NON-EXISTENCE AND UNIQUENESS OF LIMIT CYCLES FOR A CLASS OF 3-DIMENSIONAL PIECEWISE LINEAR DIFFERENTIAL SYSTEMS

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Abstract. In this paper we study the non-symmetric limit cycles for a family of 3-dimensional piecewise linear differential systems with three zeros separated by two parallel planes. For a class of these differential systems we study the non-existence, existence and uniqueness of their limit cycles.

1. Introduction and statement of the main results

As we know, the continuous and discontinuous piecewise smooth differential systems play an important role inside many disciplines, such as control theory, electrical engineering, mechanics, biology and economics, see for instance the papers [1, 5, 12, 13, 15] and the references quoted there.

The maximum number of *limit cycles*, i.e. the periodic orbits isolated in the set of all periodic orbits, for differential systems is the second part of Hilbert's 16th problem. In the last decades there have been extensively studied on the limit cycles of continuous and discontinuous piecewise differential systems in $\mathbb{R}^2$, see [2, 3, 4, 6, 8, 9, 10, 11, 14, 16]. Recently the authors of [17, 18, 19, 21] considered the limit cycles in several cases of the 3-dimensional (3D) piecewise linear differential systems. Freire et al. [7] considered the birth of limit cycles in 3D piecewise linear systems for the relevant case of symmetrical oscillators with the three zones separated by two planes. Llibre et al. [20] studied a one-parameter family of symmetric 3D piecewise linear differential systems, and proved that it has at most 2 limit cycles. But there are no results about the non-symmetric limit cycles in 3D piecewise linear differential systems with three zones.

Consider the piecewise linear differential systems

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} \quad \text{in } P_+ \cup R, \\ \dot{\mathbf{x}} &= \mathbf{B}\mathbf{x} \quad \text{in } P_- \cup C \cup P_+, \\ \dot{\mathbf{x}} &= \mathbf{C}\mathbf{x} \quad \text{in } L \cup P_-, \end{aligned} \tag{1}$$

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in $\mathbb{R}^3$, so that we have $\mathbf{x}(t) = (x(t), y(t), z(t)) \in \mathbb{R}^3$. And systems (1) are divided into three zones L, C and R separated by the two parallel planes $P_\pm$, where

$$\begin{aligned} L &= \{(x,y,z) \in \mathbb{R}^3 : x < -1\}, \\ C &= \{(x,y,z) \in \mathbb{R}^3 : -1 < x < 1\}, \\ R &= \{(x,y,z) \in \mathbb{R}^3 : x > 1\}, \\ P_\pm &= \{(x,y,z) \in \mathbb{R}^3 : x = \pm 1\}. \end{aligned}$$

In this paper we consider a three–parametric family of piecewise linear differential systems (1), with one continuous parameter $\alpha \in \mathbb{R}$ and two discrete parameters r, $s \in \mathbb{Z}$. More precisely, we consider the three–parameter family of systems (1) with

$$\mathbf{A} = \begin{pmatrix} -6 & -1 & 0 \\ 11 & 0 & -1 \\ -6 & 0 & 0 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2}$$

and

$$\mathbf{C} = \begin{pmatrix} (1+r+s)\alpha & -1 & 0 \\ (r+s+rs)\alpha^2 & 0 & -1 \\ rs\alpha^3 & 0 & 0 \end{pmatrix}. \tag{3}$$

Therefore the eigenvalues of $\mathbf{A}$ are -1, -2 and -3, while the eigenvalues of $\mathbf{C}$ are α, $r\alpha$ and $s\alpha$.

The equilibrium points of systems (1)–(3) are the strip $\{(x,0,z)|x \in [-1,1]\}$ if $rs\alpha \neq 0$, and the strip $\{(x,0,z)|x \in [-1,1]\}$ union the half plane $\{(x,(1+r+s)\alpha x,(r+s+rs)\alpha^2 x)|x \in (-\infty,-1]\}$ if $rs\alpha = 0$. The differential system $\dot{\mathbf{x}} = \mathbf{Bx}$ has no limit cycles, because it is linear in its domain of definition $P_- \cup C \cup P_+$ and has no pure imaginary eigenvalues. Hence one limit cycle must intersect the three zones L, C and R, and has at least two points on the plane P_+.

Our main result is to study the non-symmetric limit cycles in the class of piecewise linear systems (1)–(3) by the algebraic approach, which is based on the closing equations for periodic orbits, for more details see [1]. In fact we determine for every value of $\alpha \in \mathbb{R}$ the non-existence and the uniqueness of limit cycles having two points in the plane P_+, as studied in the following results.

Theorem 1. *For $r = 2$ and $s = 3$ the piecewise linear differential systems* (1) *satisfying* (2) *and* (3) *have no limit cycles for all $\alpha \in \mathbb{R}$.*

Theorem 2. *For $r = 2$ and $s = -3$ the following statements hold for the piecewise linear differential systems* (1) *satisfying* (2) *and* (3)*:*

(a) If $\alpha < \alpha_{N_1} \approx 2.936$ systems (1) *have no limit cycles having exactly two points in the plane P_+.*

(b) If $\alpha_{N_1} < \alpha < \alpha_{N_2} = 3$ systems (1) *have one non-symmetric limit cycle having exactly two points in the plane P_+.*

(c) If $\alpha > \alpha_{N_2}$ systems (1) *have no limit cycles having exactly two points in the plane P_+.*

The above results are proved in Section 3, where algebraic expressions of determining the non-existence and uniqueness of the limit cycles are provided. From Theorem 2 we obtain that the non-symmetric limit cycles are organized in one branch which exist for different ranges of parameter α. However, this three–parametric family of piecewise linear differential systems (1) is not yet complete. We conjecture that in this bifurcation there also appear limit cycles for other values of r and s, as the following.

Conjecture *For all $\alpha \in \mathbb{R}$ and $(r-1)(s-1)(r-s) \neq 0$ for all $r, s \in \mathbb{Z}$ such that the non-symmetric piecewise linear differential systems* (1)–(3) *satisfying $\alpha = 0$ or $rs \geq 0$ have no limit cycles, and satisfying $rs < 0$ have a unique non-symmetric limit cycle.*

We note that if we fix the integers r and s then we can prove this conjecture. For instance in the above theorems we prove the conjecture for some cases $r = 2$ and $s = 3$, or $r = 2$ and $s = -3$. The arguments are given in that proof can be used for proving the conjecture for other values of r and s, but of course the computations increase when r and s increase. On the other hand, we can prove that systems (1)–(3) satisfying $(r-1)(s-1)(r-s) = 0$ have no limit cycles for all $\alpha \in \mathbb{R}$. Since solving the polynomial equations based on the closing equations becomes extremely complex in this case, we do not provide it here.

2. Preliminaries

In this section we provide the algebraic procedure which helps us to determine the limit cycles for the 3D piecewise linear differential systems (1)–(3).

The solution $(x(t), y(t), z(t))$ of systems (1) satisfying (2) in the region $P_+ \cup R$ starting at the point $(1, y_0, z_0)$ when $t = 0$ is

$$(x(t), y(t), z(t))^{\mathrm{T}} = e^{\mathbf{A}t}(1, y_0, z_0)^{\mathrm{T}}, \tag{4}$$

where

$$e^{\mathbf{A}t} = \frac{u}{2}\begin{pmatrix} 1-8u+9u^2 & (u-1)(-1+3u) & (u-1)^2 \\ (1-u)(27u-5) & -5+16u-9u^2 & (1-u)(3u-5) \\ 6(u-1)(3u-1) & 6(u-1)^2 & 2(3-3u+u^2) \end{pmatrix}, \tag{5}$$

and $u = e^{-t}$. Note that for all positive values of t we will have $0 < u < 1$.

