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Symmetric Phase Portraits of Homogeneous Polynomial Hamiltonian Systems of Degree 1, 2, 3, 4, and 5 with Finitely Many Equilibria

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Abstract: Roughly speaking, the Poincaré disc $\mathbb{D}^2$ is the closed disc centered at the origin of the coordinates of $\mathbb{R}^2$, where the whole of $\mathbb{R}^2$ is identified with the interior of $\mathbb{D}^2$ and the circle of the boundary of $\mathbb{D}^2$ is identified with the infinity of $\mathbb{R}^2$, because in the plane $\mathbb{R}^2$, we can go to infinity in as many directions as points have the circle. The phase portraits of the quadratic Hamiltonian systems in the Poincaré disc were classified in 1994. Since then, no new interesting classes of Hamiltonian systems have been classified on the Poincaré disc. In this paper, we determine the phase portraits in the Poincaré disc of five classes of homogeneous Hamiltonian polynomial differential systems of degrees 1, 2, 3, 4, and 5 with finitely many equilibria. Moreover, all these phase portraits are symmetric with respect to the origin of coordinates. We showed that these polynomial differential systems exhibit precisely 2, 2, 3, 3, and 4 topologically distinct phase portraits in the Poincaré disc. Of course, the new results are for the homogeneous Hamiltonian polynomial differential systems of degrees 3, 4, and 5. The tools used here for obtaining these phase portraits also work for obtaining any phase portrait of a homogeneous Hamiltonian polynomial differential system of arbitrary degree.

Keywords: homogeneous Hamiltonian system; phase portrait; Poincaré disc

Citation: Benterki, R.; Llibre, J. Symmetric Phase Portraits of Homogeneous Polynomial Hamiltonian Systems of Degree 1, 2, 3, 4, and 5 with Finitely Many Equilibria. *Symmetry* **2023**, *15*, 1476. https://doi.org/10.3390/sym15081476

Academic Editors: Tomohiro Inagaki and Olga Kodolova

Received: 14 June 2023
Revised: 20 July 2023
Accepted: 24 July 2023
Published: 25 July 2023

1. Introduction and Statement of the Main Results

The centers of the polynomial differential systems of the form

$$\dot{x} = -y + P_n(x, y), \quad \dot{y} = x + Q_n(x, y), \tag{1}$$

with P_n and Q_n homogeneous polynomials of degree n have been studied for $n = 2, 3, 4$, and 5. Furthermore, for $n = 2$, see refs.[1–6], for $n = 3$, see refs.[7,8], for $n = 4$, see refs.[9], and for $n = 5$, see ref.[10]. While the centers of systems (1) of degrees 2 and 3 have been completely classified, this is not the case for the centers of degrees 4 and 5. Moreover, for systems (1) having a center of degrees 2 and 3, their phase portraits in the Poincaré disc have been classified in refs. [5,6] and in [11], respectively.

In a similar way to the study completed for the centers of systems (1), in this paper we classify the phase portraits in the Poincaré disc of the homogeneous Hamiltonian systems of degrees 1, 2, 3, 4, and 5, i.e., of the systems

$$\dot{x} = -\frac{\partial H_n(x, y)}{\partial y}, \qquad \dot{y} = \frac{\partial H_n(x, y)}{\partial x},$$

where $H_n(x, y)$ is a homogeneous polyonomial of degree n for $n \in \{2, 3, 4, 5, and6\}$. We recall that the phase portraits of the quadratic Hamiltonian systems in the Poincaré disc were classified in [12].