JULIA SETS WITH A WANDERING BRANCHING POINT

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Abstract. According to the Thurston No Wandering Triangle Theorem, a branching point in a locally connected quadratic Julia set is either preperiodic or precritical. Blokh and Oversteegen proved that this theorem does not hold for higher degree Julia sets: there exist cubic polynomials whose Julia set is a locally connected dendrite with a branching point which is neither preperiodic nor precritical. In this article, we reprove this result, constructing such cubic polynomials as limits of cubic polynomials for which one critical point eventually maps to the other critical point which eventually maps to a repelling fixed point.

Notations

- $\mathbb{C}$ is the complex plane,
- $\mathbb{D}$ is the unit disk,
- $\mathbb{S}^1$ is the unit circle.
- $\mathbb{T} := \mathbb{R}/\mathbb{Z}$.

Introduction

In this article, we consider polynomials $f : \mathbb{C} \to \mathbb{C}$ of degree at least 2 as dynamical systems: the orbit of a point $z \in \mathbb{C}$ is the set

$$\mathcal{O}(z) := \{f^{\circ n}(z)\}_{n\geq 0}.$$

The orbit of a point is finite if and only if the point is *preperiodic*. More precisely, a point $\alpha \in \mathbb{C}$ is preperiodic if $f^{\circ(r+s)}(\alpha) = f^{\circ r}(\alpha)$ for some integers $r \geq 0$ and $s \geq 1$. If r and s are minimal integers such that $f^{\circ(r+s)}(\alpha) = f^{\circ r}(\alpha)$, then r is the *preperiod* and s is the *period*. The point α is *periodic* if the preperiod is 0. In this case, the point is *repelling* if $\left|(f^{\circ s})'(\alpha)\right| > 1$.

The filled-in Julia set $\mathcal{K}_f$ is the set of points with bounded orbit and the Julia set $\mathcal{J}_f$ is its topological boundary. The sets $\mathcal{K}_f$ and $\mathcal{J}_f$ are compact subsets of $\mathbb{C}$. They are completely invariant: $f^{-1}(\mathcal{K}_f) = f(\mathcal{K}_f) = \mathcal{K}_f$ and $f^{-1}(\mathcal{J}_f) = f(\mathcal{J}_f) = \mathcal{J}_f$. Preperiodic points are contained in $\mathcal{K}_f$. Repelling periodic points are contained in $\mathcal{J}_f$. In fact, $\mathcal{J}_f$ is the closure of the set of repelling periodic points (see e.g. [1, 5]).

A point $\omega \in \mathbb{C}$ is a critical point if the derivative of f vanishes at ω. The topology of $\mathcal{K}_f$ and $\mathcal{J}_f$ is related to the behavior of critical orbits. For example, $\mathcal{K}_f$ and $\mathcal{J}_f$ are connected if and only if the critical points of f belong to $\mathcal{K}_f$ (see e.g. [1, 5]).

A *dendritic polynomial* is a polynomial f for which $\mathcal{J}_f$ is a *dendrite*, i.e., $\mathcal{J}_f$ is connected and locally connected and contains no simple closed curve. This is the case whenever each critical point is preperiodic to a repelling periodic point (see [3, Th. V.4.2]).

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