Limit cycles on piecewise smooth vector fields with coupled rigid centers

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We provide an upper bound for the maximum number of limit cycles of the classes of discontinuous piecewise differential systems formed by two differential systems separated by a straight line presenting rigid centers. These two rigid centers are polynomial differential systems with a linear part and a nonlinear homogeneous part. We study the maximum number of limit cycles that such a class of piecewise differential systems can exhibit.

Keywords: Piecewise smooth vector field; rigid centers; limit cycle.

1. Introduction and main results

The search for limit cycles is one of the most important studies in the qualitative theory of the planar ordinary differential equations. Such importance is evidenced by the 16th Hilbert's problem (see [Hilbert, 1902]) that seeks the determination of an upper bound for the number of limit cycles for the class of planar polynomial vector fields of degree n, a problem that remains unsolved for $n \geq 2$. Recently the study of the limit cycles also is relevant in the discontinuous piecewise differential systems.

In the last decades the study of discontinuous piecewise vector fields has had a great growth in the mathematical community, since such vector fields can be used as important models in applied science.

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Indeed, several models used in applied problems are described by systems that are not completely differentiable, but in different parts, where a law of evolution is suddenly interrupted by another law of evolution that will begin to govern such system. The modeling of such systems consists of different vector fields defined in distinct regions separated by a switching manifold and are known as piecewise smooth vector fields, discontinuous piecewise systems, or Filippov systems.

Pioneering studies initiated by Andronov [Andronov *et al.*, 1966] and Filippov [Filippov, 1988] led to a theoretical foundation for this kind of differential systems. Nowadays a vast literature on these vector fields is available. See for instance [di Bernardo *et al.*, 2008] for the main theory and some applications, [Chua *et al.*, 1986] for applications in electrical circuits, [Brogliato, 1999; Leine & Nijmeijer, 2004] for applications in mechanical models, [di Bernardo *et al.*, 2001; Jacquemard & Tonon, 2012] for applications in relay systems, among others. As in the regular case the study of the existence and location of limit cycles in piecewise smooth vector fields is also of great importance, because in addition to the smooth dynamic elements, there are new ones which did not exist in the smooth world.

Let $p \in \mathbb{R}^2$ be a singularity of an analytic differential system in the plane. The singularity p is a *center* if there exists an open neighborhood U of p such that all the solutions in $U \setminus \{p\}$ are periodic. Without loss of generality we may assume that the equilibrium point is at the origin. Denote by $\mathcal{T}_q$ the period of the periodic orbit through $q \in U \setminus \{p\}$. We say that p is an *isochronous center* if $\mathcal{T}_q$ is constant for all $q \in U \setminus \{p\}$. An isochronous center is *uniform* or *rigid* if the angular velocity of the vector field is the same for all periodic orbits in $U \setminus \{p\}$, that is if in polar coordinates (r, θ) defined by $x = r\cos\theta$, $y = r\sin\theta$ it can be written as $\dot{r} = G(r, \theta)$, $\dot{\theta} = k$, where $k \neq 0$ is a constant. See [Chavarriga & Sabatini, 1999; Conti, 1994] for details. After scaling the time (if necessary) it is always possible to consider $\dot{\theta} = 1$ in the previous expression.

Isochronicity in the regular case has been extensively studied in the last decades, see for instance [Algaba & Reyes, 2003, 2009; Dias & Mello, 2012; Gasull *et al.*, 2005; Gasull & Torregrosa, 2005; Han & Romanovski, 2012; Llibre & Rabanal, 2015] and references therein. Such importance is due to its applications in physical phenomena and its relation with the famous center-focus problem, a classical open problem in the qualitative theory of planar differential equations to distinguish an equilibrium point between a focus and a center. In recent years isochronicity has also been explored for the discontinuous piecewise differential systems, see for instance [Benterki & Llibre, 2020; Itikawa *et al.*, 2017; Llibre & Teixeira, 2018], by considering the coupling of two or more rigid systems and investigating their dynamics.

The main goal of this paper is to provide the maximum number of limit cycles that can bifurcate from a discontinuous piecewise differential systems formed by the coupling of two rigid centers whose switching manifold is the straight line $\{x = 0\}$. More precisely, in one half-plane we consider a rigid center of degree n of the form

$$\dot{x} = -y + x\sum_{i=0}^{n-1} s_i x^{n-i-1} y^i, \qquad \dot{y} = x + y\sum_{i=0}^{n-1} s_i x^{n-i-1} y^i, \tag{1}$$

and in the other half-plane we consider either an arbitrary linear rigid center, that is,

$$\dot{x} = -y, \qquad \dot{y} = x, \tag{2}$$

or a rigid center analogous to (1) but with degree m.

The linear-linear case was studied in [Itikawa & Llibre, 2015] and has no limit cycles. Our first main result is the following one.

Theorem 1. *The discontinuous piecewise differential systems separated by the straight line $x = 0$ having the linear rigid center* (2) *in $x \leq 0$ and the nonlinear rigid center* (1) *in $x \geq 0$, have no limit cycles.*

Our second main result is the next one.

Theorem 2. *Consider the discontinuous piecewise differential systems formed by system* (2) *in $x \leq 0$ and system* (1) *in $x \geq 0$ after an affine change of variables. Then such discontinuous piecewise differential systems have at most one limit cycle when the degree of system* (1) *is* 2 *or* 3. *Furthermore there are examples of such systems with one limit cycle.*

The third main result is the next.

Theorem 3. *Consider the discontinuous piecewise differential systems formed by system* (1)*, of degree* n*, in* $x \geq 0$ *and system* (1)*, of degree* m*, in* $x \leq 0$ *after an affine change of variables. Then such discontinuous piecewise differential systems have at most one limit cycle when the degree of systems are* 2 *or* 3*, i.e.,* $m, n \in \{2, 3\}$*. Furthermore there are examples of such systems with one limit cycle.*

The paper is organized as follows. In Section 2 we recall the basic theory of the piecewise smooth vector fields that we need for proving our results. Section 3 brings some considerations about the rigid centers considered in this work besides the proof of Theorem 1. Theorem 2 is proved in Section 4. Theorem 3 is proved in Section 5. Finally Section 6 closes the paper with concluding remarks.

2. Preliminary definitions and results

In this section we present the basic results of the theory on piecewise smooth vector fields that we need. A piecewise smooth vector field on an open set $U \subset \mathbb{R}^2$ is a pair of C^r-vector fields X and Y with $r \geq 1$, defined on U separated by a smooth codimension one manifold Σ. The *switching manifold* Σ is obtained by considering $\Sigma = h^{-1}(0)$, where $h : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a differentiable function having 0 as a regular value. Note that Σ is the separating boundary of the regions $\Sigma^+ = \{(x, y) \in U \,|\, h(x, y) > 0\}$ and $\Sigma^- = \{(x, y) \in U \,|\, h(x, y) < 0\}$. So a piecewise smooth vector field is provided by

$$Z(x,y) = \begin{cases} X(x,y), & h(x,y) \geq 0, \\ Y(x,y), & h(x,y) \leq 0. \end{cases} \tag{3}$$

As usual system (3) is denoted by $Z = (X, Y, \Sigma)$ or simply by $Z = (X, Y)$, when the separation line Σ is well understood. In order to establish a definition for the trajectories of Z and investigate its behavior we need a criterion for the transition of the orbits between Σ^+ and Σ^- across Σ. The contact between the vector field X (or Y) and the switching manifold Σ is characterized by the derivative of h in the direction of the vector field X (also known as the Lie derivative of h with respect to X), that is by the expression

$$Xh(p) = \langle \nabla h(p), X(p) \rangle ,$$

and for $i \geq 2$ we define $X^i h(p) = \left\langle \nabla X^{i-1} h(p), X(p) \right\rangle$, where $\langle ., . \rangle$ is the usual inner product in $\mathbb{R}^2$. The basic results of the discontinuous piecewise differential systems in this context were stated by Filippov [Filippov, 1988]. We can divide the switching manifold Σ in the following sets:

(a) Crossing set: $\Sigma^c : \{p \in \Sigma : Xh(p) \cdot Yh(p) > 0\}$;
(b) Escaping set: $\Sigma^e : \{p \in \Sigma : Xh(p) > 0$ and $Yh(p) < 0\}$;
(c) Sliding set: $\Sigma^s : \{p \in \Sigma : Xh(p) < 0$ and $Yh(p) > 0\}$.

The *escaping* Σ^e or *sliding* Σ^s regions are respectively defined on points of Σ where both vector fields X and Y simultaneously point outwards or inwards from Σ while the interior of its complement in Σ defines the *crossing region* Σ^c (see Figure 1). The complementary of the union of these regions is the set formed by the *tangency* points between X or Y with Σ.

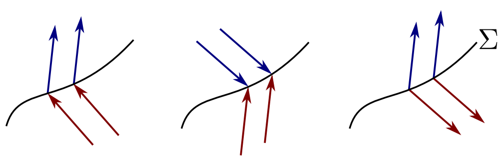


Fig. 1. Crossing, sliding and escaping regions, respectively.

A point $p \in \Sigma$ is called a *tangency* point of X (resp. Y) if it satisfies $Xh(p) = 0$ (resp. $Yh(p) = 0$). A tangency point is called a *fold* point of X if $X^2 h(p) \neq 0$. Moreover, $p \in \Sigma$ is a visible (resp. invisible) fold point of X if $X^2 h(p) > 0$ (resp. $X^2 h(p) < 0$).