Regarding the solution $(\mathrm{x}(\tau), \mathrm{y}(\tau), \mathrm{z}(\tau))$ of systems (1) satisfying (2) in the region $P_- \cup C \cup P_+$ starting at the point $(1, \mathrm{y}_0, \mathrm{z}_0)$ when $\tau = 0$, and it satisfies

$$(\mathrm{x}(\tau), \mathrm{y}(\tau), \mathrm{z}(\tau))^{\mathrm{T}} = e^{\mathbf{B}\tau}(1, \mathrm{y}_0, \mathrm{z}_0)^{\mathrm{T}}. \tag{6}$$

In other wards, any points $(1, \mathrm{y}_0, \mathrm{z}_0)$ of system (1) in P_+ will map to the point $(-1, \mathrm{y}_0, \mathrm{z}_0)$ in P_-.

While the solution $(X(T), Y(T), Z(T))$ of systems (1) satisfying (3) in the region $P_- \cup C$ starting at the point $(-1, Y_0, Z_0)$ when $T = 0$ satisfies

$$(X(T), Y(T), Z(T))^{\mathrm{T}} = e^{\mathbf{C}T}(-1, Y_0, Z_0)^{\mathrm{T}}. \tag{7}$$

Assuming that there exists one of the periodic orbits for systems (1)–(3) having two points in the plane P_+. Let $(1, y_0, z_0) \in P_+$ be the point where this periodic orbit exists the zone C enter the zone $P_+ \cup R$ and let $(1, Y_0, Z_0) \in P_+$ be the point where this periodic orbit enters into the zone C. Then this periodic orbit will enter into the zone $L \cup P_-$ through the point $(-1, Y_0, Z_0) \in P_-$. Let t be the time for this periodic orbit in going from the point $(1, y_0, z_0)$ to the point $(1, Y_0, Z_0)$, and

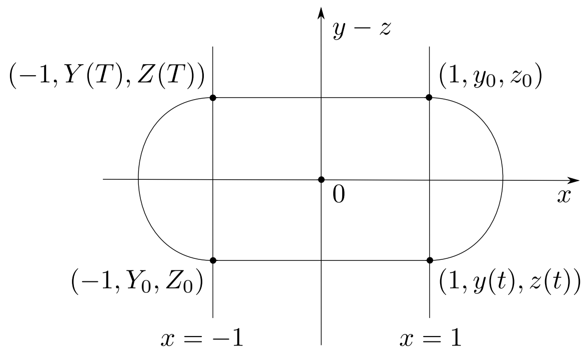

FIGURE 1. The periodic orbits passing through the points $(1, y_0, z_0)$ and $(1, Y_0, Z_0)$ in the plant P_+.

let T be the elapsed time for this periodic orbit to go from the point $(-1, Y_0, Z_0)$ to the point $(-1, y_0, z_0)$, see Figure 1. Hence we have the closing equations

$$(x(t), y(t), z(t)) = (1, Y_0, Z_0), \qquad (X(T), Y(T), Z(T)) = (-1, y_0, z_0). \tag{8}$$

The specific case $\alpha = 0$ of systems (1)–(3) will be considered with the above way. Now we assume $\alpha \neq 0$ and we can integrate backwards in time the solution from the point $(-1, y_0, z_0)$ to $(-1, Y_0, Z_0)$ within $L \cup P_-$, by defining

$$(\overline{X}(-T), \overline{Y}(-T), \overline{Z}(-T))^{\mathrm{T}} = e^{-\mathbf{C}T}(-1, y_0, z_0)^{\mathrm{T}}. \tag{9}$$

We have that the exponential $e^{-\mathbf{C}T}$ is the matrix

$$\frac{1}{(r-1)(r-s)(s-1)\alpha^2}\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \tag{10}$$

where

$$\begin{aligned}
b_{11} =&((r-s)v + r^2 v^r (s-1) + (1-r)s^2 v^s)\alpha^2, \\
b_{12} =&-((r-s)v + (s-1)rv^r + (1-r)sv^s)\alpha, \\
b_{13} =&(r-s)v + (s-1)v^r + (1-r)v^s, \\
b_{21} =&((r^2-s^2)v + (s^2-1)r^2 v^r + (1-r^2)s^2 v^s)\alpha^3, \\
b_{22} =&((s^2-r^2)v + (1-s^2)rv^r + (r^2-1)sv^s)\alpha^2, \\
b_{23} =&-((s^2-r^2)v + (1-s^2)v^r + (r^2-1)v^s)\alpha, \\
b_{31} =&((r-s)v + (s-1)rv^r + (1-r)sv^s)rs\alpha^4, \\
b_{32} =&-((r-s)v + (s-1)v^r + (1-r)v^s)rs\alpha^3, \\
b_{33} =&-((s-r)rsv + (1-s)sv^r + (r-1)rv^s)\alpha^2,
\end{aligned} \tag{11}$$

and $v = e^{-\alpha T}$. Hence we must have $(\overline{X}(-T), \overline{Y}(-T), \overline{Z}(-T)) = (-1, Y_0, Z_0)$.

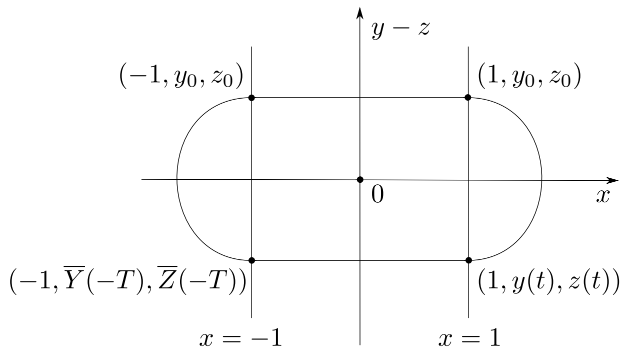

FIGURE 2. The periodic orbits passing through the points $(1, y_0, z_0)$ and $(1, Y_0, Z_0)$ in the plane P_+.

Therefore the periodic orbits that we are looking for must satisfy the following four equations

$$x(t)-1=0, \quad \overline{X}(-T)+1=0, \quad y(t)-\overline{Y}(-T)=0, \quad z(t)-\overline{Z}(-T)=0, \tag{12}$$

see Figure 2. Then we change the variables (t, T) by the variables (u, v), through $t=-\ln u>0$ and $T=-(\ln v)/\alpha>0$, the four equations (12) become

$$e_i(y_0, z_0, u, v)=0 \quad \text{for } i=1,2,3,4. \tag{13}$$

For every periodic orbits having two points $(1, y_0, z_0)$ and $(1, Y_0, Z_0)$ in the plane P_+ we can associate one solution (y_0, z_0, u, v) of equations (13) with $0<u<1$ and $v>1$ $(0<v<1)$ if $\alpha<0$ $(\alpha>0)$. Also since $\dot{x}=-6-y$ when $x=1$, the two conditions $-6-y_0>0$ and $-6-Y_0<0$ must be fulfilled. In fact the parameterization $x(u)$ is a cubic polynomial in its parameter, then the cubic equation $x(u)=1$ can not have four roots with y_0 and z_0 given. Thus these periodic orbits can not have at least other two points in P_+, i.e. they have two unique points in P_+. Similarly to Proposition 2 of [20] we give the following result to establish an equivalence relation between solutions of equations (13) and periodic orbits of systems (1)–(3) with two points in P_+.

