in several classes of Bautin switching systems. For more results on the focus-center problem of the piecewise smooth systems with the linear type singular points, see [14, 21].

The focus-center problem for the nilpotent singular points is much more challenging compared to the study for the linear type singular points. Computationally efficient methods have been developed for studying the focus-center problem of the planar smooth systems with nilpotent singular points, see [18, 29, 30, 38]. However there are no work for studying the nilpotent focus-center problem in piecewise smooth polynomial systems.

In this paper we will study the global dynamics of a class of piecewise $\mathbb{Z}_2$-equivariant differential systems formed by two cubic Hamiltonian systems separated by the straigh line $y = 0$, and having nilpotent bi-centers at the points $(\pm 1, 0)$. In section 3 we prove that such class of piecewise differential systems can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -a_{21}y + 3b_{03}y^2 + a_{21}x^2y - 3b_{03}xy^2 - 2(1+a_{21}^2)y^3 \\ -\dfrac{1}{2}x + \dfrac{1}{2}x^3 - a_{21}xy^2 + b_{03}y^3 \end{pmatrix} & \text{if} \quad y > 0, \\ \begin{pmatrix} -a_{21}y - 3b_{03}y^2 + a_{21}x^2y - 3b_{03}xy^2 - 2(1+a_{21}^2)y^3 \\ -\dfrac{1}{2}x + \dfrac{1}{2}x^3 - a_{21}xy^2 + b_{03}y^3 \end{pmatrix} & \text{if} \quad y < 0, \end{cases} \tag{3}$$

where $b_{03} < 0$ and the singular point $(1,0)$ of the first system of (3) is a third-order singular point, see section 3 for the definition of third-order singular point. The Hamiltonian functions for these two Hamiltonian systems are

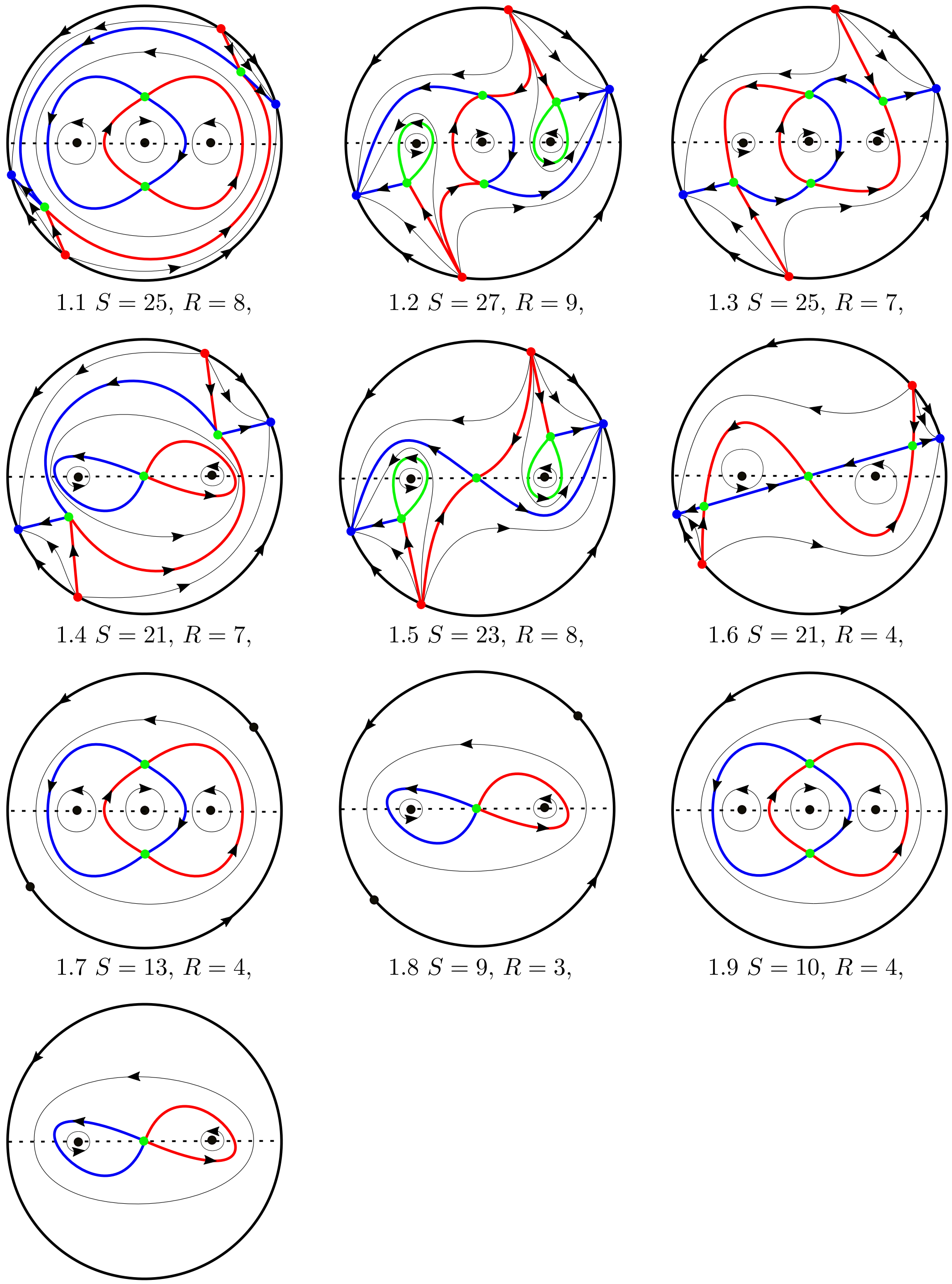
$$H(x,y)^{+} = \frac{1}{4}x^2 - \frac{1}{8}x^4 - \frac{1}{2}a_{21}y^2 + b_{03}y^3 + \frac{1}{2}a_{21}x^2y^2 - b_{03}xy^3 - \frac{1}{2}(1+a_{21}^2)y^4,$$

for the Hamiltonian system in $y > 0$, and

$$H(x,y)^{-} = \frac{1}{4}x^2 - \frac{1}{8}x^4 - \frac{1}{2}a_{21}y^2 - b_{03}y^3 + \frac{1}{2}a_{21}x^2y^2 - b_{03}xy^3 - \frac{1}{2}(1+a_{21}^2)y^4,$$

for the Hamiltonian system in $y < 0$.

Note that the piecewise differential systems (3) only are continuous on the straight line $y = 0$, so they are non-smooth piecewise differential systems. We also remark that the piecewise differential system (3) only depends on two parameters a_{21} and b_{03}.



1.1 $S = 25$, $R = 8$,
1.2 $S = 27$, $R = 9$,
1.3 $S = 25$, $R = 7$,
1.4 $S = 21$, $R = 7$,
1.5 $S = 23$, $R = 8$,
1.6 $S = 21$, $R = 4$,
1.7 $S = 13$, $R = 4$,
1.8 $S = 9$, $R = 3$,
1.9 $S = 10$, $R = 4$,
1.10 $S = 6$, $R = 3$.

FIGURE 1. The ten topological phase portraits in the Poincaré disc of Theorem 1.1.

Theorem 1.1. *In the Poincaré disc the phase portraits of the continuous piecewise $\mathbb{Z}_2$-equivariant cubic Hamiltonian systems* (3) *with a nilpotent bi-center at* $(\pm 1, 0)$ *are topologically equivalent to one of the* 10 *phase portraits showed in Figure* 1.

In section 2 we provide a brief introduction to the Poincaré compactification, a summary on how to determine the phase portrait using the separatrix skeleton, and some basic results on the topological indices that we shall need for proving Theorem 1.1. In section 3 we show how to obtain the continuous piecewise $\mathbb{Z}_2$-equivariant cubic Hamiltonian systems (3). Finally in section 4 we characterize the global phase portraits of systems (3) in the Poincaré disc, that is we prove Theorem 1.1.

2. Preliminaries

2.1. **Poincaré compactification.** In order to classify the global dynamics of the piecewise differential systems (3), the first crucial step is to characterize their finite and infinite singular points in the Poincaré compactification, as we shall see such compactification is possible due to the fact that our Hamiltonian systems are polynomial. This tool is described in chapter 5 of [16]. The second main step for determining the global flow in the Poincaré disc of polynomial differential systems is the characterization of their separatrices. For the polynomial differential systems in the Poincaré disc it is known that the separatrices are all the infinite orbits, all the finite singular points, the separatrices of the hyperbolic sectors of the finite and infinite singular points, and the limit cycles. If Σ denotes the set of all separatrices in the Poincaré disc $\mathbb{D}^2$, Σ is a closed set and the components of $\mathbb{D}^2 \setminus \Sigma$ are called the canonical regions. We denote by S and R the number of separatrices and canonical regions, respectively.