In order to define a trajectory of a discontinuous piecewise differential system passing through a crossing point, it is enough to concatenate the trajectories of X and Y through that point. However in the sliding and escaping sets we need to define an auxiliary vector field. So we consider the Filippov's convention (see [Filippov, 1988]) and a new vector field is defined on $\Sigma^s \cup \Sigma^e$. This new vector field, called *sliding vector field*, is a convex linear combination of $X(p)$ and $Y(p)$ in a way that Z^s is tangent to Σ in the cone generated by $X(p)$ and $Y(p)$. Furthermore given a discontinuous piecewise vector field $Z = (X, Y)$ we say that an equilibrium point p of X is *real* if $p \in \Sigma^+$ and it is *virtual* if $p \in \Sigma^-$. In this scenario the trajectories of Z are considered as a concatenation of trajectories of X, Y and Z^s.

Given a vector field $F(x, y) = (F_1(x, y), F_2(x, y))$, defined on an open set $U \subset \mathbb{R}^2$, we consider the corresponding ordinary differential equations

$$\dot{x} = \frac{dx}{dt} = F_1(x, y), \qquad \dot{y} = \frac{dy}{dt} = F_2(x, y), \tag{4}$$

where the independent variable t is called the *time*. Denote the flow of (4) by φ_F or simply by φ when there is no danger of confusion. Also denote by $\varphi_F(t, p)$ or by $\varphi(t, p)$ the solution of system (4) by the point p such that $\varphi(0, p) = p$. When the trajectory of the vector field X through $p \in \Sigma$ returns to Σ (by the first time) after a positive time $t_1(p)$, called X–flight time, we define the *half return map associated with* X by $\pi_X(p) = \phi_X(t_1(p), p) = p_1 \in \Sigma$. When the trajectory of Y through $p_1 \in \Sigma$ returns to Σ (by the first time) after a positive time $t_2(p_1)$, called Y–flight time, we define the *half return map associated with* Y by $\pi_Y(p_1) = \phi_Y(t_2(p_1), p_1) \in \Sigma$. The *first return map* associated with $Z = (X, Y)$ is defined by the composition of these two transition maps, that is,

$$\pi_Z(p) = \pi_Y \circ \pi_X(p) = \phi_Y(t_2(p_1), \phi_X(t_1(p), p)) \tag{5}$$

or the reverse, applying first the flow of Y and after the flow of X. See Figure 2.

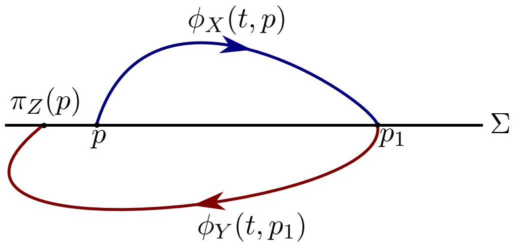


Fig. 2. First return map of a discontinuous piecewise differential system.

When the vector fields X and Y associated to $Z = (X, Y)$ have a first integral (see [Arnold, 1992]), the solution curves of the respective differential equations are contained in the level sets of the first integrals. In this scenario the first return map can be handily computed by seeking for points in Σ that are on the same level curves of these first integral functions. In this case we avoid working with flight times.

3. Proof of Theorem 1

The paper [Llibre & Teixeira, 2018] provides the normal form

$$\dot{x} = -bx - \frac{4b^2 + \omega^2}{4a} y + d, \qquad \dot{y} = ax + by + c, \tag{6}$$

with $a \neq 0$ and $\omega > 0$, for an arbitrary planar linear differential system having a linear center. Let us consider the equilibrium point of system (6) at the origin, that is, consider $c = 0$ and $d = 0$. Under these conditions. in polar coordinates (r, θ) defined by $x = r\cos\theta$ and $y = r\sin\theta$, system (6) gets the form

$$\begin{aligned} \dot{r} &= r\left(\frac{\sin\theta\cos\theta\left(4a^2 - 4b^2 - \omega^2\right)}{4a} - b\cos(2\theta)\right)\theta \\ \dot{\theta} &= a\cos^2\theta + 2b\sin\theta\cos(\theta) + \frac{\left(4b^2 + \omega^2\right)\sin^2\theta}{4a} \end{aligned} \tag{7}$$

So we have that system (6) is a rigid center if and only if $b = 0$ and $\omega = 2a$. Since a is nonzero we can take $a = 1$ rescaling the time. Therefore system (6) becomes system (2), that is system $\dot{x} = -y$, $\dot{y} = x$. We call $X(x, y) = (-y, x)$ the vector field associated with system (2). Equivalently in polar coordinates system (2) can be rewritten as

$$\dot{r} = 0, \qquad \dot{\theta} = 1.$$

We may notice that system (2) has the first integral

$$F(x, y) = \sqrt{x^2 + y^2} \qquad \text{or, in polar coordinates} \qquad F(r, \theta) = r. \tag{8}$$

Now we consider a polynomial differential system in $\mathbb{R}^2$, $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ of degree n. Assume that it has a center at the origin of coordinates. Then it is well known, see for instance [Conti, 1994; Itikawa & Llibre, 2015], that this center is a uniform isochronous if and only if by doing a linear change of variables and a rescaling of time it can be written as

$$\dot{x} = -y + xf(x, y), \qquad \dot{y} = x + yf(x, y) \tag{9}$$

where $f(x, y)$ is a polynomial of degree $n - 1$ in the variables x and y, and $f(0, 0) = 0$. Consider system (9). Conti [Conti, 1994] proved the following result.

Proposition 1. *Let $f(x, y) = \sum_{i=0}^{n-1} a_i x^{n-i-1} y^i$ be a homogeneous polynomial of degree $n - 1$. Then the origin is a uniform isochronous center of system* (9) *if either n is even, or n is odd and*

$$\sum_{i=0}^{n-1} a_i \int_0^{2\pi} \cos^{n-i-1}\theta \sin^i \theta \, d\theta = \sum_{i=0}^{n-1} a_i I_{n-i-1,i} = 0. \tag{10}$$

When n is even, $\sum_{i=0}^{n-1} a_i \, I_{n-i-1,i} = 0$ always holds, because $f(\cos(\theta + \pi), \sin(\theta + \pi)) = -f(\cos\theta, \sin\theta)$, where

$$f(\cos\theta, \sin\theta) = \sum_{i=0}^{n-1} a_i \cos^{n-i-1}\theta \sin^i \theta.$$

In the sections 2.511 and 2.512 of [Gradshteyn & Ryzhik, 2007] we obtain the following expressions for integrals of power of trigonometric functions:

$$\int_{t_0}^{t_1} \sin^p x \cos^{2m} x dx = \left(\frac{\sin^{p+1} x}{2m + p} \left[\cos^{2m-1} x + \sum_{k=1}^{m-1} \frac{(2m-1)(2m-3)\dots(2m-2k+1)\cos^{2m-2k-1} x}{(2m+p-2)(2m+p-4)\dots(2m+p-2k)} \right] \right)_{x=t_0}^{x=t_1}$$
$$+ \frac{(2m-1)!!}{(2m+p)(2m+p-2)\dots(p+2)} \int_{t_0}^{t_1} \sin^p x dx,$$

for arbitrary real p, except for the negative even integers -2, -4, $\dots$, $-2n$. We also have,

$$\int_{t_0}^{t_1} \sin^{2l} x dx = \left(-\frac{\cos x}{2l} \left[\sin^{2l-1} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\dots(2l-2k+1)\sin^{2l-2k-1} x}{2^k(l-1)(l-2)\dots(l-k)} \right] \right)_{x=t_0}^{x=t_1}$$
$$+ \frac{(2l-1)!!}{2^l l!}(t_1 - t_0)$$

and

$$\int_{t_0}^{t_1} \sin^{2l+1} x dx = \left(-\frac{\cos x}{2l+1} \left[\sin^{2l} x + \sum_{k=0}^{l-1} \frac{2^{k+1} l(l-1)\dots(l-k)\sin^{2l-2k-2} x}{(2l-1)(2l-3)\dots(2l-2k-1)} \right] \right)_{x=t_0}^{x=t_1}.$$

For arbitrary real p, except for the negative odd integers -1, -3, $\dots$, $-(2n + 1)$, we have

$$\int_{t_0}^{t_1} \sin^p x \cos^{2m+1} x dx = \left(\frac{\sin^{p+1} x}{2m + p + 1} \left[\cos^{2m} x + \sum_{k=1}^{m} \frac{2^k m(m-1)\dots(m-k+1)\cos^{2m-2k} x}{(2m+p-1)(2m+p-3)\dots(2m+p-2k+1)} \right] \right)_{x=t_0}^{x=t_1}$$

Also, for real p, except for the negative even integers -2, -4, …, $-2n$, we have

$$\int_{t_0}^{t_1} \cos^p x \sin^{2m} x dx = \left(-\frac{\cos^{p+1} x}{2m+p} \left[\sin^{2m-1} x + \sum_{k=1}^{m-1} \frac{(2m-1)(2m-3)\dots(2m-2k+1)\sin^{2m-2k-1} x}{(2m+p-2)(2m+p-4)\dots(2m+p-2k)} \right] \right)_{x=t_0}^{x=t_1}$$
$$+ \frac{(2m-1)!!}{(2m+p)(2m+p-2)\dots(p+2)} \int_{t_0}^{t_1} \cos^p x dx.$$

Moreover,

$$\int_{t_0}^{t_1} \cos^{2l} x dx = \left(\frac{\sin x}{2l} \left[\cos^{2l-1} x + \sum_{k=1}^{l-1} \frac{(2l-1)(2l-3)\dots(2l-2k+1)\cos^{2l-2k-1} x}{2^k(l-1)(l-2)\dots(l-k)} \right] \right)_{x=t_0}^{x=t_1}$$
$$+ \frac{(2l-1)!!}{2^l l!}(t_1 - t_0)$$

and

$$\int_{t_0}^{t_1} \cos^{2l+1} x dx = \left(\frac{\sin x}{2l+1} \left[\cos^{2l} x + \sum_{k=0}^{l-1} \frac{2^{k+1} l(l-1)\dots(l-k)\cos^{2l-2k-2} x}{(2l-1)(2l-3)\dots(2l-2k-1)} \right] \right)_{x=t_0}^{x=t_1}.$$