Proposition 3. *Let (y_0, z_0, u, v) be one solution of corresponding equations* (13) *and let $Y_0=y(t)$, $Z_0=z(t)$ computed from* (4). *If $\alpha<0$ $(\alpha>0)$ and the four conditions $-6-y_0>0$,$-6-Y_0<0$, $0<u<1$ and $v>1$ $(0<v<1)$ are fulfilled, then systems* (1)–(3) *have a periodic orbit passing through the points $(1, y_0, z_0)$ and $(1, Y_0, Z_0)$ of P_+.*

3. The proof of Theorems 1 and 2

Now we study the non-existence, existence and uniqueness of limit cycles for the piecewise linear systems (1)–(3) in the cases $\alpha=0$ and $\alpha(r-1)(s-1)(r-s)\neq 0$, as the following subsections.

3.1. **The case $\alpha=0$.** For the solution $(x(t), y(t), z(t))$ of systems (1)–(2) in the region $P_+\cup R$ passing through the point $(1, y_0, z_0)$ with $y_0<-6$, we have that $\dot{x}=-6-y_0>0$. Then we remain for a time t in the right zone until arriving at $(1, Y_0, Z_0)$, where from (4) we have $(1, Y_0, Z_0)^{\mathrm{T}} = e^{\mathbf{A}t}(1, y_0, z_0)^{\mathrm{T}}$. The solution $(X(T), Y(T), Z(T))$ of systems (1) in the region $L \cup P_-$ starting at the point $(-1, Y_0, Z_0)$ is

$$\begin{pmatrix} X(T) \\ Y(T) \\ Z(T) \end{pmatrix} = e^{\mathbf{C_0}T} \begin{pmatrix} -1 \\ Y_0 \\ Z_0 \end{pmatrix} = \begin{pmatrix} 1 & -T & \frac{T^2}{2} \\ 0 & 1 & -T \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ Y_0 \\ Z_0 \end{pmatrix}. \tag{14}$$

Using the flow in the left zone and starting from $(-1, Y_0, Z_0)$, after a time T we have a periodic orbit if the equations

$$(1, Y_0, Z_0)^{\mathrm{T}} = e^{\mathbf{A}t}(1, y_0, z_0)^{\mathrm{T}}, \quad (-1, y_0, z_0)^{\mathrm{T}} = e^{\mathbf{C_0}T}(-1, Y_0, Z_0)^{\mathrm{T}} \tag{15}$$

hold. After eliminating $Y_0 = y_0 + z_0 T$ and $Z_0 = z_0$, by using the last two equations of (15), we obtain the algebraic equations

(16)
$$\begin{aligned}
&-2y_0 - Tz_0 = 0, \qquad 2 + u + 9u^2 + (-u + 3u^2)y_0 + (-u + u^2)z_0 = 0,\\
&-3u + 9u^2 + (-3u + 3u^2)y_0 + (1 - 2u + u^2)z_0 = 0,\\
&-5u + 32u^2 - 27u^3 - (2 + 5u - 16u^2 + 9u^3)y_0 - (2T + 5u - 8u^2 + 3u^3)z_0 = 0,
\end{aligned}$$

where $u = e^{-t}$. From the first equation of (16) we have $T = -2y_0/z_0$. Then we observe that the second and third equations of (16) are linear in the variables (y_0, z_0) and do not depend on T. And the determinant of this linear system is $-(u-1)^2 u \neq 0$, so it has one solution $y_0 = (2 - u + 5u^2)/((1-u)u)$ and $z_0 = 6(1+u^2)/(u-1)^2$. Therefore substituting T, y_0 and z_0 in the forth equations of (16) we have $4(u-1)^2/u = 0$, but this is in contradiction with $0 < u < 1$. Hence we get no solutions from the algebraic equations (16). This proves the parts of Theorems 1 and 2 when $\alpha = 0$ and the conjecture when $\alpha = 0$.

3.2. **The case $\alpha(r-1)(s-1)(r-s) \neq 0$.** Equivalently, we can get a dual approach to write the closing equations. We define the solution of systems (1)–(3) in the region $P_+ \cup R$ backwards in time starting from the point $(1, Y_0, Z_0)$ when $t = 0$ as

$$(x(t), y(t), z(t))^{\mathrm{T}} = e^{-\mathbf{A}t}(1, Y_0, Z_0)^{\mathrm{T}}, \tag{17}$$

where the matrix $e^{-\mathbf{A}t}$ has the same expression that $e^{\mathbf{A}t}$ in (5) if we change u by $U = 1/u = e^t$. Then we can take $\bar{y}(t) = y_0$ and $\bar{z}(t) = z_0$ to write the closing equations in the four unknowns (Y_0, Z_0, U, V)

$$\bar{x}(t) - 1 = 0, \quad X(T) + 1 = 0, \quad y(t) - Y(T) = 0, \quad z(t) - Z(T) = 0, \tag{18}$$

where $(X(T), Y(T), Z(T))^{\mathrm{T}} = e^{\mathbf{C}T}(1, Y_0, Z_0)^{\mathrm{T}}$. The matrix $e^{\mathbf{C}T}$ comes from $e^{-\mathbf{C}T}$ in (10) by $V = 1/v = e^{\alpha T}$. We will have $U > 1$ for all $t > 0$ and $0 < V < 1$ ($V > 1$) for $\alpha < 0$ ($\alpha > 0$) and $T > 0$.

Remark 4. *We can obtain four equations of the form $e_i(Y_0, Z_0, U, V) = 0$, which are exactly the same that in* (13) *if we take $U = 1/u = e^t$ and $V = 1/v = e^T$, from equations* (18). *Thus, if for $\alpha(r-1)(s-1)(r-s) \neq 0$ there exists a periodic orbit with two points $(1, y_0, z_0)$ and $(1, Y_0, Z_0)$ at P_+, then both (y_0, z_0, u, v) and (Y_0, Z_0, U, V) are solutions of equations* (13).

From Remark 4, note that if (u, v) are both different from zero and satisfy the polynomial systems (13) for a given (r, s, α), the same is true for the pair $(1/u, 1/v)$.

By doing the changes $u \to 1/U$ and $v \to 1/V$, we can get Y_0 and Z_0 from y_0 and z_0, respectively.