Roughly speaking this compactification identifies the plane $\mathbb{R}^2$ with the interior of the closed unit disc $\mathbb{D}^2$ centered at the origin of $\mathbb{R}^2$, and extends analytically the differential system to its boundary, usually called the circle of the infinity. Now we shall describe the equations of the Poincaré compactification for a polynomial differential system in $\mathbb{R}^2$.

We consider the set of all polynomial vector fields in $\mathbb{R}^2$ of the form

$$(\dot{x_1}, \dot{x_2}) = X(x_1, x_2) = (P(x_1, x_2), Q(x_1, x_2)), \tag{4}$$

where P and Q are real polynomials in the variables x_1 and x_2 of degree d_1 and d_2, respectively. Taking $d = \max\{d_1, d_2\}$.

Denote by $T_p\mathbb{S}^2$ be the tangent space to the 2-dimensional sphere $\mathbb{S}^2 = \{\mathbf{s} = (s_1, s_2, s_3) \in \mathbb{R}^3 : s_1^2 + s_2^2 + s_3^2 = 1\}$ at the point p. Assume that $\mathbf{X}$ is defined in the plane $T_{(0,0,1)}\mathbb{S}^2 = \mathbb{R}^2$. Consider the central projection $f\colon T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2$. This map defines two copies of $\mathbf{X}$, one in the open northern hemisphere and other in the open southern hemisphere. Denote by $\mathbf{X}'$ the vector field $Df \circ \mathbf{X}$ defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. Clearly $\mathbb{S}^1$ is identified to the infinity of $\mathbb{R}^2$. If $\mathbf{X}$ is a planar polynomial vector field of degree d, then $p(\mathbf{X})$ is the only analytic extension of $y_3^{d-1}\mathbf{X}'$ to $\mathbb{S}^2$, the vector field $p(\mathbf{X})$ is called the *Poincaré compactification* of the vector field $\mathbf{X}$, for more details see Chapter 5 of [16].

On the Poincaré sphere $\mathbb{S}^2$ we use the following six local charts to do the calculations, which are given by $U_i = \{\mathbf{s} \in \mathbb{S}^2 : \ s_i > 0\}$ and $V_i = \{\mathbf{s} \in \mathbb{S}^2 : \ s_i < 0\}$, for $i = 1, 2, 3$, with the corresponding diffeomorphisms

$$\varphi_i : \ U_i \to \mathbb{R}^2, \qquad \psi_i : \ V_i \to \mathbb{R}^2, \tag{5}$$

defined by $\varphi_i(\mathbf{s}) = -\psi_i(\mathbf{s}) = (s_m/s_i, s_n/s_i) = (u, v)$ for $m < n$ and $m, n \neq i$. Thus (u, v) will play different roles in the distinct local charts. The expression of the vector field $p(\mathbf{X})$ are

$$(\dot{u}, \dot{v}) = \left(v^d\left(Q\left(\frac{1}{v}, \frac{u}{v}\right) - uP\left(\frac{1}{v}, \frac{u}{v}\right)\right), -v^{d+1}P\left(\frac{1}{v}, \frac{u}{v}\right)\right) \qquad \text{in } U_1,$$

$$(\dot{u}, \dot{v}) = \left(v^d\left(P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right)\right), -v^{d+1}Q\left(\frac{u}{v}, \frac{1}{v}\right)\right) \qquad \text{in } U_2,$$

$$(\dot{u}, \dot{v}) = (P(u, v), Q(u, v)) \qquad \text{in } U_3.$$

We note that the expressions of the vector field $p(\mathbf{X})$ in the local chart (V_i, ψ_i) is equal to the expression in the local chart (U_i, ϕ_i) multiplied by $(-1)^{d-1}$ for $i = 1, 2, 3$.

The orthogonal projection under $\pi(y_1, y_2, y_3) = (y_1, y_2)$ of the closed northern hemisphere of $\mathbb{S}^2$ onto the plane $s_3 = 0$ is a closed disc $\mathbb{D}^2$ of radius one centered at the origin of coordinates called the *Poincaré disc*. Since a copy of the vector field $\mathbf{X}$ on the plane $\mathbb{R}^2$ is in the open northern hemisphere of $\mathbb{S}^2$, the interior of the Poincaré disc $\mathbb{D}^2$ is identified with $\mathbb{R}^2$ and the boundary of $\mathbb{D}^2$, the equator of $\mathbb{S}^2$, is identified with the infinity of $\mathbb{R}^2$. Consequently the phase portrait of the vector field $\mathbf{X}$ extended to the infinity corresponds to the projection of the phase portrait of the vector field $p(\mathbf{X})$ on the Poincaré disc $\mathbb{D}^2$.

The singular points of $p(\mathbf{X})$ in the Poincaré disc lying on $\mathbb{S}^1$ are the *infinite singular points* of the corresponding vector field $\mathbf{X}$. The singular points of $p(\mathbf{X})$ in the interior of the Poincaré disc, i.e. on $\mathbb{S}^2 \setminus \mathbb{S}^1$, are the *finite singular points*.

For polynomial vector fields (4) if $s \in S_1$ is an infinite singular point, then $-s \in S_1$ is another infinite singular point. Thus the number of infinite singular points is even and the local behavior of one is that of the other multiplied by $(-1)^{d+1}$. This symmetry property in general does not hold for piecewise smooth differential systems (2) because the singular points at infinity are not diametrically opposite. But in our case systems (3) are symmetry with respect to the origin, so we just need to analyze the phase portraits of the infinite singular points in the local chart $U_1|_{v=0}$ and at the origin of the local chart U_2.

2.2. **Separatrix skeleton.** Given a flow $(\mathbb{D}^2, \phi)$ by the *separatrix skeleton* we mean the union of all the separatries of the flow together with one orbit from each one of the canonical regions. Let C_1 and C_2 be the separatrix skeletons of the flows $(\mathbb{D}^2, \phi_1)$ and $(\mathbb{D}^2, \phi_2)$ respectively. We say that C_1 and C_2 are *topologically equivalent* if there exists a homeomorphism $h : \mathbb{D}^2 \to \mathbb{D}^2$ which sends orbits to orbits preserving or reversing the direction of all orbits. From Markus [33], Neumann [34] and Peixoto [35] it follows the next theorem which shows that is enough to describe the separatrix skeleton in order to determine the topological equivalence class of a differential system in the Poincaré disc $\mathbb{D}^2$.

Theorem 2.1 (Markus–Neumann–Peixoto Theorem)**.** *Assume that $(\mathbb{D}^2, \phi_1)$ and $(\mathbb{D}^2, \phi_2)$ are two continuous flows with only isolated singular points. Then these flows are topologically equivalent if and only if their separatrix skeletons are equivalent.*

2.3. **Topological index.** Next we introduce the topological index of the singular points, which is one useful tool to determine the type of the singular points. Here we will present two important theorems, the Index Poincaré Formula and the Poincaré–Hopf Theorem, for more details see Chapter 6 of [16].

Theorem 2.2. *We denote by p an isolated singular point with the finite sectorial decomposition property. Let q, h and e be the number of parabolic, hyperbolic and elliptic sectors of p, respectively. Then the topological index of the singular point p equals* $1 + (e - h)/2$.

Corollary 2.3. *The topological indices of a center, a cusp, a saddle and a node equal* 1, 0, -1 *and* 1*, respectively.*

Theorem 2.4. *For any continuous vector field on the sphere $\mathbb{S}^2$ with finitely many singular points, the sum of their topological indices is* 2.

Remark 2.5. *Since the flow of Hamiltonian smooth systems preserves the area, we have that any finite singular point of a Hamiltonian smooth system must be either a center, or union of an even number of hyperbolic sectors. In particular, the finite nilpotent singular points of Hamiltonian planar differential systems are either saddles, centers, or cusps, for more details see Theorem 3.5 of* [16].

