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Phase Portraits of Families VII and VIII of the Quadratic Systems

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Abstract: The quadratic polynomial differential systems in a plane are the easiest nonlinear differential systems. They have been studied intensively due to their nonlinearity and the large number of applications. These systems can be classified into ten classes. Here, we provide all topologically different phase portraits in the Poincaré disc of two of these classes.

Keywords: quadratic vector fields; quadratic systems; phase portraits

MSC: Primary 34C05; 34A34; 34C14

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1. Introduction and Statement of the Main Results

A *quadratic polynomial differential system* (or simply, a *quadratic system*) is a differential system of the following form:

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

where P and Q are real polynomials in variables x and y and the maximum degree of the polynomials P and Q is two.

At the beginning of the 20th century, the study of quadratic systems began. In [1], Coppel noted how Büchel [2], in 1904, published the first work on quadratic systems. Two short surveys on quadratic systems were published, i.e., by Coppel [1] in 1966 and by Chicone and Tian [3] in 1982.

In recent decades, quadratic systems were intensively studied and many good results were obtained, see references [4–6]. In the second reference, one can find many applications for quadratic systems. Although quadratic systems have been studied in more than one thousand papers, we do not have a complete understanding of these systems.

In [7], the authors prove that any quadratic system is affine-equivalent, scaling the time variable, if necessary, to a quadratic system of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y) = d + ax + by + \ell x^2 + mxy + ny^2,$$

where $\dot{x} = P(x, y)$ is one of the following ten:

$$\begin{array}{llll} \text{(I)} & \dot{x} = 1 + xy, & \text{(VI)} & \dot{x} = 1 + x^2, \\ \text{(II)} & \dot{x} = xy, & \text{(VII)} & \dot{x} = x^2, \\ \text{(III)} & \dot{x} = y + x^2, & \text{(VIII)} & \dot{x} = x, \\ \text{(IV)} & \dot{x} = y, & \text{(IX)} & \dot{x} = 1, \\ \text{(V)} & \dot{x} = -1 + x^2, & \text{(X)} & \dot{x} = 0. \end{array}$$

Roughly speaking, the Poincaré disc is the disc centered at the origin of $\mathbb{R}^2$ and the radius, where the interior of this disc is identified with the whole plane $\mathbb{R}^2$ and its boundary