Now we compute the four equations $e_i(y_0, z_0, u, v)$ $(i = 1, 2, 3, 4)$ from (12). By removing the factor $(u-1)/2$ in the equation $e_1 = 0$, we obtain the new first equation $\widetilde{e}_1 = 0$, where

$$\widetilde{e}_1 = 2 + u + 9u^2 + (-u + 3u^2)y_0 + (-u + u^2)z_0. \tag{19}$$

The second equation, after multiplying it by $\alpha^2(r-1)(s-1)(r-s)$ becomes $\widetilde{e}_2 = 0$, where

$$\begin{aligned}\widetilde{e}_2 =&(r - s + r^2(s-1) - (r-1)s^2 + (-r+s)v - r^2(s-1)v^r \\ &+ (r-1)s^2v^s)\alpha^2 + ((-r+s)v - r(s-1)v^r + (r-1)sv^s)y_0\alpha \\ &+ ((r-s)v + (s-1)v^r + (1-r)v^s)z_0.\end{aligned} \tag{20}$$

To obtain a simplified third equation we define the polynomial $\widetilde{e}_3 = 3(r-s)\alpha(e_3 + (13-9u)\alpha\widetilde{e}_1/6 + 2(r+s)\widetilde{e}_2/((-1+r)(r-s)(-1+s)\alpha))$ that the equation becomes $\widetilde{e}_3 = 0$, where

$$\begin{aligned}\widetilde{e}_3 =& -\alpha(13(r-s) + 5(r-s)u + 6(r-s)u^2 + 3(-r^2+s^2)\alpha \\ &+ 3(r^2v^r - s^2v^s)\alpha) + ((s-r)u - 3rv^r + 3sv^s)y_0\alpha \\ &+ ((-r+s)u\alpha + (r-s)u^2\alpha + 3v^r - 3v^s)z_0.\end{aligned} \tag{21}$$

Finally to simplify the fourth equation we have

$$\begin{aligned}\widetilde{e}_4 =&(r-s)(3e_4 + (5-3u)\widetilde{e}_1 + 3rs\widetilde{e}_2/((r-1)(r-s)(s-1))) \\ =&10(r-s) + 8(r-s)u + 6(r-s)u^2 + (3r(r-s)s - 3rs(rv^r - sv^s))\alpha^2 \\ &+ (4(r-s)u - 3rs(v^r - v^s)\alpha)y_0 + (4(r-s)u + (-r+s)u^2 + 3sv^r \\ &- 3rv^s)z_0.\end{aligned} \tag{22}$$

We obtain that the above equations $\widetilde{e}_i$ $(i = 1, 2, 3, 4)$ are linear in y_0 and z_0. Furthermore $\widetilde{e}_1$ is a quadratic polynomial with respect to u and independent on v, $\widetilde{e}_2$ is the polynomial with respect to v and independent on u, while $\widetilde{e}_3$ and $\widetilde{e}_4$ are both the polynomials with respect to u and v without mixed terms of the form $u^m v^n$. From the two equations $\widetilde{e}_1 = \widetilde{e}_2 = 0$ we have

$$y_0 = \frac{F_1}{F_3}, \qquad z_0 = \frac{F_2}{F_3}, \tag{23}$$

where F_i $(i = 1, 2, 3)$ are polynomials in the variables u and v. Substituting y_0 and z_0 into $\widetilde{e}_3 = \widetilde{e}_4 = 0$ we obtain the polynomial expressions of $E_j(u, v)$ for $j = 3, 4$ in function of the variables u and v. Since we do not directly these expression in the proof, we do not list them here.

Note that since we are assuming that $\alpha(r-1)(s-1)(r-s) \neq 0$, $u = e^{-t}$ and $v = e^{-\alpha T}$ with $t > 0$ and $T > 0$, we have $u \in (0, 1)$ and $v \in (0, 1) \cup (1, +\infty)$. We claim that the polynomial system $E_3(u, v) = E_4(u, v) = 0$ in the open square $W = \{(u, v) : u \in (0, 1), v \in (0, 1) \cup (1, +\infty)\}$ with $F_3(u, v) \neq 0$ has at most one solution satisfying Proposition 3. If our claim holds than the conjecture will have a positive answer.

We have proved this claim for the following values of $(r,s)=(2,3),(2,4),(2,10),(3,4),(2,-3),(2,-4),(3,-2)$. Here we provide all the details for the case $(r,s)=(2,3)$ and $(r,s)=(2,-3)$, all the other cases follow in a similar way.

The proof of Theorem 1: The polynomials $E_3(u,v)$ and $E_4(u,v)$ for $(r,s)=(2,3)$ are

$$
\begin{aligned}
E_3 =&(1-v)v(-5u+11u^2+6u^4+2v^2+uv^2+9u^2v^2)-u(u-v)v(12+3u\\
&-5u^2+3v-14uv+15u^2v-5v^2+15uv^2)\alpha+(u-1)u(2u^2-5v+u^2v\\
&+11v^2+9u^2v^2+6v^4)\alpha^2,\\
E_4 =&6(u-1)u(1+u^2)(v-1)v+v(6u+11u^3-5u^4-18uv-33u^3v+15u^4v\\
&+6v^2+3uv^2+27u^2v^2-2v^3-uv^3-9u^2v^3)\alpha-u(-6u^2+2u^3-6v\\
&+18uv-3u^2v+u^3v-27u^2v^2+9u^3v^2-11v^3+33uv^3+5v^4-15uv^4)\alpha^2\\
&+6(u-1)u(v-1)v(1+v^2)\alpha^3,
\end{aligned}
$$

and the numerators and denominator of y_0 and z_0 in this case are

$$
\begin{aligned}
F_1 =&(2+u+9u^2)(1-v)v-(u-1)u(2+v+9v^2)\alpha^2,\\
F_2 =&-\alpha(2+u+9u^2)v(-1+3v)+u(-1+3u)(2+v+9v^2)\alpha^2,\\
F_3 =&uv((-1+3u)(v-1)+(u-1)(-1+3v)\alpha).
\end{aligned}
$$

By computing the resultant polynomial of the elimination of α from the equations $E_3=0$ and $E_4=0$, we get a long expression in the form $4(u-1)u^2(u-v)(v-1)v^2p(u,v)q(u,v)$, where $p(u,v)$ is

$$
\begin{aligned}
p(u,v)=&-(1-3v)^2v+u(-1+v-7v^2+9v^3)+9u^3(-1+v-7v^2+9v^3)\\
&+u^2(6+v(-7+48v-63v^2)),
\end{aligned}
$$

and $q(u,v)$ is given in Appendix 1. If $u=v$ we have

$$
\begin{aligned}
E_3 =&(v-1)v^2(-1+3v)(5+2v+5v^2)(-1+\alpha)(1+\alpha),\\
E_4 =&6(v-1)^2v^2(1+v^2)(1+\alpha)(1+\alpha^2).
\end{aligned}
$$

The equations $E_3=E_4=0$ have only one common solution $\alpha=-1$, but the denominator $F_3=0$ when $u=v$ and $\alpha=-1$. Then we need to analyze the solutions of $p(u,v)$ and $q(u,v)$ with (u,v) in the open rectangle W.

Proposition 5. *The number of solutions of the equations $p(u,v)=0$ or $q(u,v)=0$ in $W'=\{(u,v): u\in(0,1), v\in(0,+\infty)\}$ appears in Table 1.*

Proof. The following computations are based in the root determination of polynomials in only one variable. We can select any value of v in $(0,+\infty)$ and get the roots $u\in(0,1)$ of the corresponding polynomials $p(u,v)$ and $q(u,v)$. It is easy get that the roots of these polynomials are continuous complex functions. By varying v, the number of real roots of the polynomials $p(u,v)$ and $q(u,v)$ with $u\in(0,1)$ can change only at values of v where the curve $p(u,v)=0$ or $q(u,v)=0$ has a horizontal tangent, a branching point, or one solution escapes through the boundary of the rectangle W'.