For arbitrary real p, except for the negative odd integers -1, -3, …, $-(2n+1)$ we have

$$\int_{t_0}^{t_1} \cos^p x \sin^{2m+1} x dx = \left(-\frac{\cos^{p+1} x}{2m+p+1} \left[\sin^{2m} x + \sum_{k=1}^{m} \frac{2^k m(m-1)\dots(m-k+1)\sin^{2m-2k} x}{(2m+p-1)(2m+p-3)\dots(2m+p-2k+1)} \right] \right)_{x=t_0}^{x=t_1}.$$

If n odd, from the formulas above we get that $I_{n-i-1,i} = 0$ when i odd, and consequently,

$$\begin{aligned} \sum_{i=0}^{n-1} a_i \int_0^{2\pi} \cos^{n-i-1}\theta \sin^i \theta \, d\theta = a_0 \left[\frac{2\pi(n-2)!!}{2^{\frac{n-1}{2}} \left(\frac{n-1}{2}\right)!} \right] + \\ + \sum_{k=1}^{\frac{n-3}{2}} \frac{a_{2k} 2\pi(2k-1)!!(n-2k-2)!!}{(n-1)(n-3)\cdots(n-2k+1)\, 2^{\frac{n-2k-1}{2}} \left(\frac{n-2k-1}{2}\right)!} \\ + a_{n-1} \left[\frac{2\pi(n-2)!!}{2^{\frac{n-1}{2}} \left(\frac{n-1}{2}\right)!} \right], \end{aligned} \tag{11}$$

where $(2p)!! = 2p(2p-2)(2p-4)\cdots 2$ and $(2p+1)!! = (2p+1)(2p-1)\cdots 3\cdot 1$. Thus, hypothesis (10) is equivalent to supposing

$$a_{n-1} = -a_0 - \frac{2^{\frac{n-1}{2}} \left(\frac{n-1}{2}\right)!}{(n-2)!!} \sum_{j=1}^{\frac{n-3}{2}} a_{2j} \left[\frac{(2j-1)!!(n-2j-2)!!}{2^{\frac{n-2j-1}{2}} \left(\frac{n-2j-1}{2}\right)! \prod_{i=1}^{j}(n-2i+1)} \right]. \tag{12}$$

So in this paper under the conditions of Proposition 1 we shall work with rigid centers of the form (1), i.e.

$$\dot{x} = -y + x\sum_{i=0}^{n-1} a_i x^{n-i-1} y^i, \qquad \dot{y} = x + y\sum_{i=0}^{n-1} a_i x^{n-i-1} y^i.$$

We denote by

$$Y_n(x,y) = \left(-y + x\sum_{i=0}^{n-1} a_i x^{n-i-1} y^i, x + y\sum_{i=0}^{n-1} a_i x^{n-i-1} y^i \right)$$

the vector field associated to system (1). Equivalently in polar coordinates system (1) writes

$$\begin{aligned}\dot{r} &= r^n \sum_{i=0}^{n-1} a_i \cos^{n-i-1}\theta \sin^i \theta \\ \dot{\theta} &= 1.\end{aligned} \tag{13}$$

System (13) has the first integral

$$H_n(r,\theta) = \frac{1}{(1-n)r^{n-1}} - \sum_{i=0}^{n-1} a_i \int \cos^{n-i-1}\theta \sin^i \theta d\theta.$$

So in Cartesian coordinates the first integral of system (1) becomes

$$H_n(x,y) = \frac{1}{(1-n)(x^2+y^2)^{\frac{n-1}{2}}} - \sum_{i=0}^{n-1} a_i \int \frac{x^{n-i}y^i}{(x^2+y^2)^{\frac{n+1}{2}}} dy.$$

By coupling the two rigid centers (2) and (1) we can consider the piecewise smooth vector field

$$Z(x,y) = \begin{cases} Y_n(x,y), & x \geq 0, \\ X(x,y), & x \leq 0. \end{cases} \tag{14}$$

Observe that the switching manifold is the straight line $\Sigma = h^{-1}(0)$, where $h(x,y) = x$. Since the vector fields X and Y_n associated with the piecewise smooth vector field $Z = (Y_n, X)$ are integrable, then the solution curves of the respective differential equation are contained in the level sets of their respective first integrals. Thus the first return map associated with the discontinuous piecewise vector field Z can be computed by seeking for points in the switching manifold Σ that are on the same level curves of these first integrals. In this way for a limit cycle of the discontinuous piecewise differential system associated with Z given by (14) which has two intersecting points $(0, y_1)$ and $(0, y_2)$ with the line of discontinuity $\Sigma = \{x = 0\}$, its coordinates y_1 and y_2 must satisfy the set of equations

$$\begin{cases} F(0,y_1) = F(0,y_2), \\ H_n(0,y_1) = H_n(0,y_2), \end{cases} \text{or, in polar coordinates, } \begin{cases} F\left(r_0, \frac{\pi}{2}\right) = F\left(R_0, \frac{3\pi}{2}\right), \\ H_n\left(r_0, \frac{\pi}{2}\right) = H_n\left(R_0, \frac{3\pi}{2}\right), \end{cases} \tag{15}$$

where $(r_0, \pi/2)$ and $(R_0, 3\pi/2)$ are the respective intersecting points $(0, y_1)$ and $(0, y_2)$ in polar coordinates. The first equation of (15) produces $r_0 = R_0$. Substituting $r_0 = R_0$ in the second equation, we get

$$\sum_{i=0}^{n-1} a_i \int_0^{\frac{\pi}{2}} \cos^{n-i-1}\theta \sin^i \theta d\theta = \sum_{i=0}^{n-1} a_i \int_0^{\frac{3\pi}{2}} \cos^{n-i-1}\theta \sin^i \theta d\theta. \tag{16}$$

Using the formulas of [Gradshteyn & Ryzhik, 2007] described above for the power of trigonometric functions we conclude that

(i) If $a_i = 0$ for all i then the equality (16) holds for every r_0, i.e. we have a continuum of period orbits. Therefore there are no limit cycles and the origin is a piecewise rigid center.

(ii) If $a_i \neq 0$ for some i then, from equality (16) we obtain an expression of the form

$$P(a) = k_0 a_0 + k_1 a_1 + \ldots + k_{n-1} a_{n-1}$$

where $a = (a_0, \ldots, a_{n-1})$ and $k_i \in \mathbb{R}$. In this way the vectors a such that $P(a) = 0$ correspond to values of a_i such that the equality (16) holds, for all $r_0 > 0$. Therefore we also get a continuum of period orbits. The vectors a such that $P(a) \neq 0$ correspond to values of a_i such that the equality (16) does not hold. Therefore the origin behave like a focus. So there are no periodic orbits and then no limit cycles.

In summary there are no limit cycles in the piecewise smooth vector field Z and Theorem 1 is proved.

Example 3.1. We present a numerical example of Theorem 1. Consider the discontinuous piecewise vector field

$$W(x,y) = \begin{cases} (W_1(x,y), W_2(x,y)), & x \geq 0, \\ (-y, x), & x \leq 0, \end{cases} \tag{17}$$

where

$$W_1(x,y) = -y + x\left(a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3\right)$$

and

$$W_2(x,y) = x + y\left(a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3\right).$$

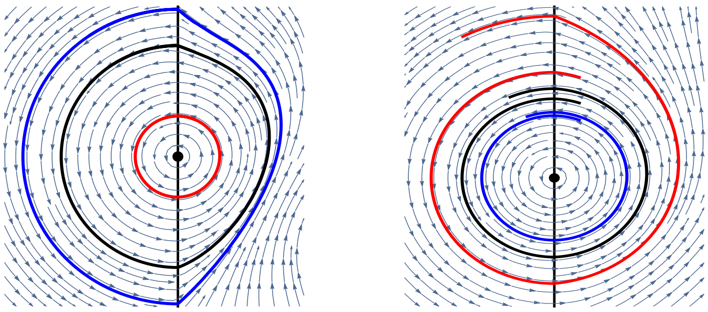
A first integral of the linear vector field is given by $F(r,\theta) = r$. So $F(r_0, \pi/2) = F(R_0, 3\pi/2)$ wields $r_0 = R_0$. A first integral of the nonlinear system in polar coordinates is

$$H(r,\theta) = \frac{1}{12}\left(-\frac{4}{r^3} - 4a_2\sin^3\theta - a_0(9\sin\theta + \sin(3\theta)) + 4a_1\cos^3\theta + 9a_3\cos\theta - a_3\cos(3\theta)\right). \tag{18}$$

Then we obtain that

$$P(a) = H(r_0, \pi/2) - H(r_0, 3\pi/2) = -\frac{2}{3}(2a_0 + a_2).$$

Therefore if $a_2 = -2a_0$ we have a continuum of periodic orbits around the singularity at the origin, that is the singularity is a center. If $a_2 \neq -2a_0$ we have no periodic orbits and the origin is a repulsive focus. See Figure 3.



(a) (b)

Fig. 3. In (a) we have the phase portrait of the discontinuous piecewise vector field W near the origin when $a_0 = 1$, $a_1 = 1$, $a_2 = -2$ and $a_3 = 1$. In (b) we have the phase portrait of the discontinuous piecewise vector field W when $a_0 = 1$, $a_1 = 1$, $a_2 = 1$ and $a_3 = 1$.

