ON THE LIMIT CYCLES OF DISCONTINUOUS PIECEWISE LINEAR DIFFERENTIAL SYSTEMS FORMED BY CENTERS AND SEPARATED BY IRREDUCIBLE CUBIC CURVES III

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ABSTRACT. In this paper we study the distribution and the number of limit cycles of the class of planar discontinuous piecewise linear differential systems formed by centers and separated by an irreducible algebraic cubic curve. More precisely we study the existence of simultaneous limit cycles with two and four intersection points with the cubic of separation. In previous papers [3, 4] we already have studied this problem for the limit cycles having only either two, or four intersection points with the cubic of separation.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

1.1. **Introduction.** Probably the most difficult problem in the qualitative theory of a planar differential system is the determination of the existence and the number of its limit cycles. We recall that a *limit cycle* is an isolated periodic orbit in the set of all periodic orbits.

In the last two decades many authors have been interested in piecewise linear differential systems due to their simplicity and their applications to many natural phenomena. Andronov, Vitt and Khaikin [1] were the first who started to study this kind of differential systems in 1920's. For more recent applications of the piecewise linear differential systems see the books [6, 18], and the hundreds of references quotted therein.

But although of the apparently simplicity of the piecewise linear differential systems only recently have been studied the existence and the number of their crossing limit cycles. Thus in 2010 Han and Zhang [8] gave an example of discontinuous piecewise linear differential systems separated by a straight line with two crossing limit cycles, and the first example with three crossing limit cycles was given in 2012, numerically by Huan and Yang [10], and analytically by Llibre and Ponce [16].

In this paper we study the distribution and the number of crossing limit cycles that the planar discontinuous piecewise linear differential systems formed by centers and separated by an irreducible cubic curve can exhibit. See subsection 1.3 for the precise definition of a planar discontinuous piecewise differential and of a crossing limit cycle.

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This paper is a continuation of two previous papers of the authors. In the first one, see [3], we proved that these piecewise differential systems can exhibit at most three crossing limit cycles having two intersection points with the cubic of separation. While in the second paper, see, [4] we proved that these piecewise differential systems can have at least four crossing limit cycles having four intersection points with the cubic of separation which are contained in three pieces of the piecewise differential system, and that they can have at least two crossing limit cycles having four intersection points with the cubic of separation which are contained in two pieces of the piecewise differential system.

In summary with the results of this paper and the ones of the papers [3, 4] we have studied the extension of the 16th Hilbert problem (see [9, 11, 14]) about the number of crossing limit cycles and their distribution that the discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubic curves can have.

We must mention that the number of crossing limit cycles and their distribution of the discontinuous piecewise linear differential systems formed by centers and separated by conics have been studied in [12].

1.2. **Classification of the irreducible cubic polynomials.** A *cubic curve* is the set of points $(x, y) \in \mathbb{R}^2$ satisfying $P(x, y) = 0$ for some polynomial $P(x, y)$ of degree three. This cubic is *irreducible* (respectively *reducible*) if the polynomial $P(x, y)$ is irreducible (respectively reducible) in the ring of all real polynomials in the variables x and y.

A point (x_0, y_0) of a cubic $P(x, y) = 0$ is *singular* if $P_x(x_0, y_0) = 0$ and $P_y(x_0, y_0) = 0$, as usual here P_x and P_y denote the partial derivaties of P with respect to the variables x and y respectively. A cubic curve is *singular* if it has some singular point.

A *flex* of an algebraic curve C is a point p of C such that C is nonsingular at p and the tangent at p intersects C at least three times. The next theorem characterizes all the irreducible cubic algebraic curves.