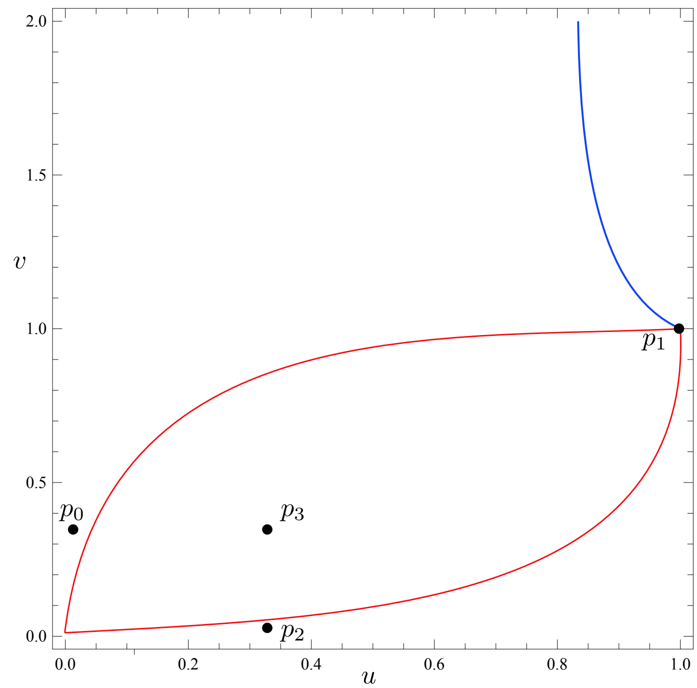

FIGURE 3. The zero level set of $p(u,v)q(u,v) = 0$ in the rectangle W'. Points p_0, p_1, p_2 are in the boundary of the open rectangle.

TABLE 1. Number of solutions of $p(u,v)q(u,v) = 0$ in the open rectangle W'.

Range	Solutions of $p(u,v)$	Solutions of $q(u,v)$	Periodic orbits
$0 < v < 1/3$	0	2	0
$v = 1/3$	2	2	0
$1/3 < v < 1$	0	2	0
$v = 1$	1	1	0
$1 < v$	1	0	0

Starting at the left edge of the rectangle W', the polynomial $p(0,v) = -v(-1+3v)^2$ has $v = 1/3$ as the unique solution. Hence we get the point p_0. In the right edge the situation is $p(1,v) = 2(v-1)(2+v+9v^2)$ has one solution $v = 1$, then we have the point p_1. Similar we have that $p(u,0) = -u(-1+3u)^2$ has $u = 1/3$ as the unique solution, which is a point p_2 of horizontal tangent for the curve $p(u,v) = 0$. We also obtain $p(u,1) = 2(u-1)(2+u+9u^2)$, which has no solution in W'.

Then we study other points belonging to the curve $p(u,v) = 0$ with horizontal tangent. Taking derivatives with respect to u in $p(u,v) = 0$, so that these points must be satisfied with

$$p(u,v)=0, \qquad p_u(u,v)=0 \tag{24}$$

with $p_v(u,v)\neq 0$. We compute the resultant of the polynomials $p(u,v)$ and $p_u(u,v)$ with respect to the variable v. And we have the following polynomial $900(-1+3u)^3(-1+9u+u^2+3u^3)^3$. The factor $-1+3u$ gives the values $v=0$ and $v=1/3$, but the first is not in W'. We obtain the point $p_3=(1/3,1/3)$ is a branching point, since then both p_u and p_v vanish. The other roots of $-1+9u+u^2+3u^3$ in $(0,1)$ are approximately 0.109346, but it has no values for v in the range W'. Hence, equations (24) have only one solution $(1/3,1/3)$.

On the other hand, at the rectangle W' we have the polynomials

$$\begin{aligned}
q(0,v)=&2v^7(9+v+2v^2)^2,\\
q(1,v)=&288(-1+v)^6(1+v)(1+v^2)(1+v+v^2),\\
q(u,0)=&2u^7(9+u+2u^2)^2,\\
q(u,1)=&288(u-1)^6(1+u)(1+u^2)(1+u+u^2).
\end{aligned}$$

Only the second polynomial has one solution $(1,1)$ in W'. Then we search for other points belonging to the curve $q(u,v)=0$ with horizontal tangent. Similarly the wanted points must be satisfied with

$$q(u,v)=0, \qquad q_u(u,v)=0 \tag{25}$$

with $q_v(u,v)\neq 0$. Then we compute the resultant of the polynomials $q(u,v)$ and $q_u(u,v)$ with respect to the variable v, and we have the following polynomial

$$\begin{aligned}
\text{Resultant}[q(u,v),q_u(u,v),v]=&A_1(u-1)^{16}u^{26}(9+u+2u^2)(5+2u+5u^2)^2\\
&\times(2+u+9u^2)(2+u+6u^2+u^3+2u^4)^2\\
&\times R_{24}^2(u)R_{88}(u),
\end{aligned} \tag{26}$$

where A_1 is an integer. The factors R_{24} and R_{88} have the degrees indicated in their subscripts and are not explicitly given for sake of brevity. The factors of (26) have no roots in $(0,1)$ except R_{88}. And the root of R_{88} is approximately 0.059872, which has no solutions for v in the range $(0,+\infty)$. Therefore, it has no solutions to satisfy the curves $q(u,v)=q_u(u,v)=0$.

The first three columns of Table 1 take intermediate values of v in the corresponding range and provide the number of solutions of $p(u,v)$ and $q(u,v)$ in $(0,1)$, respectively. □

Next we check the sign of α for such solutions. From the polynomials E_3 and E_4, we see that in the rectangle W' the value of α only can vanish in the line $v=1$. Thus the sign of α only can change if $v-1$ vanishes. Along the branch of solutions we have α being negative when $0<v<1$ and being positive when $v>1$, which is in contradiction with $v>1$ $(0<v<1)$ if $\alpha<0$ $(\alpha>0)$. Then we discard all branches for counting periodic solutions, i.e. systems (1)–(3) has no limit cycles when $r=2$ and $s=3$. Therefore we have concluded the proof of Theorem 1.

□

The proof of Theorem 2: The polynomials $\widetilde{E}_3(u,v)$ and $\widetilde{E}_4(u,v)$ for $(r,s)=(2,-3)$ are

$$\begin{aligned}\widetilde{E}_3 =&(1-v)(-5u+11u^2+6u^4+8v-6uv+58u^2v+12u^4v+6v^2-12uv^2\\&+60u^2v^2+18u^4v^2+4v^3-18uv^3+62u^2v^3+24u^4v^3+2v^4+uv^4+9u^2v^4)\\&-u(-18+18u-9u^2+15u^3-10v-6uv-9u^2v+15u^3v-15v^2+9uv^2\\&-9u^2v^2+15u^3v^2-15v^3+9uv^3-9u^2v^3+15u^3v^3-45v^4+39uv^4-24u^2v^4\\&+40u^3v^4+3v^5-9uv^5)+(u-1)u(v-1)(-9+9u^2-30v+18u^2v-21v^2\\&+27u^2v^2-12v^3+16u^2v^3+2v^4)\alpha^2,\end{aligned}$$