4. Systems (2) and (1) after an affine change of variables

The discontinuous piecewise vector field (14) has the centers of both smooth vector fields placed at the origin of coordinates. Now, we give the expression of the differential Systems (2) and (1) and their first integrals after the respective general affine change of variables

$$(X,Y) = (b_1x + b_2y + d_1, b_3x + b_4y + d_2), \quad b_i, d_j \in \mathbb{R},\ i = 1,2,3,4 \text{ and } j = 1,2, \tag{19}$$

and

$$(X,Y) = (c_1x + c_2y + M_1, c_3x + c_4y + M_2), \quad c_i, M_j \in \mathbb{R},\ i = 1,2,3,4 \text{ and } j = 1,2. \tag{20}$$

We want to investigate the number of limit cycles of the discontinuous piecewise vector field (14) after this change of variables. However after this change of variables we still want that the centers of each smooth vector field are rigid. First we consider the change of variables (19) with $d_1 = d_2 = 0$. Since $X = b_1x + b_2y$ and $Y = b_3x + b_4y$ we have that the linear system in the new variables is of the form

$$\begin{aligned} \dot{X} &= \frac{(-b_1b_3 - b_2b_4)\,X}{b_2b_3 - b_1b_4} + \frac{\left(b_1^2 + b_2^2\right)Y}{b_2b_3 - b_1b_4}, \\ \dot{Y} &= \frac{\left(-b_3^2 - b_4^2\right)X}{b_2b_3 - b_1b_4} + \frac{(b_1b_3 + b_2b_4)\,Y}{b_2b_3 - b_1b_4}. \end{aligned} \tag{21}$$

Equivalently in polar coordinates we obtain

$$\begin{aligned} \dot{r} &= \frac{r\left(b_1^2\sin(2\theta) + b_2^2\sin(2\theta) - \left(b_3^2 + b_4^2\right)\sin(2\theta) - 2b_3b_1\cos(2\theta) - 2b_2b_4\cos(2\theta)\right)}{2b_2b_3 - 2b_1b_4}, \\ \dot{\theta} &= \frac{-b_1^2\sin^2(\theta) - b_2^2\sin^2(\theta) + b_1b_3\sin(2\theta) + b_2b_4\sin(2\theta) - \left(b_3^2 + b_4^2\right)\cos^2(\theta)}{b_2b_3 - b_1b_4}. \end{aligned} \tag{22}$$

Using polar coordinates we conclude that (21) has a rigid center at the origin if and only if $b_1 = b_4$ and $b_2 = -b_3$ and (21) writes

$$\dot{X} = -Y \qquad \dot{Y} = X. \tag{23}$$

Now with $(d_1, d_2) \neq (0, 0)$ we obtain

$$\begin{cases} \dot{X} = -Y + d_2, \\ \dot{Y} = X - d_1, \end{cases} \quad \text{or, in polar coordinates,} \quad \begin{cases} \dot{r} = d_2\cos\theta - d_1\sin\theta, \\ \dot{\theta} = \dfrac{r - d_2\sin\theta - d_1\cos\theta}{r}. \end{cases} \tag{24}$$

Observe that a first integral of (24) is

$$F_1(X, Y) = \sqrt{(X - d_1)^2 + (Y - d_2)^2}, \tag{25}$$

or, in polar coordinates,

$$F_1(r, \theta) = \sqrt{(r\cos\theta - d_1)^2 + (r\sin\theta - d_2)^2}.$$

On the other hand we consider in $x \geq 0$ system (1) with the affine change of variables (20) given by $(X, Y) = (c_1x + c_2y + M_1, c_3x + c_4y + M_2)$. First we consider $M_1 = M_2 = 0$. So we get

$$\begin{aligned} \dot{X} &= \frac{\left(c_1^2 + c_2^2\right)Y - (c_1c_3 + c_2c_4)\,X}{c_2c_3 - c_1c_4} + X\left(c_2c_3 - c_1c_4\right)^{1-n}\cdot S, \\ \dot{Y} &= \frac{(c_1c_3 + c_2c_4)\,Y - \left(c_3^2 + c_4^2\right)X}{c_2c_3 - c_1c_4} + Y\left(c_2c_3 - c_1c_4\right)^{1-n}\cdot S, \end{aligned} \tag{26}$$

where

$$S = \sum_{i=0}^{n-1} a_i\left(c_3X - c_1Y\right)^i\left(c_2Y - c_4X\right)^{n-i-1},$$

or equivalently in polar coordinates

$$\begin{aligned} \dot{r} &= \frac{r\left(\left(c_1^2 + c_2^2 - c_3^2 - c_4^2\right)\sin(2\theta) - 2\left(c_1c_3 + c_2c_4\right)\cos(2\theta) + 2\left(c_2c_3 - c_1c_4\right)^{2-n}\bar{S}\right)}{2c_2c_3 - 2c_1c_4}, \\ \dot{\theta} &= -\frac{c_1^2\sin^2\theta - c_3c_1\sin(2\theta) + c_3^2\cos^2\theta + \left(c_2\sin\theta - c_4\cos\theta\right)^2}{c_2c_3 - c_1c_4}, \end{aligned} \tag{27}$$

where

$$\bar{S} = \sum_{i=0}^{n-1} a_i\left(c_3r\cos\theta - c_1r\sin\theta\right)^i\left(c_2r\sin\theta - c_4r\cos\theta\right)^{n-i-1}.$$

Again using polar coordinates we conclude that this system has a rigid center at the origin if and only if $c_4 = c_1$ and $c_3 = -c_2$. Under these conditions system (26) writes

$$\begin{aligned}\dot{X} &= -Y + X\left(c_1^2+c_2^2\right)^{1-n}\sum_{i=0}^{n-1} a_i\left(c_2X+c_1Y\right)^i\left(c_1X-c_2Y\right)^{n-i-1},\\ \dot{Y} &= X + Y\left(c_1^2+c_2^2\right)^{1-n}\left(\sum_{i=0}^{n-1} a_i\left(c_2X+c_1Y\right)^i\left(c_1X-c_2Y\right)^{n-i-1}\right).\end{aligned} \tag{28}$$

or equivalently

$$\begin{aligned}\dot{r} &= r^n\left(c_1^2+c_2^2\right)^{1-n}\sum_{i=0}^{n-1} a_i\left(c_1\sin\theta+c_2\cos\theta\right)^i\left(c_1\cos\theta-c_2\sin(\theta)\right)^{n-i-1},\\ \dot{\theta} &= 1.\end{aligned} \tag{29}$$

Note that

$$H_n(r,\theta) = \frac{r^{1-n}\left(c_1^2+c_2^2\right)^{n-1}}{1-n} - \sum_{i=0}^{n-1} a_i\int\left(c_1\sin\theta+c_2\cos\theta\right)^i\left(c_1\cos\theta-c_2\sin(\theta)\right)^{n-i-1}d\theta \tag{30}$$

is a first integral of (29). Then a first integral of system (28) has the expression

$$H_n(X,Y) = \frac{\left(c_1^2+c_2^2\right)^{n-1}\left(X^2+Y^2\right)^{\frac{1-n}{2}}}{1-n} - \sum_{i=0}^{n-1} a_i\int\frac{(c_1Y+c_2X)^i(c_1X-c_2Y)^{n-i-1}}{(X^2+Y^2)^{\frac{n+1}{2}}}X\,dY \tag{31}$$

Now if we apply the change of variables $(\mathcal{X},\mathcal{Y}) = (X+M_1, Y+M_2)$ to system (28) with $(M_1,M_2)\neq(0,0)$, and rewriting it again in the variables X and Y we get

$$\begin{aligned}\dot{X} &= M_2 - Y + \left(c_1^2+c_2^2\right)^{1-n}\left(X-M_1\right)\cdot S_M,\\ \dot{Y} &= X - M_1 + \left(c_1^2+c_2^2\right)^{1-n}\left(Y-M_2\right)\cdot S_M,\end{aligned} \tag{32}$$

where

$$S_M = \sum_{i=0}^{n-1} a_i\left(c_2\left(X-M_1\right)+c_1\left(Y-M_2\right)\right)^i\left(c_1\left(X-M_1\right)-c_2\left(Y-M_2\right)\right)^{n-i-1}.$$

whose first integral is

$$H_n(x,y) = \frac{\left(c_1^2+c_2^2\right)^{n-1}\left(\left(M_1-X\right)^2+\left(M_2-Y\right)^2\right)^{\frac{1-n}{2}}}{1-n} - \left(X-M_1\right)\sum_{i=0}^{n-1} a_i\int T(X,Y)\,dY \tag{33}$$

where

$$\begin{aligned}T(X,Y) = {}&\left[c_1\left(Y-M_2\right)+c_2\left(X-M_1\right)\right]^i\left[c_1\left(X-M_1\right)-c_2\left(Y-M_2\right)\right]^{n-i-1}\cdot\\ &\cdot\left[\left(X-M_1\right)^2+\left(Y-M_2\right)^2\right]^{\frac{1}{2}(-n-1)}\end{aligned}$$

In polar coordinates, the corresponding first integral is

$$H_n(r,\theta) = \frac{\left(c_1^2+c_2^2\right)^{n-1}\left(\left(M_2-r\sin\theta\right)^2+\left(M_1-r\cos(\theta)\right)^2\right)^{\frac{1-n}{2}}}{1-n} - \sum_{i=0}^{n-1} a_i\int T(r,\theta)\,d\theta, \tag{34}$$

where

$$\begin{aligned}T(r,\theta) = {}&\frac{r\sec\theta\left(r\cos\theta-M_1\right)\left[c_1\left(r\sin\theta-M_2\right)+c_2\left(r\cos\theta-M_1\right)\right]^i}{\left[\left(M_2-r\sin\theta\right)^2+\left(M_1-r\cos(\theta)\right)^2\right]^{\frac{n+1}{2}}}\cdot\\ &\cdot\left[c_1\left(r\cos\theta-M_1\right)-c_2\left(r\sin\theta-M_2\right)\right]^{n-i-1}\end{aligned}$$

Notice that the above expressions coincide with the expression for the first integral when $M_1 = M_2 = 0$.