3. Obtaining systems (3)

Here a vector field $\mathbf{X}(x, y)$ is $\mathbb{Z}_2$*-equivariant* if $-\mathbf{X}(x, y) = \mathbf{X}(-x, -y)$. Then $\mathbb{Z}_2$*-equivariant piecewise cubic polynomial differential systems* (3) *separated by the straight line* $y = 0$ are differential systems of the form

$$(6)\qquad \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{cases}\begin{pmatrix}a_{00}+a_{10}x+a_{01}y+a_{20}x^2+a_{11}xy+a_{02}y^2+a_{30}x^3\\ +a_{21}x^2y+a_{12}xy^2+a_{03}y^3\\ b_{00}+b_{10}x+b_{01}y+b_{20}x^2+b_{11}xy+b_{02}y^2+b_{30}x^3\\ +b_{21}x^2y+b_{12}xy^2+b_{03}y^3\end{pmatrix} & \text{if } y>0,\\ \begin{pmatrix}-a_{00}+a_{10}x+a_{01}y-a_{20}x^2-a_{11}xy-a_{02}y^2+a_{30}x^3\\ +a_{21}x^2y+a_{12}xy^2+a_{03}y^3\\ -b_{00}+b_{10}x+b_{01}y-b_{20}x^2-b_{11}xy-b_{02}y^2+b_{30}x^3\\ +b_{21}x^2y+b_{12}xy^2+b_{03}y^3\end{pmatrix} & \text{if } y<0.\end{cases}$$

Assuming that $(\pm1,0)$ are two singular points of systems (6), we have

$$(7)\qquad a_{00}=-a_{20},\quad a_{10}=-a_{30},\quad b_{00}=-b_{20},\quad b_{10}=-b_{30}.$$

The Jacobian matrices of the first and second systems of (6) evaluated at $(1,0)$ are

$$(8)\qquad J^{\pm}=\begin{pmatrix}\pm2a_{20}+2a_{30} & a_{01}\pm a_{11}+a_{21}\\ \pm2b_{20}+2b_{30} & b_{01}\pm b_{11}+b_{21}\end{pmatrix}.$$

It follows from $J^+=J^-$ that

$$(9)\qquad a_{20}=a_{11}=b_{20}=b_{11}=0.$$

If we assume that $b_{30}=0$, then $J^+=J^-$ yields a triangular matrix having the two characteristic roots

$$\lambda_1=2a_{30},\quad \lambda_2=b_{01}+b_{21}.$$

Furthermore we take $\lambda_1=\lambda_2=0$, because we want that the singular points $(\pm1,0)$ of the piecewise differential systems (6) are nilpotent, so we obtain

$$(10)\qquad a_{30}=0,\quad b_{01}=-b_{21}.$$

From (7), (9) and (10) the piecewise differential systems (6) become

$$(11)\qquad \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{cases}\begin{pmatrix}y(a_{01}+a_{21}x^2+a_{02}y+a_{12}xy+a_{03}y^2)\\ y(-b_{21}+b_{21}x^2+b_{02}y+b_{12}xy+b_{03}y^2)\end{pmatrix} & \text{if } y>0,\\ \begin{pmatrix}y(a_{01}+a_{21}x^2-a_{02}y+a_{12}xy+a_{03}y^2)\\ y(-b_{21}+b_{21}x^2-b_{02}y+b_{12}xy+b_{03}y^2)\end{pmatrix} & \text{if } y<0.\end{cases}$$

Since the polynomials in (11) have a common factor y, the singular points $(\pm1,0)$ are not isolated singular points. Hence in order to make the singular points $(\pm1,0)$ isolated nilpotent singular points of system (11) we force that

$$J^{\pm}=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},$$

consequently $b_{30}=\frac{1}{2}$. Then systems (6) can be rewritten as

$$(12)\qquad \begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{cases}\begin{pmatrix}-a_{21}y+a_{21}x^2y+a_{02}y^2+a_{12}xy^2+a_{03}y^3=X^+(x,y)\\ -\dfrac{x}{2}+\dfrac{x^3}{2}-b_{21}y+b_{21}x^2y+b_{02}y^2+b_{12}xy^2+b_{03}y^3=Y^+(x,y)\end{pmatrix} & \text{if } y>0,\\ \begin{pmatrix}-a_{21}y+a_{21}x^2y-a_{02}y^2+a_{12}xy^2+a_{03}y^3=X^-(x,y)\\ -\dfrac{x}{2}+\dfrac{x^3}{2}-b_{21}y+b_{21}x^2y-b_{02}y^2+b_{12}xy^2+b_{03}y^3=Y^-(x,y)\end{pmatrix} & \text{if } y<0.\end{cases}$$

Next, let $H^+(x,y)$ be the Hamiltonian of the first system of systems (12). To find this Hamiltonian, we integrate $X^+(x,y)$ of (12) with respect to y and obtain

$$
\begin{aligned}
H_1^+(x,y) =& f(x) + \int X^+(x,y)dy \\
=& f(x) - \frac{1}{2}a_{21}y^2 + \frac{1}{2}a_{21}x^2y^2 + \frac{1}{3}a_{02}y^3 + \frac{1}{3}a_{12}xy^3 + \frac{1}{4}a_{03}y^4,
\end{aligned} \tag{13}
$$

for some real polynomials $f(x)$. And we integrate $Y^+(x,y)$ of (12) with respect to x and obtain

$$
\begin{aligned}
H_2^+(x,y) =& g(y) - \int Y^+(x,y)dx \\
=& g(y) + \frac{1}{4}x^2 - \frac{1}{8}x^4 + b_{21}xy - \frac{1}{3}b_{21}x^2y - b_{02}xy^2 - \frac{1}{2}b_{12}x^2y^2 - b_{03}xy^3,
\end{aligned} \tag{14}
$$

for some real polynomials $g(y)$. Equating $H_1^+(x,y)$ to $H_2^+(x,y)$ we obtain

$$
b_{12} = -a_{21}, \qquad a_{12} = -3b_{03}, \qquad b_{02} = b_{21} = 0, \tag{15}
$$

$f(x) = x^2/4 - x^4/8$ and $g(y) = -a_{21}y^2/2 + a_{02}y^3/3 + a_{03}y^4/4$.

Then systems (12) become the piecewise Hamiltonian systems

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -a_{21}y + a_{21}x^2y + a_{02}y^2 - 3b_{03}xy^2 + a_{03}y^3 \\ -\dfrac{x}{2} + \dfrac{x^3}{2} - a_{21}xy^2 + b_{03}y^3 \end{pmatrix} & \text{if } \ y > 0, \\ \begin{pmatrix} -a_{21}y + a_{21}x^2y - a_{02}y^2 - 3b_{03}xy^2 + a_{03}y^3 \\ -\dfrac{x}{2} + \dfrac{x^3}{2} - a_{21}xy^2 + b_{03}y^3 \end{pmatrix} & \text{if } \ y < 0, \end{cases} \tag{16}
$$

where systems (16) have the Hamiltonian

$$
H(x,y)^+ = \frac{1}{4}x^2 - \frac{1}{8}x^4 - \frac{1}{2}a_{21}y^2 + \frac{1}{2}a_{21}x^2y^2 + \frac{1}{3}a_{02}y^3 - b_{03}xy^3 + \frac{1}{4}a_{03}y^4, \tag{17}
$$

for the Hamiltonian system in $y > 0$, and the Hamiltonian

$$
H(x,y)^- = \frac{1}{4}x^2 - \frac{1}{8}x^4 - \frac{1}{2}a_{21}y^2 + \frac{1}{2}a_{21}x^2y^2 - \frac{1}{3}a_{02}y^3 - b_{03}xy^3 + \frac{1}{4}a_{03}y^4, \tag{18}
$$

for the Hamiltonian system in $y < 0$, respectively.

Introducing the transformation $x \to x+1$ into systems (16) we get

(19)

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} 2a_{21}xy + a_{21}x^2y + (a_{02} - 3b_{03})y^2 + a_{12}xy^2 + a_{03}y^3 = \Psi^+(x,y) \\ x + \dfrac{3x^2}{2} + \dfrac{x^3}{2} + a_{21}y^2 - a_{21}xy^2 + b_{03}y^3 = x + \Phi^+(x,y) \end{pmatrix} & \text{if } y > 0, \\ \begin{pmatrix} 2a_{21}xy + a_{21}x^2y - (a_{02} + 3b_{03})y^2 + a_{12}xy^2 + a_{03}y^3 = \Psi^-(x,y) \\ x + \dfrac{3x^2}{2} + \dfrac{x^3}{2} + a_{21}y^2 - a_{21}xy^2 + b_{03}y^3 = x + \Phi^-(x,y) \end{pmatrix} & \text{if } y < 0, \end{cases}
$$

and so the singular point $(1,0)$ of systems (16) is moved to the origin of systems (19). Then we assume that

$$
f^{\pm}(y) = \sum_{k=2}^{\infty} c_k^{\pm} y^k
$$

are the unique solutions of the implicit function equations $x + \Phi^{\pm}(x,y) = 0$ in a neighborhood of the origin, respectively. In order to determine the local phase portraits of the nilpotent points $(\pm 1, 0)$ we write

$$
\begin{aligned}
\Psi^{\pm}(f^{\pm}(y), y) &= \sum_{k=2}^{\infty} \alpha_k^{\pm} y^k, \\
\left[\frac{\partial \Psi^{\pm}}{\partial x} + \frac{\partial \Phi^{\pm}}{\partial y}\right]_{(f^{\pm}(y),y)} &= \sum_{k=1}^{\infty} \beta_k^{\pm} y^k,
\end{aligned} \tag{20}
$$

where

$$\beta_n^{\pm} \equiv 0, \qquad \alpha_2^{\pm} = \pm a_{02} - 3b_{03}, \qquad \alpha_3^{\pm} = a_{03} + 2a_{21}^2. \tag{21}$$