$$\begin{aligned}\widetilde{E}_4 =&6(u-1)u(1+u^2)(v-1)(1+2v+3v^2+4v^3)+(-18u-33u^3+15u^4+16v\\&-10uv+72u^2v-33u^3v+15u^4v+6v^2-15uv^2+27u^2v^2-33u^3v^2+15u^4v^2\\&+6v^3-15uv^3+27u^2v^3-33u^3v^3+15u^4v^3+6v^4-45uv^4+27u^2v^4-88u^3v^4\\&+40u^4v^4+6v^5+3uv^5+27u^2v^5)\alpha+u(1-v)(2-6u-27u^2+9u^3-12v\\&+36uv-54u^2v+18u^3v-21v^2+63uv^2-81u^2v^2+27u^3v^2-30v^3+90uv^3\\&-48u^2v^3+16u^3v^3-9v^4+27uv^4)\alpha^2+6(1-u)u(v-1)(1+6v+6v^2+6v^3\\&+v^4)\alpha^3,\end{aligned}$$

and the numerators and denominator of y_0 and z_0 in this case are

$$\begin{aligned}F_1 =&(1-v)((2+u+9u^2)(1+2v+3v^2+4v^3)\\&+(u-1)u(9+18v+27v^2+16v^3)\alpha^2),\\F_2 =&-\alpha(2+u+9u^2)(3+3v+3v^2+3v^3+8v^4)\\&+u(3u-1)(v-1)(9+18v+27v^2+16v^3)\alpha^2,\\F_3 =&u(3u-1)(v-1)(1+2v+3v^2+4v^3)\\&+(u-1)u(3+3v+3v^2+3v^3+8v^4)\alpha.\end{aligned}$$

The calculation of the resultant of the polynomials $\widetilde{E}_3$ and $\widetilde{E}_4$ with respect to the parameter α yields $4(u-1)u^2(-1+v)^2P(u,v)Q(u,v)$ where $P(u,v)$ is

(27)
$$\begin{aligned}P(u,v)=&-(3+3v+3v^2+3v^3+8v^4)^2+(9+18v+27v^2+26v^3+15v^4\\&+6v^5-3v^6+38v^7+64v^8)u+(-63-126v-189v^2-192v^3-165v^4\\&-102v^5-39v^6-276v^7-448v^8)u^2+9(9+18v+27v^2+26v^3+15v^4\\&+6v^5-3v^6+38v^7+64v^8)u^3\end{aligned}$$

and $Q(u,v)$ is given in Appendix 1. Similarly to the proof of Theorem 1 we consider the solutions of $P(u,v)$ and $Q(u,v)$ with (u,v) in the open rectangle W.

Proposition 6. *The number of solutions of the equation $P(u,v)=0$ or $Q(u,v)=0$ in $W'=\{(u,v):u\in(0,1),v\in(0,+\infty)\}$ is as displayed in Table 2. If $r=2$ and $s=-3$ the number of limit cycles of system (1)–(3) with exactly 2 points in the plane P_+ is as indicated in last column of Table 2.*

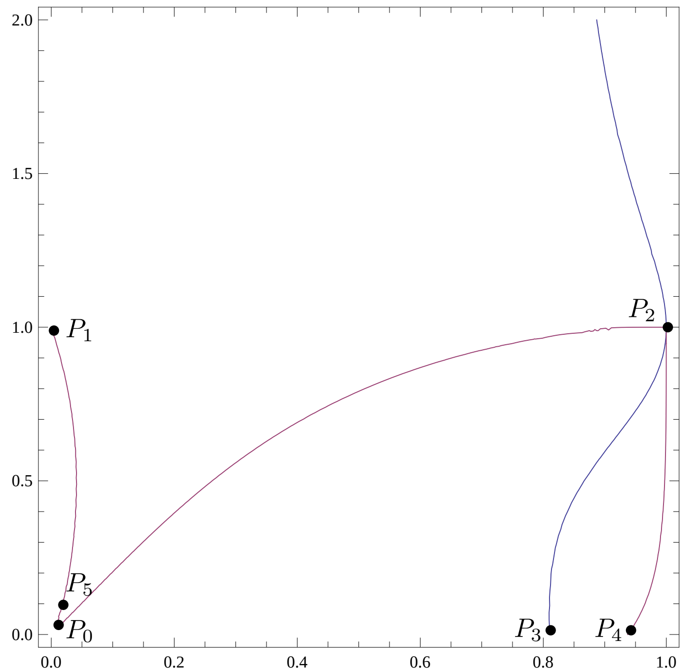

FIGURE 4. The zero level set of $P(u,v)Q(u,v)=0$ in the rectangle W'.

TABLE 2. Number of solutions of $P(u,v)Q(u,v)=0$ in the open rectangle W.

Range	Solutions of $P(u,v)$	Solutions of $Q(u,v)$	Periodic orbits
$0<v\le v_5$	1	3	1
$v_5<v<1$	1	3	0
$v=1$	1	2	0
$1<v$	1	0	0

Proof. At the left edge of the rectangle W' the polynomial $P(0,v)=-(3+3v+3v^2+3v^3+8v^4)^2$ has no solutions. For the right edge the situation we get that $P(1,v)=2(-1+v)^2(1+2v+3v^2+4v^3)(9+18v+27v^2+16v^3)$ has a double root $v=1$, hence we have the point P_2. Similar we obtain that $P(u,0)=9(-1+u-7u^2+9u^3)$ has $u=0.809964$ as the unique solution, it corresponds the point P_3. We also obtain $P(u,1)=200(-1+u)(2+u+9u^2)$, which has no real solutions in W'.

Now we study the other points belonging to the curve $P(u,v)=0$ with horizontal tangent. Taking derivatives with respect to u in $P(u,v)=0$, so that the points must satisfy with $P(u,v)=0$, $P_u(u,v)=0$ and $P_v(u,v)\neq 0$. We compute the resultant of the polynomials $P(u,v)$ and $P_u(u,v)$ with respect to the variable v, then we have $A_2(3u-1)^8(-1+9u+u^2+3u^3)^8$ where A_2 is an integer. The two factors give no corresponding values v in W'.

Next we consider the solutions of $Q(u,v)$ at the rectangle W' and have

$$\begin{aligned}
Q(0,v) &= -2(-1+v)v^3(4+3v+2v^2+v^3)(16+27v+18v^2+9v^3)^2,\\
Q(1,v) &= -2880(-1+v)^6(1+v)(1+v+v^2)(1+6v+6v^2+6v^3+v^4),\\
Q(u,0) &= -u^4(-2704-2587u-8173u^2+4011u^3+5232u^4+4149u^5\\
&\qquad +2367u^6+423u^7+162u^8),\\
Q(u,1) &= -24000(-1+u)^6u(5+2u+5u^2)^2.
\end{aligned}$$

The first polynomial $Q(0,v)$ has two solutions $v=1$ and $v=0$, the corresponding point $P_1=(0,1)$ is in W' but $P_0=(0,0)$ is not in W'. The second polynomial $Q(1,v)$ has only one solution $v=0$ in W', while the solution of the third one corresponds the point $P_4=(0.937203,0)$ in W'. And the polynomial $Q(u,1)$ has no solution in W'.

Then we study the points belonging to the curve $Q(u,v)=0$ with horizontal tangent. Taking derivatives with respect to u in $Q(u,v)=0$, so that the remaining points must satisfy with

$$Q(u,v)=0,\ Q_u(u,v)=0 \text{ and } Q_v(u,v)\neq 0. \tag{28}$$

To study the solutions of (28) we compute the resultant of $Q(u,v)$ and $Q_u(u,v)$ with respect to the variable v. And we have the following polynomial $A_3(u-1)^{18}u^7(5+2u+5u^2)R_{54}^2(u)R_{144}(u)$ where A_3 is an integer. The polynomials R_{54} and R_{144} have the degrees indicated in their subscripts. The root of R_{54} in $(1,0)$ are approximately 0.208334, 0.294286 and 0.840437. We must guarantee that the corresponding point is in W' for these values of u. After checking these values we conclude that it has no points of the curve $Q(u,v)$ with horizontal tangent in W'. The root of R_{144} in $(1,0)$ are approximately 0.122432, 0.283114, 0.389696 and 0.399156. Similar we obtain no points of the curve $Q(u,v)$ with horizontal tangent in W'.