4.1. *Case linear-quadratic*

We investigate the number of limit cycles of the discontinuous piecewise vector field formed by the translated linear system (24) and system (32) when $n = 2$, that is we consider the consider

$$Z_2(x,y) = \begin{cases} Y_2(x,y) = \left(Y_2^1(x,y), Y_2^2(x,y)\right), & x \geq 0, \\ X(x,y) = (-y + d_2, x - d_1), & x \leq 0, \end{cases} \tag{35}$$

where

$$Y_2^1(x,y) = -y + M_2 + \frac{(x - M_1) \cdot S_2}{c_1^2 + c_2^2}, \tag{36}$$

$$Y_2^2(x,y) = x - M_1 + \frac{(y - M_2) \cdot S_2}{c_1^2 + c_2^2}. \tag{37}$$

and

$$S_2 = a_1 \left(c_2 \left(x - M_1\right) + c_1 \left(y - M_2\right)\right) + a_0 \left(c_1 \left(x - M_1\right) - c_2 \left(y - M_2\right)\right).$$

From equation (25) we have that $F_1(x,y) = \sqrt{(x - d_1)^2 + (y - d_2)^2}$ is a first integral of X, and from (33) we get that

$$H_2(x,y) = \frac{a_0 \left(c_2 \left(M_1 - x\right) + c_1 \left(M_2 - y\right)\right) + a_1 \left(c_1 \left(x - M_1\right) + c_2 \left(M_2 - y\right)\right) - c_1^2 - c_2^2}{\sqrt{(x - M_1)^2 + (y - M_2)^2}} \tag{38}$$

is a first integral of Y_2. So the limit cycles for the discontinuous piecewise vector field Z_2 are given by the solutions of the following system of equations

$$F_1(0,y_1) = F_1(0,y_2), \qquad H_2(0,y_1) = H_2(0,y_2). \tag{39}$$

From the first equation of (39) we obtain that $y_2 = -y_1 + 2d_2$, for every $y_1, y_2 \in \mathbb{R}$ with $y_1 \neq y_2$. Substituting this expression into the second equation of (39) we have that the solutions of $H_2(0,y_1) = H_2(0, -y_1 + 2d_2)$ are given by $y_1 = d_2$ and $y_1 = d_2 \pm A$ where $A = \sqrt{\upsilon_1\upsilon_2\upsilon_3\upsilon_4}/\upsilon_5$ with $\upsilon_1 = -a_0c_1 - a_1c_2$,

$$\upsilon_2 = -a_1c_1M_1 + c_2 \left(a_0M_1 - c_2\right) - c_1^2,$$

$$\upsilon_3 = c_1 \left(a_0 \left(d_2 - M_2\right) + a_1M_1\right) + c_2 \left(a_1 \left(d_2 - M_2\right) - a_0M_1 + c_2\right) + c_1^2,$$

$$\upsilon_4 = -c_1M_1 \left(a_1 \left(M_2 - d_2\right) + a_0M_1\right) + c_1^2 \left(d_2 - M_2\right) + c_2 \left(c_2 \left(d_2 - M_2\right) - M_1 \left(a_0 \left(d_2 - M_2\right) + a_1M_1\right)\right),$$

and

$$\upsilon_5 = \left(a_0c_1 + a_1c_2\right) \left(M_1 \left(a_1c_1 - a_0c_2\right) + c_1^2 + c_2^2\right).$$

Observe that the solution $y_1 = d_2$ corresponds to the periodic orbit of X that is tangent to the separation line given by the y-axis. Therefore this solution does not correspond to a limit cycle. Moreover the solutions $y_{11} = d_2 + A$ and $y_{12} = d_2 - A$ are on the same level curve of the first integral. Therefore they correspond to the same limit cycle, because each limit cycle corresponds to a pair of intersections between the curves given by (39) and the switching manifold. As a consequence, we conclude that there are at most one limit cycle associated to the piecewise smooth vector field Z_2.

Example 4.1. We now present a numerical example. Consider the discontinuous piecewise vector field

$$Q(x,y) = \begin{cases} \left(-\frac{17y}{5} + 5 + \frac{2}{5}x\left(x + 3y - 7\right), \frac{1}{5}\left(3x + 2(-8 + x)y + 6y^2\right)\right), & x \geq 0, \\ \left(-y + 2, x + 2\right), & x \leq 0. \end{cases} \tag{40}$$

Observe that (40) corresponds to system (35) when $d_1 = -2$, $d_2 = 2$, $c_2 = -1$, $c_1 = 2$, $M_1 = 2$, $M_2 = 1$, $a_0 = 2$ and $a_1 = 2$. By solving the first equation of the respective system (39) we get $y_2 = 4 - y_1$. Then we have that

$$H_2(x, y) = \frac{6x - 2y - 15}{\sqrt{x^2 - 4x + y^2 - 2y + 5}}$$

is a first integral of the nonlinear system of the vector field Q. So the solutions of $H_2(0, y_1) = H_2(0, 4-y_1)$ are given by $y_1 = 2$ and $y_1 = 2 \pm 3\sqrt{19/34}$. Therefore there is a limit cycle through the points $\left(0, 2 - 3\sqrt{19/34}\right)$ and $\left(0, 2 + 3\sqrt{19/34}\right)$, see Figure 4.

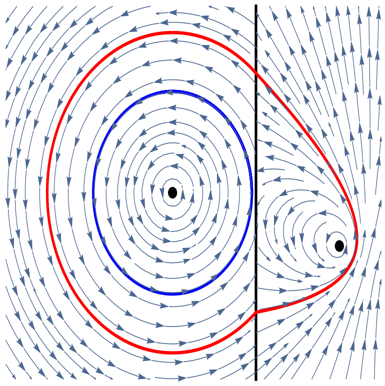


Fig. 4. The limit cycle of the discontinuous piecewise vector field Q in red. In blue we have the periodic orbit of the vector field Q that is tangent to the separation line and corresponds to the solution $y_1 = d_2$.

4.2. *Case linear-cubic*

Now we investigate the number of limit cycles of the discontinuous piecewise differential system formed by the translated linear system (24) and system (32) when $n = 3$, i.e. of the discontinuous piecewise vector field

$$Z_3(x, y) = \begin{cases} Y_3(x, y) = \left(Y_3^1(x, y), Y_3^2(x, y)\right), & x \geq 0, \\ X(x, y) = \left(-y + d_2, x - d_1\right), & x \leq 0, \end{cases} \tag{41}$$

where

$$Y_3^1(x, y) = -y + M_2 + (x - M_1)P(x, y),$$

$$Y_3^2(x, y) = x - M_1 + (y - M_2)P(x, y),$$

and

$$P(x, y) = \frac{a_2\left(c_2\left(x - M_1\right) + c_1\left(y - M_2\right)\right)^2 - a_2\left(c_1\left(x - M_1\right) - c_2\left(y - M_2\right)\right)^2}{\left(c_1^2 + c_2^2\right)^2} + \frac{a_1\left(c_1\left(x - M_1\right) - c_2\left(y - M_2\right)\right)\left(c_2\left(x - M_1\right) + c_1\left(y - M_2\right)\right)}{\left(c_1^2 + c_2^2\right)^2}.$$

Note that the vector field Y_3 takes into account hypothesis (10) of Proposition 1 to have a rigid center. In this case hypothesis (10) becomes $\pi(a_0 + a_2) = 0$. Therefore we are assuming that $a_0 = -a_2$. Again Equation (25) assures that $F_1(x, y) = \sqrt{(x - d_1)^2 + (y - d_2)^2}$ is a first integral of X. From (33) we get that

$$H_3(x,y) = \frac{a_1(x-M_1)\left(c_1^2(x-M_1)+c_2^2(M_1-x)+2c_2c_1(M_2-y)\right)}{2\left(-2M_1x-2M_2y+M_1^2+M_2^2+x^2+y^2\right)} + \frac{2a_2(x-M_1)\left(2c_2c_1(x-M_1)+c_1^2(y-M_2)+c_2^2(M_2-y)\right)-\left(c_1^2+c_2^2\right)^2}{2\left(-2M_1x-2M_2y+M_1^2+M_2^2+x^2+y^2\right)}$$

is a first integral of Y_3. So the limit cycles of the discontinuous piecewise vector field Z_3 are given by the solutions of the system of equations

$$F_1(0,y_1) = F_1(0,y_2), \qquad H_3(0,y_1) = H_3(0,y_2). \tag{42}$$

From the first equation of (42) we obtain that $y_2 = -y_1+2d_2$, for every $y_1, y_2 \in \mathbb{R}$ with $y_1 \neq y_2$. Substituting this expression in the second equation of (42) we have that the solutions of $H_3(0,y_1) = H_3(0,-y_1+2d_2)$ are given by $y_1 = d_2$ and $y_1 = d_2 \pm B$ where

$$B = \frac{\sqrt{M_1\left(a_1c_1c_2+a_2\left(c_2^2-c_1^2\right)\right)B_1}}{M_1\left(a_1c_1c_2+a_2\left(c_2^2-c_1^2\right)\right)}$$

with

$$\begin{aligned} B_1 = {} & c_1^2\left(M_1\left(a_1M_1(d_2-M_2)-a_2\left((d_2-M_2)^2-M_1^2\right)\right)+2c_2^2(M_2-d_2)\right)+ \\ & + c_2c_1M_1\left(a_1\left((d_2-M_2)^2-M_1^2\right)+4a_2M_1(d_2-M_2)\right)+c_2^4(M_2-d_2) \\ & + c_2^2M_1\left(a_2\left((d_2-M_2)^2-M_1^2\right)+a_1M_1(M_2-d_2)\right)+c_1^4(M_2-d_2). \end{aligned}$$

Also the solutions $d_2 \pm B$ are on the same level of the first integral H_3 because

$$H_3(0,d_2+B) = H_3(0,d_2-B) = \frac{M_1\left(a_1c_1c_2+a_2\left(c_2^2-c_1^2\right)\right)}{2(d_2-M_2)}.$$

Since the solution $y_1 = d_2$ corresponds to the periodic orbit of X that is tangent to the separation line given by the y-axis and each limit cycle corresponds to a pair of intersections between the curves given by (42) and the switching manifold, we conclude that there is at most one limit cycle associated to the piecewise smooth vector field Z_3.