For polynomial differential systems if $\alpha_2 = \alpha_3 = \cdots = \alpha_{k-1} = 0$ and $\alpha_k \neq 0$, then the multiplicity of the nilpotent singular point is exactly k, for more detail see [30]. It follows from Theorem 3.5 in [16] that if $\beta_n = 0$ and $\alpha_m \neq 0$ this nilpotent singular point is a

$$\begin{cases} \text{a cusp if } m = 2k, \\ \text{a saddle if } m = 2k+1 \text{ and } \alpha_m > 0, \\ \text{a center or a focus if } m = 2k+1 \text{ and } \alpha_m < 0. \end{cases} \tag{22}$$

Since the multiplicity of a nilpotent center or focus (i.e. of a monodromic singular point) of a differential system is an odd positive integer greater than one, it follows that the smallest multiplicity of $(1,0)$ must be 3 if the singular point $(1,0)$ is a nilpotent focus or a center in the first system of (16). For convenience, we will call this singular point a *third-order singular point*. More precisely, we have the following statement: The singular point $(1,0)$ of the first system of (16) is a monodromic critical point with multiplicity 3 if and only if

$$\alpha_2^+ = 0, \quad \alpha_3^+ < 0,$$

namely,

$$a_{02} = 3b_{03}, \quad a_{03} + 2a_{21}^2 < 0.$$

Setting $\alpha_3^+ = -2$ yields $a_{03} = -2a_{21}^2 - 2$. Then we have that the singular point $(1,0)$ of the first system of (16) is monodromic. Therefore we obtain systems (3) and we have

$$\alpha_2^- = -6b_{03}, \quad \alpha_3^- = -2.$$

If $b_{03} = 0$, i.e., $\alpha_2^- = 0$, then the piecewise differential systems (3) are smooth. If $b_{03} \neq 0$, i.e.,$\alpha_2^- \neq 0$, then the singular point $(1,0)$ of the second differential system of (3) is a cusp. But the singular points $(\pm 1,0)$ of the piecewise differential systems (3) cannot be monodromic when $b_{03} > 0$, so we only consider $b_{03} < 0$.

In summary we have obtained the continuous piecewise differential system (3).

Furthermore from Proposition 2.1 of [8] we have that the Hamiltonians of the first and second systems of (3) satisfy with $H^+(x,0) \equiv H^-(x,0)$. Hence systems (3) have nilpotent bi-centers at $(\pm 1,0)$. Remark that when $\alpha_2^+ \neq 0$, i.e. $a_{03} \neq 3b_{03}$ systems (3) can also have nilpotent bi-centers at $(\pm 1,0)$, but this case becomes more complicated we do not provide its analysis in this paper.

4. Global phase portraits of systems (3)

Now we consider the finite singular points of systems (3). The singular points $p_{1,2} = (\pm 1,0)$ are two centers, the origin p_3 of systems (3) is also a singular point, whose Jacobian matrix is

$$\begin{pmatrix} 0 & -a_{21} \\ -\frac{1}{2} & 0 \end{pmatrix}. \tag{23}$$

From (23) we have that the origin is a saddle when $a_{21} \geq 0$ (a nilpotent saddle when $a_{21} = 0$), or a center when $a_{21} < 0$. Now we need to study if there are additional singular points.

Since systems (3) are symmetric with respect to the origin of coordinates p_3, we just need to study the phase portrait of the first system in (3).

The Jacobian matrix of the first system of systems (3) at a finite singular point (x,y) is

$$\begin{pmatrix} y(2a_{21}x - 3b_{03}y) & N_1 \\ \frac{1}{2}(-1 + 3x^2 - 2a_{21}y^2) & -y(2a_{21}x - 3b_{03}y) \end{pmatrix}, \tag{24}$$

where

$$N_1 = -a_{21} + a_{21}x^2 + 6b_{03}y - 6b_{03}xy - 6y^2 - 6a_{21}^2y^2.$$

We claim that there are no finite singular points for the first system of system (3) whose linear part be identically zero. Indeed, we obtain that $-1+3x^2-2a_{21}y^2$ and $y(2a_{21}x-3b_{03}y)$ have no common solutions, because the Gröbner basis for the polynomials $\dot{x}$, $\dot{y}$, $-1+3x^2-2a_{21}y^2$ and $y(2a_{21}x-3b_{03}y)$ is 1. We again calculate the Gröbner basis for four polynomials $\dot{x}$, $\dot{y}$, $y(2a_{21}x-3b_{03}y)$ and N_1, then we obtain seven polynomials $a_{21}y$, $b_{03}y^2$, y^3, $-a_{21}+a_{21}x^2+6b_{03}y-6b_{03}xy-6y^2$, $xy-b_{03}+b_{03}x+y$, $x(1+x)y^2$ and $(-1+x)x(1+x)$. It means that there are no other nilpotent singular points different from p_k for $k=1,2,3$, such these four polynomials be zero. Hence all the remaining finite singular points are hyperbolic, or semi-hyperbolic, or centers and by Theorems 2.15 and 2.19 of [16] the remaining finite singular points must be saddles or centers because the system is Hamiltonian.

The explicit expressions of the finite singular points different from p_k for $k=1,2,3$, and their eigenvalues in terms of parameters a_{21} and b_{03} are complicated, it is hard to study their existence and their types. Thus we need to present more algebraic tools for solving this problem.

From the first system in (3) we compute the Gröbner basis for $\dot{x}$ and $\dot{y}$ and we obtain eight polynomials, where the following two polynomials

$$y^2\Big[3a_{21}b_{03}-3a_{21}b_{03}x-9b_{03}^2y+(2a_{21}+9b_{03}^2)xy+2b_{03}(3+4a_{21}^2)y^2\Big]$$

and

$$\begin{aligned}&y^3\Big[6a_{21}b_{03}+(-2a_{21}-18b_{03}^2+15a_{21}^2b_{03}^2)y+(18b_{03}+12a_{21}^2b_{03}-54a_{21}b_{03}^3)y^2\\&+(-4-4a_{21}^2+36a_{21}b_{03}^2+32a_{21}^3b_{03}^2+27b_{03}^4)y^3\Big]=y^3f(y).\end{aligned}\tag{25}$$

are enough for our analysis. We note that polynomial (25) is not identically zero, because in order that it be identically zero we need that $a_{21}=b_{03}=0$, but then the resultant reduces to $-4y^6\neq 0$. Now in order to study the number of the real roots of the polynomial $f(y)$ we shall use the method of the *discriminant sequence* associated to $f(y)$ developed in [42].

We associate to the polynomial

$$f(y)=a_0+a_1y+\cdots+a_ky^k\tag{26}$$

the $(2k+1)\times(2k+1)$ matrix

$$M=\begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_k & & & & \\ 0 & ka_0 & (k-1)a_1 & \cdots & a_{k-1} & & & & \\ & a_0 & a_1 & \cdots & a_{k-1} & a_k & & & \\ & & na_0 & \cdots & 2a_{k-2} & a_{k-1} & & & \\ & & & \cdots & \cdots & & & & \\ & & & \cdots & \cdots & & & & \\ & & & & a_0 & a_1 & \cdots & a_k & \\ & & & & 0 & ka_0 & \cdots & a_{k-1} & \\ & & & & & a_0 & a_1 & a_{k-1} & a_k \end{pmatrix}.$$

We define d_j as the determinant of the submatrix of M constructed with the first j rows and columns of the matrix M for $j=1,\ldots,2k+1$. Thus we have the sequence

$$\{d_1,d_2,\ldots,d_{2k+1}\}.\tag{27}$$

Consider the discriminant sequence $\{d_2,d_4,\cdots,d_{2k}\}$ and the sequence of its signs

$$[\mathrm{sign}(d_2),\mathrm{sign}(d_4),\cdots,\mathrm{sign}(d_{2k})],$$

called *sign list*, where as usual the sign function is

$$\mathrm{sign}(x)\ =\begin{cases}-1 & \text{if } x<0,\\ \ \ 0 & \text{if } x=0,\\ \ \ 1 & \text{if } x>0.\end{cases}\tag{28}$$

For a sign list $[s_1,s_2,\cdots,s_n]$ of $f(y)$ we define its *revised sign list* $[l_1,l_2,\cdots,l_n]$ as follows:

1. If $s_k \neq 0$ we write $l_k = s_k$.
2. If subsection $[s_i, s_{i+1}, \cdots, s_{i+j}]$ of this sign list, which satisfies with $s_{i+1} = \cdots = s_{i+j-1} = 0$ and $s_i s_{i+j} \neq 0$, we replace the subsection $[s_{i+1}, s_{i+2}, \cdots, s_{i+j-1}]$ with $[-s_i, -s_i, s_i, s_i, -s_i, -s_i, s_i, s_i, -s_i, \cdots]$ keeping the number of terms.