Now we have to check whether all the solutions of Proposition 6 satisfy with Proposition 3 to guarantee that a solution of $P(u,v)Q(u,v)$ corresponds to a periodic orbit with only two points in P_+. First we check the sign of α for such solutions. From the polynomials $\widetilde{E}_3$ and $\widetilde{E}_4$, we see that the common factor of terms of degree zero in α is $v-1$. Hence, for a branch of solutions the sign of α only can change if this factor vanishes. Accordingly, in the rectangle W' the value of α only can vanish in the line $v=1$. Along the branch of solutions from P_0 to P_1 we have α being positive. And we have that α being negative for all the branches from P_0 to P_2, P_2 to P_3 and P_2 to P_4. On the other hand, α being positive for all the branch from P_2 to finite. In short only the branch from P_0 to P_1, the sign of α is consistent with $v>1$ $(0<v<1)$ if $\alpha<0$ $(\alpha>0)$.

To prove the second statement of Proposition 6 we need to show the inequalities $-6-y_0>0$. We start by looking for possible values of (u,v) in the branch from P_0 to P_1, where such inequality is not true any longer, that is

$$\begin{aligned}
H(u,v,\alpha) &= -(2+9u)(v-1)(1+2v+3v^2+4v^3)-6u(3+3v+3v^2+3v^3\\
&\qquad +8v^4)\alpha+u(v-1)(9+18v+27v^2+16v^3)\alpha^2.
\end{aligned} \tag{29}$$

Computing the resultants of the polynomials $\widetilde{E}_3$ and H, $\widetilde{E}_4$ and H, respectively, then we have the following polynomials

$$\begin{aligned} h_1(u,v) =& \text{Resultant}[\widetilde{E}_3, H, \alpha] = 2u^2(-1+v)^2 P(u,v) M_1(u,v),\\ h_2(u,v) =& \text{Resultant}[\widetilde{E}_4, H, \alpha] = 2u^2(-1+v)^2(1+2v+3v^2+4v^3) P(u,v) M_2(u,v), \end{aligned} \tag{30}$$

where the expression of $P(u,v)$ is as displayed in (27) and

$$\begin{aligned} M_1(u,v) =& -2(v-1)^4(3+4v+3v^2)^2 + u(414+4824v+13824v^2+21748v^3\\ &+22685v^4+14616v^5+6106v^6+636v^7-153v^8)+4u^2(72+432v\\ &+1071v^2+1880v^3+2110v^4+1488v^5+623v^6+24v^7)-6u^3(18\\ &+120v+297v^2+520v^3+580v^4+392v^5+165v^6+8v^7)-4u^4(1+2v\\ &+3v^2+4v^3)(9+18v+27v^2+16v^3)+u^5(1+2v+3v^2+4v^3)(9+18v\\ &+27v^2+16v^3),\\ M_2(u,v) =& 4(v-1)(-2+12v+21v^2+30v^3+9v^4)^2+12u(-48+624v+2070v^2\\ &+2690v^3+1789v^4+1143v^5+1476v^6+2460v^7+1845v^8+351v^9)\\ &+3u^2(-3168-1440v+19728v^2+31856v^3+27109v^4-2817v^5\\ &-5970v^6+762v^7+16209v^8+4131v^9)+6u^4(-162-486v-1377v^2\\ &-1515v^3+840v^4+5112v^5+6671v^6+4597v^7+720v^8)-12u^5(v\\ &-1)(9+18v+27v^2+16v^3)^2+u^6(v-1)(9+18v+27v^2+16v^3)^2. \end{aligned}$$

It is easy to get that $P(u,v)\neq 0$ on the branch P_0-P_1, see Figure 4. By computing the resultant we eliminate v between polynomials M_1 and M_2, then we obtain a polynomial in u whose roots in the interval W' are 0.00483442, 0.0163653, 0.0995178, 0.194119. Going back with these values to $M_1=0$ and $M_2=0$, we obtain the points $(0.004834, 0.221817)$, $(0.016365, 0.087319)$, $(0.099517, -1.342034)$ and $(0.194119, -0.860593)$. Clearly, only the second point $P_5=(0.016365, 0.087319)$ is in W' being in fact on the branch P_0-P_1. The conclusion is that, by continuity, the sign of $H(u,v,\alpha)$ changes along the branch P_0-P_1.

We select $v=0.05$ and have the values of $u\approx 0.010455$ and $\alpha\approx 2.96596$ for the corresponding point on the branch P_0-P_1, so that the value of $-6-y_0$ is approximately 8.55439. Next we select $v=0.1$ and compute the values of $u\approx 0.018116$ and $\alpha\approx 2.943606$ for the corresponding point, then we obtain the value of $-6-y_0\approx -1.45632$. Then we choose $v=0.3$ and obtain the values of $u\approx 0.035656$ and $\alpha\approx 3.00209$ for the corresponding point, and the value of $-6-y_0$ is approximately -8.26739. Hence we conclude from Proposition 3 that all the points of the branch P_0-P_5 correspond to periodic orbits of systems (1)–(3). □

We consider the branch from P_0 to P_5. By continuity the value of α reaches a maximum positive value and a minimum positive value. These values correspond to α_{N_2} and α_{N_1}, respectively, which can be calculated by standard procedures with as much accuracy as possible. Hence the proof of Theorem 2 is complete. □

4. Appendix 1

The polynomials $q(u,v)$:

$$
\begin{aligned}
q(u,v) =& 2u^7(9+u+2u^2)^2 + u^5(855-1626u+1156u^2-1592u^3+225u^4-178u^5\\
&+8u^6)v + u^3(-2700+420u-1986u^2+1707u^3+1510u^4+1796u^5\\
&+682u^6+225u^7+74u^8)v^2 + 2u^2(-1350+5565u-4791u^2+11352u^3\\
&-10267u^4+5941u^5-7290u^6+898u^7-796u^8+18u^9)v^3 + 2u^2(210\\
&-4791u-43u^2-6811u^3+5518u^4-862u^5+5941u^6+755u^7+578u^8\\
&+81u^9)v^4 - u(-855+1986u-22704u^2+13622u^3-32762u^4+31872u^5\\
&-11036u^6+20534u^7-1707u^8+1626u^9)v^5 + u(-1626+1707u\\
&-20534u^2+11036u^3-31872u^4+32762u^5-13622u^6+22704u^7\\
&-1986u^8+855u^9)v^6 + 2(81+578u+755u^2+5941u^3-862u^4+5518u^5\\
&-6811u^6-43u^7-4791u^8+210u^9)v^7 - 2(-18+796u-898u^2+7290u^3\\
&-5941u^4+10267u^5-11352u^6+4791u^7-5565u^8+1350u^9)v^8 + (74\\
&+225u+682u^2+1796u^3+1510u^4+1707u^5-1986u^6+420u^7\\
&-2700u^8)v^9 + (8-178u+225u^2-1592u^3+1156u^4-1626u^5\\
&+855u^6)v^{10} + 2(2+u+9u^2)^2v^{11}
\end{aligned}
$$