Example 4.2. Consider the discontinuous piecewise vector field

$$Q_3(x,y) = \begin{cases} \left(Y_3^1(x,y), Y_3^2(x,y)\right), & x \geq 0, \\ (-y+2, x+2), & x \leq 0. \end{cases} \tag{43}$$

where

$$Y_3^1(x,y) = \frac{2x^3}{25} + \frac{22x^2y}{25} - \frac{19x^2}{25} - \frac{2xy^2}{25} + \frac{26xy}{25} - \frac{9x}{10} - \frac{y^2}{25} - \frac{7y}{10} + \frac{3}{4}$$

$$Y_3^2(x,y) = \frac{2x^2y}{25} - \frac{2x^2}{25} + \frac{22xy^2}{25} - \frac{42xy}{25} + \frac{9x}{5} - \frac{2y^3}{25} + \frac{17y^2}{25} - \frac{11y}{10} + 1.$$

Observe that (43) corresponds to system (41) when $d_1 = -2$, $d_2 = 2$, $c_2 = -1$, $c_1 = 2$, $M_1 = -1/2$, $M_2 = 1$, $a_0 = 2$, $a_1 = 2$, and $a_2 = -2$. By solving the first equation of the respective system (42) we get $y_2 = 4-y_1$. In this case, we have that

$$H_3(x,y) = \frac{44x^2+x(52-8y)-4y-35}{4x^2+4x+4y^2-8y+5}$$

is a first integral of the right-hand side system of the vector field Q_3. Then the solutions of $H_3(0,y_1) = H_3(0,4-y_1)$ are given by $y_1 = 2$, $y_1 = -5/2$, and $y_1 = 13/2$. Note that $H_3(0,-5/2) = H_3(0,13/2) = -1/2$. Therefore there exists a limit cycle through the points $(0,-5/2)$ and $(0,13/2)$, see Figure 5.

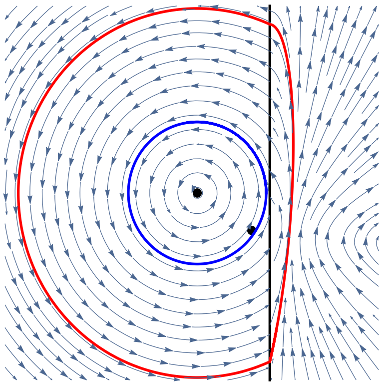


Fig. 5. The limit cycle of the discontinuous piecewise vector field Q_3 in red. In blue we have the periodic orbit of the vector field Q that is tangent to the separation line and corresponds to the solution $y_1 = d_2$. The black dots represent the equilibrium points of the two centers of Q_3.

5. Study of cases where none vector field is linear

In order to obtain a complete characterization of the $n \leq 3$ case, in this section, we analyze the maximum number of limit cycles that can bifurcate from the piecewise smooth vector field

$$Z_{n,m}(x,y) = \begin{cases} U_n(x,y), & x \geq 0, \\ V_m(x,y), & x \leq 0, \end{cases} \tag{44}$$

with U_n and V_m, $n, m \in \{2,3\}$, being rigid centers of the form (1) after the respective change of variables (19) and (20), with $a_i = s_i$ for the vector field U_n. We will call these cases (n,m) with $n, m \in \{2,3\}$. In other words, we have that V_m is of the vector field associated with the differential system (32) and U_n is the vector field associated with the differential system

$$\begin{aligned} \dot{X} &= d_2 - Y + \left(b_1^2 + b_2^2\right)^{1-n} (X - d_1) \cdot S_d, \\ \dot{Y} &= X - d_1 + \left(b_1^2 + b_2^2\right)^{1-n} (Y - d_2) \cdot S_d, \end{aligned} \tag{45}$$

where

$$S_d = \sum_{i=0}^{n-1} s_i \left(b_2 (X - d_1) + b_1 (Y - d_2)\right)^i \left(b_1 (X - d_1) - b_2 (Y - d_2)\right)^{n-i-1}.$$

Moreover, we found examples that ensure that the upper bound is reached in each case.

5.1. *Case (2,2)*

Consider the piecewise smooth vector field

$$Z_{2,2}(x,y) = \begin{cases} U_2(x,y), & x \geq 0, \\ V_2(x,y), & x \leq 0, \end{cases} \tag{46}$$

where $V_2(x,y) = \left(Y_2^1(x,y), Y_2^2(x,y)\right)$ is given by Equations (36) and (37), and $U_2(x,y) = (U_2^1(x,y), U_2^2(x,y))$ has components

$$U_2^1(x,y) = -y + d_2 + \frac{(x - d_1)\, S_{22}}{b_1^2 + b_2^2},$$

and

$$U_2^2(x,y) = x - d_1 + \frac{(y - d_2)\, S_{22}}{b_1^2 + b_2^2},$$

with

$$S_{22} = s_0\left(b_1\left(x-d_1\right)-b_2\left(y-d_2\right)\right)+s_1\left(b_2\left(x-d_1\right)+b_1\left(y-d_2\right)\right).$$

From (33) we get that H_2 given by (38) is a first integral of V_2 and

$$F_2(x,y) = \frac{b_1\left(s_1\left(x-d_1\right)+s_0\left(d_2-y\right)\right)-b_2\left(b_2+s_0\left(x-d_1\right)+s_1\left(y-d_2\right)\right)-b_1^2}{\sqrt{\left(x-d_1\right)^2+\left(y-d_2\right)^2}}$$

is a first integral of U_2. So the limit cycles for the discontinuous piecewise vector field $Z_{2,2}$ are given by the solutions of the following system of equations

$$F_2(0,y_0)=F_2(0,y_1), \qquad H_2(0,y_0)=H_2(0,y_1). \tag{47}$$

From the second equation of (47) we obtain $y_1=y_0$ and

$$y_1=y_{1e}=\frac{\eta_1+\eta_2+\eta_3+2a_0(\eta_4+\eta_5)}{\eta_6+\eta_7+\eta_8} \tag{48}$$

where

$$\eta_1 = 2a_1\left(c_1^2+c_2^2\right)\left(c_1M_1\left(2M_2-y_0\right)+c_2\left(M_2\left(y_0-M_2\right)+M_1^2\right)\right)+\left(c_1^2+c_2^2\right)^2\left(2M_2-y_0\right),$$

$$\eta_2 = a_0^2M_1\left(c_1^2M_1\left(y_0-2M_2\right)-2c_2c_1\left(M_2\left(y_0-M_2\right)+M_1^2\right)+c_2^2M_1\left(2M_2-y_0\right)\right),$$

$$\eta_3 = a_1^2M_1\left(c_1^2M_1\left(2M_2-y_0\right)+2c_2c_1\left(M_2\left(y_0-M_2\right)+M_1^2\right)+c_2^2M_1\left(y_0-2M_2\right)\right),$$

$$\eta_4 = c_2^2M_1\left(c_2\left(y_0-2M_2\right)-a_1\left(M_2\left(y_0-M_2\right)+M_1^2\right)\right)+c_1^3\left(M_2\left(y_0-M_2\right)+M_1^2\right),$$

$$\begin{aligned}\eta_5 = {}& c_1^2M_1\left(a_1\left(M_2\left(y_0-M_2\right)+M_1^2\right)+c_2\left(y_0-2M_2\right)\right)+\\ &+c_2c_1\left(2a_1M_1^2\left(y_0-2M_2\right)+c_2\left(M_2\left(y_0-M_2\right)+M_1^2\right)\right),\end{aligned}$$

$$\eta_6 = c_2^2\left(-2c_2\left(a_1\left(M_2-y_0\right)+a_0M_1\right)+M_1\left(2a_1a_0\left(M_2-y_0\right)+a_0^2M_1-a_1^2M_1\right)+c_2^2\right)+c_1^4,$$

$$\begin{aligned}\eta_7 = {}& 2c_1^3\left(a_0\left(y_0-M_2\right)+a_1M_1\right)-\\ &-2c_1c_2\left(a_0\left(M_2-y_0\right)\left(c_2-a_0M_1\right)+a_1M_1\left(2a_0M_1-c_2\right)+a_1^2M_1\left(M_2-y_0\right)\right),\end{aligned}$$

$$\eta_8 = c_1^2\left(-2c_2\left(a_1\left(M_2-y_0\right)+a_0M_1\right)\right)+c_1^2\left(M_1\left(2a_1a_0\left(y_0-M_2\right)+a_0^2\left(-M_1\right)+a_1^2M_1\right)+2c_2^2\right).$$