For convenient we denote by RSL and SL the revised sign list and the sign list of the discriminant sequence, respectively. Then the RSL $[l_1, l_2, \cdots, l_n]$ has no zeros between two nonzero members.

From Theorems 2.1 and 3.3 of [42] we obtain the following two theorems.

Theorem 4.1. *Let $f(y)$ be the polynomial* (26) *with real coefficient. If the number of the sign changes of the RSL $[d_2, d_4, \ldots, d_{2k}]$ is equal to m, and the number of nonzero elements of this RSL is equal to ℓ, then the number of the distinct real roots of the polynomial $f(y)$ is $\ell - 2m$.*

Theorem 4.2. *Let $f(y)$ be the polynomial* (26) *with real coefficient such that $f(0) \neq 0$. If the number of the sign changes of the RSL $[d_1d_2, d_2d_3, \cdots, d_{2k}d_{2k+1}]$ is equal to m, and the number of nonzero elements of this RSL is equal to l, the number of the negative roots of the polynomial $f(y)$ is $l/2 - m$.*

We separate the study of the roots of the polynomial $f(y)$ of (25) in two cases.

Case 1: The coefficient of the cubic term of $f(y)$ in (25) is zero, i.e.

$$N_2 = -4 - 4a_{21}^2 + 36a_{21}b_{03}^2 + 32a_{21}^3 b_{03}^2 + 27b_{03}^4 = 0.$$

Then we have

$$b_{03}^2 = \frac{2}{27}\Big(- 9a_{21} - 8a_{21}^3 + \sqrt{(3 + 4a_{21}^4)^3}\Big). \tag{29}$$

Now we calculate the resultant of the coefficient of y^2 of $f(y)$ in (25) with N_2 with respect to the variable a_{21} and obtain $6912b_{03}^2(1 + 27b_{03}^4)^3 \neq 0$. Thus the coefficient of y^2 in $f(y)$ is nonzero when the coefficient of y^3 is zero. Multiply this quadratic coefficient and the constant term of $f(y)$ we obtain

$$\begin{aligned} &\frac{8}{3}a_{21}\Big(- 9a_{21} - 8a_{21}^3 + \sqrt{27 + 108a_{21}^2 + 144a_{21}^4 + 64a_{21}^6}\Big)\times \\ &\Big[3 + 2a_{21}^2 - \frac{2}{3}a_{21}\Big(- 9a_{21} - 8a_{21}^3 + \sqrt{27 + 108a_{21}^2 + 144a_{21}^4 + 64a_{21}^6}\Big)\Big] \leq 0. \end{aligned} \tag{30}$$

Hence $f(y)$ has at most one positive root. Actually if $a_{21} = 0$ we have $b_{03} = -\sqrt{2}/(3^{3/4})$, and $f(y)$ has no positive roots.

Case 2: The coefficient of the cubic term of $f(y)$ in (25) is nonzero, i.e. $N_2 \neq 0$. Then finding the number of the positive roots of $f(y)$ in (25) is equivalent to find the number of the negative roots of $-f(-y)$. Now we shall compute the negative roots of the polynomial $-f(-y)$ using Theorem 4.2. So we consider the sequence

$$\{d_1d_2, d_2d_3, d_3d_4, d_4d_5, d_5d_6, d_6d_7\} \tag{31}$$

associated to $-f(-y)$, and we have

$$\begin{aligned} &d_1 = N_2, \qquad d_2 = 3N_2^2, \qquad d_3 = -6b_{03}(3 + 2a_{21}^2 - 9a_{21}b_{03}^2)N_2^2, \\ &d_4 = -6N_2^2 N_3, \qquad d_5 = -4N_2^2 N_4, \\ &d_6 = -4(a_{21} + 9b_{03}^2 + 6a_{21}^2 b_{03}^2)^2 N_2^2 N_5, \\ &d_7 = 24a_{21}b_{03}(a_{21} + 9b_{03}^2 + 6a_{21}^2 b_{03}^2)^2 N_2^2 N_5, \end{aligned} \tag{32}$$

where

$$
\begin{aligned}
N_3 =&8a_{21}+8a_{21}^3-36b_{03}^2-204a_{21}^2b_{03}^2-172a_{21}^4b_{03}^2-54a_{21}b_{03}^4+396a_{21}^3b_{03}^4\\
&+480a_{21}^5b_{03}^4-486b_{03}^6-567a_{21}^2b_{03}^6,\\
N_4 =&-16a_{21}^2-16a_{21}^4+198a_{21}b_{03}^2+852a_{21}^3b_{03}^2+656a_{21}^5b_{03}^2+162b_{03}^4\\
&-567a_{21}^2b_{03}^4-5724a_{21}^4b_{03}^4-5088a_{21}^6b_{03}^4+2673a_{21}b_{03}^6-486a_{21}^3b_{03}^6\\
&+3456a_{21}^5b_{03}^6+7200a_{21}^7b_{03}^6+8748b_{03}^8+5103a_{21}^2b_{03}^8-4860a_{21}^4b_{03}^8,\\
N_5 =&32a_{21}+32a_{21}^3-36b_{03}^2-240a_{21}^2b_{03}^2-208a_{21}^4b_{03}^2+2052a_{21}b_{03}^4\\
&+5040a_{21}^3b_{03}^4+3000a_{21}^5b_{03}^4-1944b_{03}^6-2025a_{21}^2b_{03}^6.
\end{aligned}
\tag{33}
$$

From Theorem 4.2 we obtain that the polynomial $-f(-y)$ has three distinct negative roots if and only if the revised sign list of (31) is $[1,1,1,1,1,1]$ or $[-1,-1,-1,-1,-1,-1]$, which we cannot be obtained varying the parameters a_{21} and b_{03}. Therefore the polynomial $-f(-y)$ has at most two negative roots.

Now we study the case when the polynomial $-f(-y)$ has distinct negative roots. We denote by $\mathrm{R}[f(\alpha),i]$ the i-th real root of the polynomial $f(\alpha)$ with respect to α, and these roots are ordered as follows $\mathrm{R}[f(\alpha),i]<\mathrm{R}[f(\alpha),j]$ if and only if $i<j$. We describe the possible revised sign lists of the associated discriminant sequences as we show in Tables 1,2,3, when the polynomial $-f(-y)$ has two negative roots, one negative root and no negative roots, respectively, where

$$
\begin{aligned}
N_6 =&-36-21a_{21}^2+20a_{21}^4,\\
N_7 =&-108-27a_{21}^2+99a_{21}^4+5a_{21}^6,\\
N_8 =&-81+1026a_{21}^2+3429a_{21}^4+3498a_{21}^6+1175a_{21}^8,\\
N_9 =&-648+837a_{21}^2+3942a_{21}^4+3468a_{21}^6+1000a_{21}^8,\\
N_{10} =&-128490624-62227804500a_{21}^2-515496116628a_{21}^4\\
&-916466231925a_{21}^6+210402679464a_{21}^8+1653908444856a_{21}^{10}\\
&+863216641008a_{21}^{12}-432308074320a_{21}^{14}-139867591104a_{21}^{16}\\
&+313035878400a_{21}^{18}+137815040000a_{21}^{20},\\
N_{11} =&-243-4536a_{21}^2-9180a_{21}^4-3168a_{21}^6+4260a_{21}^8+2540a_{21}^{10}.
\end{aligned}
\tag{34}
$$

TABLE 1. The conditions in order that the revised sign list (RSL) of (31) has two distinct negative roots.

RSL	Conditions
$[1,1,1,-1,-1,-1]$	$\mathrm{R}[N_6,1]\approx-1.40204<a_{21}<\mathrm{R}[N_7,1]\approx-1.07347$, $b_{03}\le\mathrm{R}[N_4,1]$, or $a_{21}=\mathrm{R}[N_7,1]$, $b_{03}<\mathrm{R}[N_4,1]$, or $\mathrm{R}[N_7,1]<a_{21}<0$, $b_{03}<\mathrm{R}[N_2,1]$;
$[1,1,1,1,1,-1]$	$a_{21}\le\mathrm{R}[N_6,1]$, $b_{03}<\mathrm{R}[N_2,1]$, or $\mathrm{R}[N_6,1]<a_{21}<\mathrm{R}[N_7,1]$, $\mathrm{R}[N_4,1]<b_{03}<\mathrm{R}[N_2,1]$.

In summary, the first system of the piecewise differential system (3) has at most two singular points different from p_j for $j=1,2,3$. Next we shall determine the local phase portraits of these additional finite singular points using the information provided by the infinite singular points.

In the local chart U_2 the first system of (3) becomes

$$
\begin{aligned}
u' =&\frac{1}{2}(-4-4a_{21}^2-8b_{03}u+4a_{21}u^2-u^4+6b_{03}v-2a_{21}v^2+u^2v^2),\\
v' =&-\frac{1}{2}v(2b_{03}-2a_{21}u+u^3-uv^2).
\end{aligned}
\tag{35}
$$

TABLE 2. The conditions in order that the revised sign list (RSL) of (31) has one distinct negative roots.