$$
\begin{aligned}
Q(u,v) =& -2(v-1)v^3(4+3v+2v^2+v^3)(16+27v+18v^2+9v^3)^2 - uv^2(12352\\
&+73648v+136805v^2+171365v^3+138730v^4+68846v^5+16704v^6\\
&-7980v^7-8490v^8-2430v^9+27v^{10}+423v^{11}) - u^2v(-16640-129584v\\
&-332928v^2-578585v^3-731935v^4-677770v^5-458242v^6-206364v^7\\
&-36240v^8+16410v^9+17010v^{10}+12501v^{11}+2367v^{12}) - u^3v(12432\\
&+210790v+812054v^2+1522535v^3+1975495v^4+1684186v^5+976910v^6\\
&+302672v^7-26860v^8-49120v^9-11268v^{10}+2025v^{11}+4149v^{12})\\
&-2u^4(-1352-42764v-264806v^2-679324v^3-1124475v^4-1362011v^5\\
&-1231697v^6-831833v^7-414017v^8-117325v^9-13037v^{10}+15561v^{11}\\
&+16464v^{12}+2616v^{13}) + u^5(2587-65653v-571412v^2-1665416v^3\\
&-2831825v^4-3434849v^5-2891784v^6-1905256v^7-1171223v^8-864575v^9\\
&-697988v^{10}-375888v^{11}-82707v^{12}-4011v^{13}) + u^6(1+v)(8173+92982v\\
&+492568v^2+968910v^3+1253915v^4+1288284v^5+1150336v^6+1288284v^7\\
&+1253915v^8+968910v^9+492568v^{10}+92982v^{11}+8173v^{12}) + u^7(-4011\\
&-82707v-375888v^2-697988v^3-864575v^4-1171223v^5-1905256v^6\\
&-2891784v^7-3434849v^8-2831825v^9-1665416v^{10}-571412v^{11}\\
&-65653v^{12}+2587v^{13}) + 2u^8(-2616-16464v-15561v^2+13037v^3
\end{aligned}
$$

$$+ 117325v^4 + 414017v^5 + 831833v^6 + 1231697v^7 + 1362011v^8 + 1124475v^9 + 679324v^{10} + 264806v^{11} + 42764v^{12} + 1352v^{13}) + u^9(-4149 - 2025v + 11268v^2 + 49120v^3 + 26860v^4 - 302672v^5 - 976910v^6 - 1684186v^7 - 1975495v^8 - 1522535v^9 - 812054v^{10} - 210790v^{11} - 12432v^{12}) + u^{10}(-2367 - 12501v - 17010v^2 - 16410v^3 + 36240v^4 + 206364v^5 + 458242v^6 + 677770v^7 + 731935v^8 + 578585v^9 + 332928v^{10} + 129584v^{11} + 16640v^{12}) + u^{11}(-423 - 27v + 2430v^2 + 8490v^3 + 7980v^4 - 16704v^5 - 68846v^6 - 138730v^7 - 171365v^8 - 136805v^9 - 73648v^{10} - 12352v^{11}) + 2u^{12}(v-1)(1 + 2v + 3v^2 + 4v^3)(9 + 18v + 27v^2 + 16v^3)^2.$$

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References

[1] A. Andronov, A. Vitt, S. Khaikin, *Theory of Oscillations*, Pergamon Press, Oxford, 1966.
[2] D. C. Braga, L. F. Mello, *Limit cycles in a family of discontinuous piecewise linear differential systems with two zones in the plane*, Nonlinear Dynam. 73 (2013) 1283–1288.
[3] C. Buzzi, C. Pessoa, J. Torregrosa, *Piecewise linear perturbations of a linear center*, Discrete Contin. Dyn. Syst. 9 (2013) 3915–3936.
[4] T. Chen, L. Huang, P. Yu, W. Huang, *Bifurcation of limit cycles at infinity in piecewise polynomial systems*, Nonlinear Anal.: Real World Appl. 41 (2018) 82–106.
[5] S. Coombes, *Neuronal networks with gap junctions: A study of piecewise linear planar neuron models*, SIAM Appl. Dyn. Syst. 7 (2008) 1101–1129.
[6] E. Freire, E. Ponce, F. Rodrigo, F. Torres, *Bifurcation sets of continuous piecewise linear systems with two zones*, Int. J. Bifurcation Chaos 8 (1998) 2073–2097.
[7] E. Freire, E. Ponce, J. Ros, *The focus-center-limit cycle bifurcation in symmetric 3D piecewise linear systems*, SIAM J. Appl. Math. 65 (2005) 1933–1951.
[8] E. Freire, E. Ponce, F. Torres, *Canonical discontinuous planar piecewise linear systems*, SIAM J. Appl. Dyn. Syst. 11 (2012) 181–211.
[9] E. Freire, E. Ponce, F. Torres, *A general mechanism to generate three limit cycles in planar Filippov systems with two zones*, Nonlinear Dynam. 78 (2014) 251–263.
[10] F. Giannakopoulos, K. Pliete, *Planar systems of piecewise linear differential equations with a line of discontinuity*, Nonlinearity 14 (2001) 1611–1632.
[11] M. Han, W. Zhang, *On Hopf bifurcation in non-smooth planar systems*, J. Differential Equations 248 (2010) 2399–2416.
[12] S. Lefschetz, *Stability of Nonlinear Control Systems*, Academic Press, New York, 1965.
[13] R. Leine, H. Nijmeijer, *Dynamics and Bifurcations of Non-Smooth Mechanical Systems*, Lect. Notes Appl. Comput. Mech., vol.18, Springer-Verlag, Berlin, 2004.
[14] S. Huan, X. Yang, *Existence of limit cycles in general planar piecewise linear systems of saddle-saddle dynamics*, Nonlinear Anal. 92 (2013) 82–85.
[15] M. Kunze, T. Kupper, *Qualitative bifurcation analysis of a non-smooth friction oscillator model*, Math. Phys. 48 (1997) 87–101.

[16] J. Llibre, M. Ordóñez, E. Ponce, *On the existence and uniqueness of limit cycles in a planar piecewise linear systems without symmetry*, Nonlinear Anal.: Real World Appl. 14 (2013) 2002–2012.

[17] J. Llibre, E. Ponce, *Global first harmonic bifurcation diagram for odd piecewise linear control systems*, Dyn. Stab. Syst. 11 (1996) 49–88.

[18] J. Llibre, E. Ponce, J. Ros, F. Torres, *On the fold-Hopf bifurcation for continuous piecewise differential systems with symmetry*, Chaos 20 (2010) 033119, pp13.

[19] J. Llibre, E. Ponce, A. Teruel, *Horseshoes near homoclinic orbits for piecewise linear differential systems in* R^3, Int. J. Bifurcation Chaos 17 (2007) 1171–1184.

[20] J. Llibre, E. Ponce, J. Ros, *Algebraic determination of limit cycles in a family of three-dimensional piecewise linear differential systems*, Nonlinear Anal. 74 (2011) 6712–6727.

[21] E. Ponce, J. Ros, *On periodic orbits of 3D symmetric piecewise linear systems with real triple eigenvalues*, Int. J. Bifurcation Chaos 19 (2009) 2391–2399.

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