From the first equation of (47) we obtain $y_1=y_0$ and

$$y_1=y_{1d}=\frac{\mu_1+2b_1b_2(\mu_2+\mu_3)+b_2^2(\mu_4+\mu_5)+b_1^2(\mu_6+\mu_7)}{\mu_8+\mu_9+\mu_{10}} \tag{49}$$

where $\mu_1 = 2b_1^3\left(d_1s_1\left(2d_2-y_0\right)+d_2s_0\left(y_0-d_2\right)+d_1^2s_0\right)+b_1^4\left(2d_2-y_0\right),$

$$\mu_2 = d_1\left(2d_1s_0s_1\left(y_0-2d_2\right)+d_2\left(s_0^2-s_1^2\right)\left(d_2-y_0\right)+d_1^2\left(s_1^2-s_0^2\right)\right),$$

$$\mu_3 = b_2\left(d_1s_1\left(2d_2-y_0\right)+d_2s_0\left(y_0-d_2\right)+d_1^2s_0\right),$$

$$\mu_4 = d_1\left(d_1\left(s_0^2-s_1^2\right)\left(2d_2-y_0\right)+2d_2s_0s_1\left(d_2-y_0\right)-2d_1^2s_0s_1\right),$$

$$\mu_5 = 2b_2\left(d_1s_0\left(y_0-2d_2\right)+d_2s_1\left(y_0-d_2\right)+d_1^2s_1\right)+b_2^2\left(2d_2-y_0\right),$$

$$\mu_6 = 2b_2\left(d_1s_0\left(y_0-2d_2\right)+d_2s_1\left(y_0-d_2\right)+d_1^2s_1\right)-2b_2^2\left(y_0-2d_2\right),$$

$$\mu_7 = d_1\left(d_1\left(s_0^2-s_1^2\right)\left(y_0-2d_2\right)+2d_2s_0s_1\left(y_0-d_2\right)+2d_1^2s_0s_1\right),$$

$$\mu_8 = b_2^2\left(-2b_2\left(s_1\left(d_2-y_0\right)+d_1s_0\right)+b_2^2+d_1\left(2s_0s_1\left(d_2-y_0\right)+d_1\left(s_0^2-s_1^2\right)\right)\right)+b_1^4,$$

$$\mu_9 = 2b_1^3\left(s_0\left(y_0-d_2\right)+d_1s_1\right)+b_1^2\left(-2b_2\left(s_1\left(d_2-y_0\right)+d_1s_0\right)+2b_2^2+d_1\left(2s_0s_1\left(y_0-d_2\right)+d_1\left(s_1^2-s_0^2\right)\right)\right),$$

$$\mu_{10} = 2b_1b_2\left(b_2\left(s_0\left(y_0-d_2\right)+d_1s_1\right)+d_1\left(s_0^2\left(d_2-y_0\right)+s_1^2\left(y_0-d_2\right)-2d_1s_1s_0\right)\right).$$

Consider $D_{2,2}$ the numerator of the difference $y_{1e}-y_{1d}$. We have that $D_{2,2}$ has two zeros in the variable y_0 whose expressions are complicated and will be omitted. Therefore we conclude that the piecewise smooth vector field $Z_{2,2}$ given by (46) has at most one crossing limit cycle.

Example 5.1. Consider the discontinuous piecewise vector field

$$Q_{2,2}(x,y) = \begin{cases} \left(U_2^1(x,y), U_2^2(x,y)\right), & x \geq 0, \\ \left(V_2^1(x,y), V_2^2(x,y)\right), & x \leq 0. \end{cases} \tag{50}$$

where

$$U_2^1(x,y) = -y+1+\frac{1}{5}(x-1)(3(x+2(y-1)-1)-2(x-1)+y-1),$$

$$U_2^2(x,y) = x-1+\frac{1}{5}(y-1)(3(x+2(y-1)-1)-2(x-1)+y-1),$$

$$V_2^1(x,y) = 1-y+\frac{1}{5}\left(x-\frac{1}{2}\right)\left(2\left(-x+2(y-1)+\frac{1}{2}\right)+2\left(2\left(x-\frac{1}{2}\right)+y-1\right)\right), \tag{51}$$

and

$$V_2^2(x,y) = x-\frac{1}{2}+\frac{1}{5}(y-1)\left(2\left(-x+2(y-1)+\frac{1}{2}\right)+2\left(2\left(x-\frac{1}{2}\right)+y-1\right)\right). \tag{52}$$

Observe that (50) corresponds to system (46) when $b_1=2$, $b_2=1$, $d_1=1$, $d_2=1$, $s_0=-1$, $s_1=3$, $c_2=-1$, $c_1=2$, $M_1=1/2$, $M_2=1$, $a_0=2$, and $a_1=2$. By solving the first and second equations of the respective system (47) we get

$$y_{1d} = \frac{286-119y_0}{24y_0+119} \quad \text{and} \quad y_{1e} = \frac{102-31y_0}{32y_0+31}.$$

Solving $y_{1e}=y_{1d}$ we get the solutions $y_0=\left(419\pm7\sqrt{386}\right)/383$ which are the intersections of the limit cycle with the ordinate axis. See Figure 6.

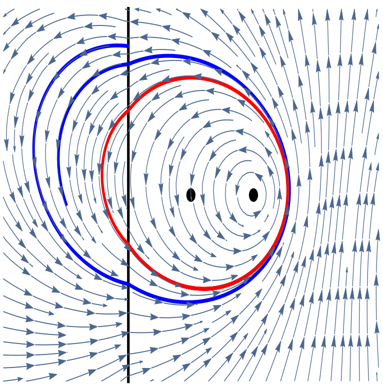


Fig. 6. The limit cycle of the discontinuous piecewise vector field $Q_{2,2}$ in red. The black dots represent the equilibrium points of the two centers of $Q_{2,2}$.

5.2. *Case (2,3)*

Consider the piecewise smooth vector field

$$Z_{2,3}(x,y)=\begin{cases}U_3(x,y), & x\geq 0,\\ V_2(x,y), & x\leq 0,\end{cases} \tag{53}$$

where $V_2(x,y) = \left(Y_2^1(x,y), Y_2^2(x,y)\right)$ is given by Equations (36) and (37), and $U_3(x,y) = (U_3^1(x,y), U_3^2(x,y))$ has components

$$U_3^1(x,y)=-y+d_2+\frac{(x-d_1)\,S_{23}}{\left(b_1^2+b_2^2\right)^{2}}, \tag{54}$$

and

$$U_3^2(x,y)=x-d_1+\frac{(y-d_2)\,S_{23}}{\left(b_1^2+b_2^2\right)^{2}}, \tag{55}$$

with

$$\begin{aligned}S_{23}=&-s_0\left(b_2\left(x-d_1\right)+b_1\left(y-d_2\right)\right)^2+s_1\left(b_1\left(x-d_1\right)-b_2\left(y-d_2\right)\right)\cdot\\ &\cdot\left(b_2\left(x-d_1\right)+b_1\left(y-d_2\right)\right)+s_0\left(b_1\left(x-d_1\right)-b_2\left(y-d_2\right)\right)^2.\end{aligned}$$

Note that the vector field U_3 is considering the hypothesis $s_2=-s_0$ given by Equation (12) so that we have a rigid center. From (33) we get that H_2 given by (38) is a first integral of V_2 and

$$\begin{aligned}F_3(x,y)=&-\frac{b_1^2\left(2b_2^2+(x-d_1)\left(s_1\left(d_1-x\right)+2s_0\left(y-d_2\right)\right)\right)}{2\left(-2d_1x-2d_2y+d_1^2+d_2^2+x^2+y^2\right)}+\\ &+\frac{2b_2b_1\left(x-d_1\right)\left(2s_0\left(x-d_1\right)+s_1\left(y-d_2\right)\right)}{2\left(-2d_1x-2d_2y+d_1^2+d_2^2+x^2+y^2\right)}+\\ &+\frac{b_2^2\left(x-d_1\right)\left(s_1\left(x-d_1\right)+2s_0\left(d_2-y\right)\right)+b_1^4+b_2^4}{2\left(-2d_1x-2d_2y+d_1^2+d_2^2+x^2+y^2\right)}\end{aligned} \tag{56}$$

is a first integral of U_3. So the limit cycles for the discontinuous piecewise vector field $Z_{2,3}$ are given by the solutions of the following system of equations

$$F_3(0,y_0)=F_3(0,y_1),\qquad H_2(0,y_0)=H_2(0,y_1). \tag{57}$$

From the second equation of (57) we obtain $y_1=y_0$ and $y_1=y_{1e}$ given by Equation (48). From the first equation of (57) we obtain $y_1=y_0$ and

$$y_1=y_{3d}=\frac{\omega_1+\omega_2+\omega_3}{\omega_4} \tag{58}$$

where $\omega_1=b_2^2d_1\left(2s_0\left(d_2\left(y_0-d_2\right)+d_1^2\right)-d_1s_1\left(y_0-2d_2\right)\right)+b_1^4\left(-\left(y_0-2d_2\right)\right)-b_2^4\left(y_0-2d_2\right)$,

$$\omega_2=2b_1b_2d_1\left(-2d_1s_0\left(y_0-2d_2\right)-s_1\left(d_2\left(y_0-d_2\right)+d_1^2\right)\right),$$

$$\omega_3=b_1^2\left(d_1\left(d_1s_1\left(y_0-2d_2\right)-2s_0\left(d_2\left(y_0-d_2\right)+d_1^2\right)\right)-2b_2^2\left(y_0-2d_2\right)\right),$$

and

$$\begin{aligned}\omega_4=&\,b_1^2\left(2b_2^2-d_1\left(2s_0\left(y_0-d_2\right)+d_1s_1\right)\right)+2b_2b_1d_1\left(s_1\left(d_2-y_0\right)+2d_1s_0\right)+\\ &+b_2^2d_1\left(2s_0\left(y_0-d_2\right)+d_1s_1\right)+b_1^4+b_2^4.\end{aligned}$$

Consider $D_{2,3}$ the numerator of the difference $y_{1e}-y_{3d}$. We have that $D_{2,3}$ has two zeros in the variable y_0 whose expressions are complicated and will be omitted. Therefore we conclude that the piecewise smooth vector field $Z_{2,3}$ given by (53) has at most one crossing limit cycle. The case (3,2) can be treated similarly.