RSL	Conditions
$[1,-1,1,1,1,1]$	$\mathrm{R}[N_{11},2]<a_{21}\le\sqrt{\frac{3}{2}}$, $\mathrm{R}[N_3,1]<b_{03}\le \mathrm{R}[N_3,2]$, or $a_{21}>\sqrt{\frac{3}{2}}$, $\mathrm{R}[N_3,1]<b_{03}<-\sqrt{\frac{3+2a_{21}^2}{9a_{21}}}$;
$[1,1,-1,1,1,1]$	$\sqrt{\frac{3}{2}}<a_{21}\le \mathrm{R}[N_6,2]$, $-\sqrt{\frac{3+2a_{21}^2}{9a_{21}}}<b_{03}\le \mathrm{R}[N_4,1]$, or $a_{21}>\mathrm{R}[N_6,2]$, $-\sqrt{\frac{3+2a_{21}^2}{9a_{21}}}<b_{03}\le \mathrm{R}[N_5,2]$;
$[1,1,1,-1,1,1]$	$\mathrm{R}[N_8,2]<a_{21}\le \mathrm{R}[N_9,2]$, $\mathrm{R}[N_5,1]<b_{03}<\mathrm{R}[N_2,1]$, or $\mathrm{R}[N_9,2]<a_{21}<\mathrm{R}[N_{10},2]$, $-\sqrt{\frac{3+2a_{21}^2}{9a_{21}}}<b_{03}<\mathrm{R}[N_2,1]$, or $\mathrm{R}[N_{10},2]\le a_{21}<\sqrt{\frac{3}{2}}$, $-\sqrt{\frac{3+2a_{21}^2}{9a_{21}}}<b_{03}<\mathrm{R}[N_2,1]$, $b_{03}\ne \mathrm{R}[N_4,2]$, or $\sqrt{\frac{3}{2}}\le a_{21}\le \mathrm{R}[N_6,2]$, $\mathrm{R}[N_4,2]<b_{03}<\mathrm{R}[N_2,1]$, or $a_{21}>\mathrm{R}[N_6,2]$, $\mathrm{R}[N_4,3]<b_{03}<\mathrm{R}[N_2,1]$;
$[1,1,1,1,-1,1]$	$\mathrm{R}[N_{10},2]<a_{21}\le\sqrt{\frac{3}{2}}$, $\mathrm{R}[N_4,1]<b_{03}<\mathrm{R}[N_4,2]$, or $\sqrt{\frac{3}{2}}<a_{21}\le \mathrm{R}[N_6,2]$, $\mathrm{R}[N_3,2]<b_{03}<\mathrm{R}[N_4,2]$, or $a_{21}>[N_6,2]$, $\mathrm{R}[N_3,2]<b_{03}<\mathrm{R}[N_4,3]$;
$[1,-1,-1,1,1,1]$	$a_{21}>\mathrm{R}[N_6,2]$, $b_{03}\le \mathrm{R}[N_4,1]$, or $a_{21}>\sqrt{\frac{3}{2}}$, $b_{03}=-\sqrt{\frac{3+2a_{21}^2}{9a_{21}}}$;
$[1,1,1,-1,-1,1]$	$0<a_{21}\le \mathrm{R}[N_8,2]$, $-\sqrt{\frac{3+2a_{21}^2}{9a_{21}}}<b_{03}<\mathrm{R}[N_2,1]$, or $\mathrm{R}[N_8,2]<a_{21}<\mathrm{R}[N_9,2]$, $-\sqrt{\frac{3+2a_{21}^2}{9a_{21}}}<b_{03}<\mathrm{R}[N_5,1]$, or $\mathrm{R}[N_{10},2]\le a_{21}<\sqrt{\frac{3}{2}}$, $b_{03}=\mathrm{R}[N_6,1]$, or $\mathrm{R}[N_{10},2]<a_{21}\le \mathrm{R}[N_6,2]$, $b_{03}=\mathrm{R}[N_6,2]$, or $a_{21}>\mathrm{R}[N_6,2]$, $b_{03}=\mathrm{R}[N_6,3]$;
$[1,-1,-1,-1,1,1]$	$\mathrm{R}[N_9,2]<a_{21}<\mathrm{R}[N_{11},2]=1.20891$, $\mathrm{R}[N_5,1]<b_{03}\le-\sqrt{\frac{3+2a_{21}^2}{9a_{21}}}$, or $a_{21}=\mathrm{R}[N_{11},2]$, $\mathrm{R}[N_5,1]<b_{03}\le-\sqrt{\frac{3+2a_{21}^2}{9a_{21}}}$, $b_{03}\ne \mathrm{R}[N_3,1]$, or $\mathrm{R}[N_{11},2]<a_{21}$, $\mathrm{R}[N_5,1]<b_{03}<\mathrm{R}[N_3,1]$, or $\mathrm{R}[N_{11},2]<a_{21}<\sqrt{\frac{3}{2}}$, $\mathrm{R}[N_3,2]<b_{03}\le-\sqrt{\frac{3+2a_{21}^2}{9a_{21}}}$;
$[1,1,-1,-1,-1,1]$	$\sqrt{\frac{3}{2}}<a_{21}\le \mathrm{R}[N_6,2]$, $\mathrm{R}[N_4,1]<b_{03}\le \mathrm{R}[N_3,2]$, or $a_{21}>\mathrm{R}[N_6,2]$, $\mathrm{R}[N_4,2]<b_{03}\le \mathrm{R}[N_3,2]$;
$[1,-1,-1,-1,-1,1]$	$0<a_{21}<\mathrm{R}[N_9,2]$, $b_{03}\le-\sqrt{\frac{3+2a_{21}^2}{9a_{21}}}$, or $a_{21}=\mathrm{R}[N_9,2]$, $b_{03}<-\sqrt{\frac{3+2a_{21}^2}{9a_{21}}}$, or $\mathrm{R}[N_9,2]<a_{21}<\mathrm{R}[N_6,2]$, $b_{03}<\mathrm{R}[N_5,1]$, or $a_{21}>\mathrm{R}[N_6,2]$, $\mathrm{R}[N_4,1]<b_{03}<\mathrm{R}[N_5,1]$;
$[1,-1,-1,-1,0,0]$	$a_{21}=\mathrm{R}[N_9,2]$, $b_{03}=-\sqrt{\frac{3+2a_{21}^2}{9a_{21}}}$, or $\mathrm{R}[N_9,2]<a_{21}$, $b_{03}=\mathrm{R}[N_5,1]$;
$[-1,1,1,-1,-1,-1]$	$\mathrm{R}[N_7,1]<a_{21}<0$, $\mathrm{R}[N_2,1]<b_{03}<\mathrm{R}[N_4,1]$;
$[-1,1,1,1,1,-1]$	$a_{21}\le \mathrm{R}[N_7,1]$, $\mathrm{R}[N_2,1]<b_{03}<0$ and $b_{03}\ne\sqrt{\frac{2a_{21}(1+a_{21}^2)}{3}}$, or $\mathrm{R}[N_7,1]<a_{21}<0$, $\mathrm{R}[N_4,1]<b_{03}<0$ and $b_{03}\ne\sqrt{\frac{2a_{21}(1+a_{21}^2)}{3}}$;
$[-1,1,1,1,0,0]$	$a_{21}<0$, $b_{03}=\sqrt{\frac{2a_{21}(1+a_{21}^2)}{3}}$.

Clearly the origin of U_2 is not a singular point. In U_1 the first system of (3) has the form

$$u' =\frac{1}{2}(1-4a_{21}u^2+8b_{03}u^3+4u^4+4a_{21}^2u^4-6b_{03}u^3v-v^2+2a_{21}u^2v^2),$$
$$v' =uv(-a_{21}+3b_{03}u+2u^2+2a_{21}^2u^2-3b_{03}uv+a_{21}v^2). \tag{36}$$

The linear part of system (36) on $v=0$ is

$$\begin{pmatrix} 4uN_{12} & -3b_{03}u^3 \\ 0 & uN_{12} \end{pmatrix}, \tag{37}$$

where $N_{12}=-a_{21}+3b_{03}u+2u^2+2a_{21}^2u^2$. By computing the resultant of

$$g(u)=u'|_{v=0}=1-4a_{21}u^2+8b_{03}u^3+(4+4a_{21}^2)u^4 \tag{38}$$

TABLE 3. The conditions in order that the revised sign list (RSL) of (31) has no negative roots.

RSL	Conditions
$[-1,1,-1,1,1,1]$	$0<a_{21}<\mathrm{R}[N_{10},2]$, $\mathrm{R}[N_3,1]<b_{03}\le \mathrm{R}[N_4,1]$, or $\mathrm{R}[N_{10},2]\le a_{21}<\mathrm{R}[N_{11},2]$, $\mathrm{R}[N_3,1]<b_{03}<\mathrm{R}[N_4,3]$, or $\mathrm{R}[N_{11},2]\le a_{21}<\mathrm{R}[N_6,2]$, $\mathrm{R}[N_3,3]<b_{03}\le \mathrm{R}[N_4,3]$, or $a_{21}>\mathrm{R}[N_6,2]$, $\mathrm{R}[N_3,1]<b_{03}<\mathrm{R}[N_4,4]$;
$[-1,1,1,-1,1,1]$	$0<a_{21}\le \mathrm{R}[N_8,2]$, $\mathrm{R}[N_5,1]<b_{03}<\mathrm{R}[N_3,1]$, or $\mathrm{R}[N_{10},2]\le a_{21}<\mathrm{R}[N_{11},2]$, $\mathrm{R}[N_2,1]<b_{03}<\mathrm{R}[N_3,1]$, or $a_{21}\ge \mathrm{R}[N_{11},2]$, $\mathrm{R}[N_2,1]<b_{03}<\mathrm{R}[N_3,3]$;
$[-1,1,-1,-1,1,1]$	$0<a_{21}<\mathrm{R}[N_{11},2]$, $b_{03}=\mathrm{R}[N_3,1]$, or $a_{21}\ge \mathrm{R}[N_{11},2]$, $b_{03}=\mathrm{R}[N_3,3]$;
$[-1,1,1,-1,-1,1]$	$0<a_{21}<\mathrm{R}[N_8,2]$, $\mathrm{R}[N_2,1]<b_{03}<\mathrm{R}[N_5,1]$;
$[-1,1,-1,-1,-1,1]$	$0<a_{21}<\mathrm{R}[N_{10},2]$, $\mathrm{R}[N_4,1]<b_{03}<0$, or $\mathrm{R}[N_{10},2]\le a_{21}\le \mathrm{R}[N_6,2]$, $\mathrm{R}[N_4,3]<b_{03}<0$, or $a_{21}>\mathrm{R}[N_6,2]$, $\mathrm{R}[N_4,4]<b_{03}<0$,
$[-1,1,1,-1,0,0]$	$0<a_{21}<\mathrm{R}[N_8,2]$, $b_{03}=\mathrm{R}[N_5,1]$.

and uN_{12} with respect to the variable u we obtain the polynomial $-(1+a_{21}^2)N_2$. And the possible singular points in U_1 are nilpotent when $N_2=0$, or nodes when $N_2\neq 0$.

Now we shall determine the local phase portraits of the infinite singular points in the chart U_1. We need to find the real solutions in $g(u)=0$. But we will be able to determine the number and the type of the remaining infinite singular points using Theorems 4.2 and 2.4. Then we do not need to calculate explicitly the coordinates of these singular points.

Remark 4.3. *When $u<0$ the infinite singular points of the first system of (3) in U_1 are virtual points, but there are corresponding infinite singular points in V_1 with $u>0$ by the symmetry. And there are no infinite singular points in U_2 and V_2 in our cases. Hence we can study all real solutions in $g(u)=0$ to study the infinite singular points.*

We compute the sequence $\{\widetilde{d}_2,\widetilde{d}_4,\widetilde{d}_6,\widetilde{d}_8\}$ of $g(u)$ from (38), and have

$$\begin{aligned}\widetilde{d}_2&=64(1+a_{21}^2)^2,\\ \widetilde{d}_4&=1024(1+a_{21}^2)^2(2a_{21}+2a_{21}^3-3b_{03}^2),\\ \widetilde{d}_6&=-16384(1+a_{21}^2)^2(2a_{21}+2a_{21}^3+3b_{03}^2+a_{21}^2b_{03}^2),\\ \widetilde{d}_8&=-65536(1+a_{21}^2)^2N_2.\end{aligned}\tag{39}$$

We cannot find the parameter values such that the corresponding RSL be $[1,1,1,1]$, $[-1,-1,-1,-1]$, $[1,1,1,-1]$, $[1,-1,-1,-1]$, $[-1,-1,-1,1]$, $[1,-1,-1,-1]$, $[-1,-1,-1,0]$ or $[1,1,1,0]$, but we know that there are at most two distinct positive roots of (38), i.e., there are at most two infinite singular points in U_1.

(a) When the polynomial $g(u)$ has two distinct roots, we obtain that the possible RSL of $g(u)$ is $[1,1,-1,-1]$, whose condition is $b_{03}<\mathrm{R}[N_2,1]$, i.e.,

$$b_{03}<-\frac{\sqrt{2(-9a_{21}-8a_{21}^3+\sqrt{(3+4a_{21}^2)^3})}}{(3\sqrt{3})}.$$

Since $N_2\neq 0$, from (37) we have that the two remaining infinite singular points are two nodes in U_1. Since the piecewise differential systems (3) are symmetric with respect to the origin, systems (3) have two corresponding infinite singular points in V_1. On the other hand, systems (3) are continuous becasue $(f^+(x,0),g^+(x,0))=(f^-(x,0),g^-(x,0))$ in these systems.

(a.1) If $a_{21}<0$, the origin p_3 is a center. Hence on the Poincaré sphere the sum of the indices of the known singular points is 10. By Theorem 2.4, the sum of the indices of the remaining finite singular points must be -8. From the previous analysis, systems (3) have at most two finite singular points in $y>0$, which are different from p_j for $j=1,2,3$. Due to the symmetry with respect to the origin of coordinates the remaining finite singular points are four saddles p_j for $j=4,5,6,7$, where the two saddles p_4 and p_6 are in $y>0$, and two saddles p_5 and p_7 are in $y<0$. From (35) and (36) we have $u'|_{u=0}=-2(1+a_{21}^2)<0$ in U_2 and $u'|_{u=0}=2>0$ in

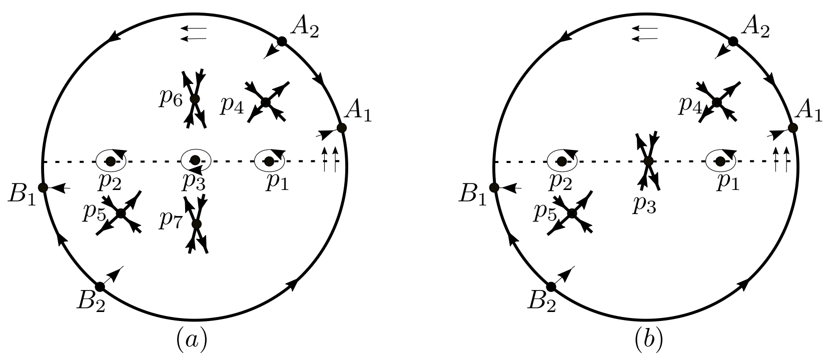


FIGURE 2. The local phase portraits at all finite and infinite singular points of the piecewise differential system (3) when $g(u)$ has two distinct roots.

U_1. Then we obtain that the local phase portraits at these singular points in the Poincaré disc are shown in Figure 2(a). Since the finite singular points of the piecewise differential systems (3) are saddles or centers there must be one saddle on the boundary of period annulus of the center.

(a.1.1) Assume that saddle p_6 is on the boundary of period annulus of center p_3, by the symmetry the saddle p_7 must be on the boundary of period annulus of the center p_3. If one repelling and one attracting separatrices of saddle p_4 of systems (3) connect with saddle p_5 and with the infinite singular point A_1 and A_2 of U_1 (see Figure 2(a)), respectively, then the saddles p_6 and p_7 are also on the boundary of the period annulus of the center p_1. By the symmetry we have that this phase portrait in the Poincaré disc is topologically equivalent to the phase portrait 1.1 of Figure 1, which for instance can be realized when $a_{21} = -1$ and $b_{03} = -2$.

(a.1.2) If the saddle p_4 is on the boundary of period annulus of the center p_1, creating a center-loop. By the symmetry the phase portrait of the piecewise differential systems (3) is topologically equivalent to the phase portrait 1.2 of Figure 1. This phase portrait for instance can be realized when $a_{21} = -1$ and $b_{03} = -5$.

(a.1.3) From the phase portraits 1.1 to 1.2 it follows by the continuity of the phase portraits with respect to the parameters that there must exist one phase portrait that the saddles p_4 and p_6 are on the boundary of period annulus of the center p_1. We have the phase portrait 1.3 of Figure 1.

(a.2) If $a_{21} \geq 0$ then the origin p_3 is a saddle. Hence on the Poincaré sphere the sum of the indices of the known singular points is 6. By Theorem 2.4 the sum of the indices of the remaining finite singular points must be -2. Hence the finite singular points other than p_j for $j = 1, 2, 3$, can be either two saddles, or four saddles and two centers. From the previous analysis we know that when $a_{12} \geq 0$ the piecewise differential systems (3) have at most one finite singular point in $y > 0$, see Tables 2 and 3. Hence the remaining finite singular points are two saddles p_4 and p_5, where p_4 is in $y > 0$ and p_5 is in $y < 0$. Then we obtain that the local phase portraits at these singular points in the Poincaré disc are shown in Figure 2(b).