Example 5.2. Consider the discontinuous piecewise vector field

$$Q_{2,3}(x,y) = \begin{cases} \left(U_3^1(x,y), U_3^2(x,y)\right), & x \geq 0, \\ \left(V_2^1(x,y), V_2^2(x,y)\right), & x \leq 0. \end{cases} \tag{59}$$

where $V_2(x,y) = \left(V_2^1(x,y), V_2^2(x,y)\right)$ is given by equations (51) and (52),

$$U_3^1(x,y) = -y + 1 + \frac{16}{289}(x+1)\,(S_e)\,, \tag{60}$$

and

$$U_3^2(x,y) = x + 1 + \frac{16}{289}(y-1)\,(S_e)\,, \tag{61}$$

with

$$S_e = \left(\frac{y-1}{2} - 2(x+1)\right)^2 + \left(\frac{x+1}{2} + 2(y-1)\right)\left(\frac{y-1}{2} - 2(x+1)\right) - \left(\frac{x+1}{2} + 2(y-1)\right)^2.$$

Observe that (59) corresponds to system (53) when $b_1 = 1/2$, $b_2 = -2$, $d_1 = -1$, $d_2 = 1$, $s_0 = -1$, $s_1 = 1$, $s_2 = -s_0 = 1$, $c_2 = -1$, $c_1 = 2$, $M_1 = 1/2$, $M_2 = 1$, $a_0 = 2$, and $a_1 = 2$. By solving the first and second equations of the respective system (57) we get

$$y_{3d} = \frac{826 - 325y_0}{88y_0 + 325} \quad \text{and} \quad y_{1e} = \frac{102 - 31y_0}{32y_0 + 31}.$$

Solving $y_{1e} = y_{3d}$ we get the solutions $y_0 = \left(1091 \pm 2\sqrt{71486}\right)/959$ which are the intersections of the limit cycle with the ordinate axis. See Figure 7.

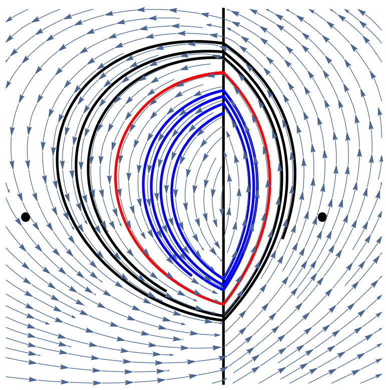


Fig. 7. The limit cycle of the discontinuous piecewise vector field $Q_{2,3}$ in red. The black dots represent the equilibrium points of the two centers of $Q_{2,3}$.

5.3. *Case (3,3)*

Consider the piecewise smooth vector field

$$Z_{3,3}(x,y) = \begin{cases} U_3(x,y), & x \geq 0, \\ V_3(x,y), & x \leq 0, \end{cases} \tag{62}$$

where $U_3(x,y) = \left(U_3^1(x,y), U_3^2(x,y)\right)$ is given by Equations (54) and (55), and $V_3(x,y) = (V_3^1(x,y), V_3^2(x,y))$ has components

$$V_3^1(x,y) = -y + M_2 + \frac{(x - M_1)\,(S_{33})}{\left(c_1^2 + c_2^2\right)^2},$$

$$V_3^2(x,y) = x - M_1 + \frac{(y-M_2)\,(S_{33})}{\left(c_1^2+c_2^2\right)^2},$$

with

$$\begin{aligned} S_{33} = &-a_0\,(c_2\,(x-M_1)+c_1\,(y-M_2))^2 + a_0\,(c_1\,(x-M_1)-c_2\,(y-M_2))^2 \\ &+ a_1\,(c_1\,(x-M_1)-c_2\,(y-M_2))\,(c_2\,(x-M_1)+c_1\,(y-M_2)). \end{aligned}$$

Note that the vector field V_3 is considering the hypothesis $a_2 = -a_0$ given by Equation (12) so that we have a rigid center. From (33) we get that

$$\begin{aligned} H_3(x,y) = &-\frac{-a_1\,(x-M_1)\left(c_1^2\,(x-M_1)+c_2^2\,(M_1-x)+2c_2c_1\,(M_2-y)\right)}{2\left(-2M_1x-2M_2y+M_1^2+M_2^2+x^2+y^2\right)} \\ &-\frac{2a_0\,(x-M_1)\left(2c_2c_1\,(x-M_1)+c_1^2\,(y-M_2)+c_2^2\,(M_2-y)\right)}{2\left(-2M_1x-2M_2y+M_1^2+M_2^2+x^2+y^2\right)} \\ &-\frac{\left(c_1^2+c_2^2\right)^2}{2\left(-2M_1x-2M_2y+M_1^2+M_2^2+x^2+y^2\right)} \end{aligned} \tag{63}$$

is a first integral of V_3 and F_3 given by (56) is a first integral of U_3. The limit cycles for the discontinuous piecewise vector field $Z_{3,3}$ are given by the solutions of the following system of equations

$$F_3(0,y_0) = F_3(0,y_1), \qquad H_3(0,y_0) = H_3(0,y_1). \tag{64}$$

From the first equation of (64) we obtain $y_1 = y_0$ and $y_1 = y_{3d}$ given by Equation (58). From the second equation of (64) we obtain $y_1 = y_0$ and

$$y_1 = y_{3e} = \frac{\nu_1-\nu_2}{\nu_3} \tag{65}$$

where

$$\nu_1 = a_1M_1\left(c_1^2M_1\,(y_0-2M_2)-2c_2c_1\left(M_2\,(y_0-M_2)+M_1^2\right)+c_2^2M_1\,(2M_2-y_0)\right)+\left(c_1^2+c_2^2\right)^2(2M_2-y_0),$$

$$\nu_2 = 2a_0M_1\left(c_1^2\left(M_2\,(y_0-M_2)+M_1^2\right)+2c_2c_1M_1\,(y_0-2M_2)\right)-2a_0M_1\left(c_2^2\left(M_2\,(y_0-M_2)+M_1^2\right)\right),$$

and

$$\begin{aligned} \nu_3 = &\, c_1^2\left(2c_2^2-M_1\,(2a_0\,(y_0-M_2)+a_1M_1)\right)+2c_2c_1M_1\,(a_1\,(M_2-y_0)+2a_0M_1) \\ &+ c_2^2M_1\,(2a_0\,(y_0-M_2)+a_1M_1)+c_1^4+c_2^4. \end{aligned}$$

Consider $D_{3,3}$ the numerator of the difference $y_{3e}-y_{3d}$. We have that $D_{3,3}$ has two zeros in the variable y_0 whose expressions are complicated and will be omitted. Therefore we conclude that the piecewise smooth vector field $Z_{3,3}$ given by (62) has at most one crossing limit cycle.

Example 5.3. Consider the discontinuous piecewise vector field

$$Q_{3,3}(x,y) = \begin{cases} \left(U_3^1(x,y), U_3^2(x,y)\right), & x \geq 0, \\ \left(V_3^1(x,y), V_3^2(x,y)\right), & x \leq 0. \end{cases} \tag{66}$$

where $U_3^1(x,y)$ and $U_3^2(x,y)$ are given by Equations (60) and (61),

$$V_3^1(x,y) = -y + 1 + \frac{1}{25}\left(x+\frac{1}{2}\right)(S_{33e}),$$

and

$$V_3^2(x,y) = x + \frac{1}{2} + \frac{1}{25}(y-1)\,(S_{33e}),$$

with

$$S_{33e} = -2\left(-x+2(y-1)-\frac{1}{2}\right)^2 -3\left(2\left(x+\frac{1}{2}\right)+y-1\right)\left(-x+2(y-1)-\frac{1}{2}\right)+2\left(2\left(x+\frac{1}{2}\right)+y-1\right)^2.$$

Note that (66) corresponds to system (62) when $b_1 = 1/2$, $b_2 = -2$, $d_1 = -1$, $d_2 = 1$, $s_0 = -1$, $s_1 = 1$, $s_2 = -s_0 = 1$, $c_2 = -1$, $c_1 = 2$, $M_1 = -1/2$, $M_2 = 1$, $a_0 = 2$, $a_1 = -3$, and $a_2 = -a_0 = -2$. By solving the first and second equations of the respective system (64) we get

$$y_{3d} = \frac{826 - 325y_0}{88y_0 + 325} \quad \text{and} \quad y_{3e} = \frac{50 - 15y_0}{16y_0 + 15}.$$

Solving $y_{3e} = y_{3d}$ we get the solutions $y_0 = \left(1102 + \sqrt{278354}\right)/970$ which are the intersections of the limit cycle with the ordinate axis. See Figure 8.

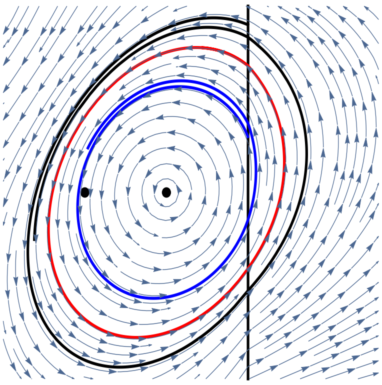


Fig. 8. The limit cycle of the discontinuous piecewise vector field $Q_{3,3}$ in red. The black dots represent the equilibrium points of the two centers of $Q_{3,3}$.

6. Conclusions

In this paper we have studied the upper bound for the maximum number of limit cycles of discontinuous piecewise differential systems formed by two differential systems separated by the straight line $x = 0$. We assume that each one of the differential systems has a rigid center formed of a linear part with a homogeneous polynomial nonlinear part of degree m and n respectively.

If both centers are localized at the origin of coordinates and $n = 1$ then for all positive integer m we have proved that such discontinuous piecewise differential systems have no limit cycles, see Theorem 1.

If the centers are not at the origin of coordinates for a subclass of these discontinuous piecewise differential systems with $n, m = 2, 3$ we have proved that they can have at most one limit cycle, see for more details the statement of Theorems 2 and 3.

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