(a.2.1) If only the saddle p_3 is on the boundary of the period annulus of the center p_1, taking into account the symmetry p_3 is also on the boundary of the period annulus of p_2, creating one eight-figure loop. Then one repelling and one attracting separatrices of the saddle p_4 of system (3) connect with the saddle p_5 and with the infinite singular point A_1 and A_2 of the local chart U_1 (see Figure 2(b)), respectively. Hence we have that this phase portrait in the Poincaré disc is topologically equivalent to the phase portrait 1.4 of Figure 1, which for instance can be realized when $a_{21} = 1$ and $b_{03} = -0.4$.

(a.2.2) The saddle p_4 on the boundary of period annulus of the center p_1 creates a center-loop. Then one repelling and one attracting separatrices of the saddle p_3 connect with the infinite singular point A_1 and A_2 of the local chart U_1. By the symmetry the phase portrait of the piecewise differential systems (3) is topologically equivalent to the phase portrait 1.5 of Figure 1. This phase portrait can be realized by taking $a_{21} = 1$ and $b_{03} = -1$.

(a.2.3) From the phase portraits 1.4 to 1.5 it follows by the continuity of the phase portraits with respect to the parameters that there must exist one phase portrait that the saddles p_3 and p_4 are on the boundary of the period annulus of the center p_1. Then this phase portrait is topologically equivalent to the phase portrait 1.6 of Figure 1.

(b) When the polynomial $g(u)$ has one distinct negative root, it shows that the possible RSL of $g(u)$ is $[1, 1, -1, 0]$, whose condition is $b_{03} = \mathrm{R}[N_2, 1]$. Since $N_2 = 0$, from (37) we have that the remaining infinite singular point in U_1 is nilpotent.

From the previous analysis we know that the infinite singular point are nilpotent. The linear part of the two systems of (3) at an infinite singular point $(u, 0)$ of the local chart U_1 is

$$\begin{pmatrix} 4u\left(2\left(a_{21}^2+1\right)u^2+3b_{03}u-a_{21}\right) & \mp 3b_{03}u^3 \\ 0 & u\left(2\left(a_{21}^2+1\right)u^2+3b_{03}u-a_{21}\right) \end{pmatrix},$$

where in $\mp 3b_{03}u^3$ we have the minus sign for the first system of (3) and the positive sign for the second system.

In order that the point $(u, 0) \in U_1$ be a nilpotent infinite singular point the two following equations must be satisfied:

$$(40) \qquad 4u\left(2\left(a_{21}^2+1\right)u^2+3b_{03}u-a_{21}\right)=0, \qquad 1-4a_{21}u^2+8^{b_{03}}u^3+4u^4+4a_{21}^2u^4=0.$$

The second equation comes from imposing that $(u, 0)$ be an infinite singular point in the local chart U_1 for both systems forming the piecewise differential system (3). This system has only one real solution

$$(u, b_{03}) = \left(\sqrt{\frac{3}{2\sqrt{4a_{21}^2+3}+2a_{21}}}, \frac{1}{3}\sqrt{\frac{2}{3\sqrt{4a_{21}^2+3}+3a_{21}}}\left(-2a_{21}^2+\sqrt{4a_{21}^2+3}a_{21}-3\right)\right).$$

Then the nilpotent infinite singular point is

$$(u, 0) = \left(\sqrt{\frac{3}{2\sqrt{4a_{21}^2+3}+2a_{21}}}, 0\right),$$

since its u-coordinate is positive it belongs to the first system of (3) defined in $y > 0$. Applying Theorem 3.5 of [16] such a nilpotent infinite singular point is formed by an elliptic sector and a hyperbolic sector, the hyperbolic sector has its two separatrices contained in the straight line of the infinity and its elliptic sector is outside the Poincaré disc, so it does not appear in the phase portrait of our piecewise differential system.

(b.1): $a_{21} < 0$. Then the origin is a center. From the previous analysis the piecewise differential systems (3) have at most two finite singular points, which are different from the three centers p_j for $j = 1, 2, 3$.

(b.1.1): We can assume that the remaining finite singular points are two saddles p_4 and p_5, where the saddle p_4 is in $y > 0$ and the saddle p_5 is in $y < 0$. These piecewise differential systems have the phase portrait 1.7 of Figure 1. For instance the piecewise differential system (3) with $a_{21} = -1$ and $b_{03} = -\frac{1}{9}\sqrt{1+\sqrt{7}}\left(5+\sqrt{7}\right)$ realizes such a phase portrait.

(b.1.2): Assume the remaining finite singular points are two centers p_4 and p_5, where p_4 is in $y > 0$ and p_5 is in $y < 0$. Then on the Poincaré sphere the sum of the indices of the known singular points is 10. By Theorem 2.4 the sum of the indices of the remaining two nilpotent infinite singular points must be -8. But the nilpotent singular point formed by one elliptic and one hyperbolic sector has index zero. In summary, it follows that the remaining finite singular points p_4 and p_5 cannot be two centers.

(b.2): $a_{21} \geq 0$. Then the origin p_3 is a saddle. Recall that p_1 and p_2 are centers, and the two nilpotent infinite singular points inside the Poincaré disc have a hyperbolic sector with their two separatrices contained in the straight line of the infinity. Then these piecewise differential systems have the phase portrait 1.8 of Figure 1. For instance the piecewise differential system (3) with $a_{21} = 1$ and $b_{03} = \left(\sqrt{7} - 5\right) \sqrt{2/(3\left(1 + \sqrt{7}\right))}/3$ realizes such a phase portrait.

(c) When the polynomial $-g(-u)$ has no distinct negative roots, its possible RSL of $g(u)$ are described in Table 4, i.e. $b_{03} > \mathrm{R}[N_2, 1]$. In this case we have that there are no infinite singular points in U_1 with $u > 0$.

TABLE 4. The conditions of the revised sign lists of $g(u)$ in (38) without real roots.

RSL	Conditions
$[1, -1, 1, 1]$	$a_{21} < 0,\ -\sqrt{\frac{-2a_{21}(1+a_{21}^2)}{3+a_{21}^2}} < b_{03} < 0;$
$[1, 1, -1, 1]$	$a_{21} \leq 0,\ \mathrm{R}[N_2, 1] < b_{03} < -\sqrt{\frac{2a_{21}(1+a_{21}^2)}{3}},$ or $a_{21} > 0,\ \mathrm{R}[N_2, 1] < b_{03} < 0;$
$[1, -1, -1, 1]$	$a_{21} < 0,\ -\sqrt{\frac{2a_{21}(1+a_{21}^2)}{3}} < b_{03} < -\sqrt{\frac{-2a_{21}(1+a_{21}^2)}{3+a_{21}^2}}.$

(c.1) If $a_{21} < 0$ then the origin p_3 is a center. Hence on the Poincaré sphere the sum of the indices of the known singular points is 6. By Theorem 2.4 the sum of the indices of the remaining finite singular points must be -4. From the previous analysis, when $a_{21} < 0$ the piecewise differential systems (3) have at most two finite singular points in $y > 0$, which are different from the p_1, p_2 and the origin. By the symmetry the remaining finite singular points are two saddles, where one saddle is in $y > 0$ and the other is in $y < 0$. Similarly to case (b.1.1) we obtain the phase portrait 1.9 of Figure 1, which is achieved when $a_{21} = -1$ and $b_{03} = -1$.

(c.2) If $a_{21} \geq 0$ then the origin p_3 is a saddle. And we know that when $a_{21} \geq 0$ the piecewise differential systems (3) have at most one finite singular point in $y > 0$, which is a center or a saddle. In fact, by the symmetry and the sum of the indices we obtain that the piecewise differential systems (3) have no other finite singular points different from the p_j, $j = 1, 2, 3$. Similarly to case (b.1.2) we obtain the phase portrait 1.10 of Figure 1, which can be realized when $a_{21} = 1$ and $b_{03} = -0.2$.

Thus we have obtained all the phase portraits of the $\mathbb{Z}_2$-equivariant cubic Hamiltonian systems (3) with a nilpotent bi-center, which are provided in Theorem 1.1. On the other hand we note that we can use these series of symbolic way to obtain the phase portraits of piecewise smooth systems having more complicated singular points.

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[1] School of Statistics and Mathematics, Guangdong University of Finance and Economics, Guangzhou, 510320, PR China

[2] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Email address: `chenting0715@126.com` (T.Chen),

Email address: `lism1983@126.com` (S.Li),

Email address: `jllibre@mat.uab.cat` (J.Llibre).

*Corresponding author