Theorem 1. *The following statements classify all the irreducible cubic algebraic curves.*

(a) *A cubic curve is nonsingular and irreducible if and only if it can be transformed with affine transformations into one of the following two curves:*

$$c_1(x, y) = y^2 - x(x^2 + bx + 1) = 0 \quad \textit{with } b \in (-2, 2), \textit{ or}$$
$$c_2(x, y) = y^2 - x(x - 1)(x - r) = 0 \quad \textit{with } r > 1.$$

(b) *A cubic curve is singular and irreducible if and only if it can be transformed with affine transformations into one of the following three curves:*

$$c_3(x, y) = y^2 - x^3 = 0, \qquad \textit{or}$$
$$c_4(x, y) = y^2 - x^2(x - 1) = 0, \quad \textit{or}$$
$$c_5(x, y) = y^2 - x^2(x + 1) = 0.$$

Statement (a) of Theorem 1 is proved in Theorem 8.3 of the book [5] under the additional assumption that the cubic has a flex, but in section 12 of that book it is shown that every nonsingular irreducible cubic curve has a flex. While statement (b) of Theorem 1 follows directly from Theorem 8.4 of [5].

For $k = 1, \ldots, 5$ let C_k be the five classes of planar discontinuous piecewise linear differential systems formed by centers and separated by the irreducible cubic curve $c_k(x, y) = 0$, or simply the irreducible cubic curve c_k.

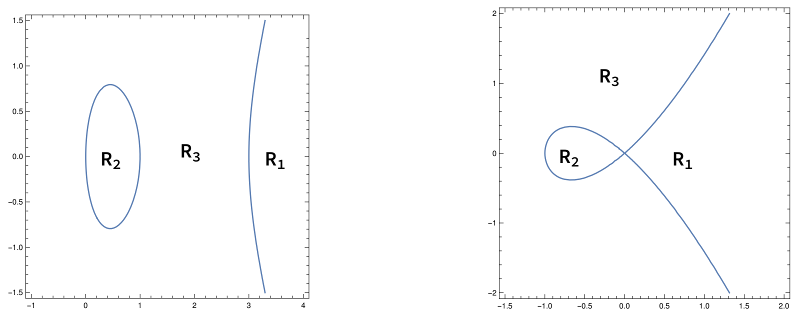


FIGURE 1. The three regions R_1, R_2 and R_3 of the plane separated by the curves c_2 to the left and c_5 to the right.

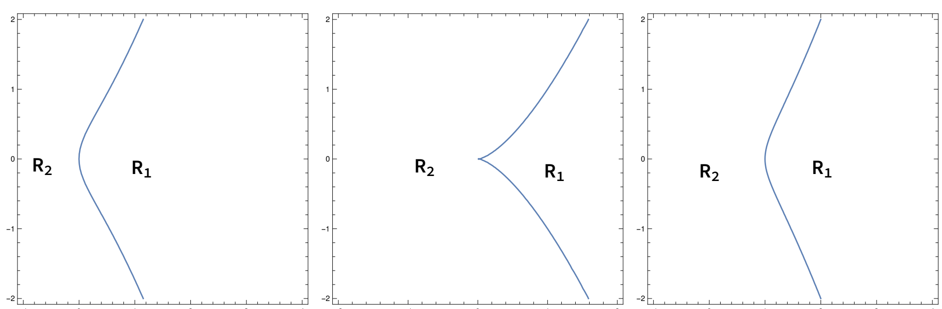


FIGURE 2. The two regions R_1 and R_2 of the plane separated by the curves c_1 to the left, c_3 to the middle and c_4 to the right.

Figures 1 and 2 show the different regions separated by the cubic curves c_i, with $i = 1 \ldots 5$.

In this work our objective is to provide a lower bound for the maximum number of simultaneous crossing limit cycles that the planar discontinuous piecewise linear differential systems formed by centers can have intersecting the irreducible cubic curve c_i, with $i = 1 \ldots 5$, in four points or in two points. First we give this lower bound for the number of the crossing limit cycles contained in two pieces of the piecewise differential system. Second we give the lower bound for the number of crossing limit cycles contained in two or in three pieces of the piecewise differential system.

1.3. **Piecewise differential systems.** A *piecewise differential system* on $\mathbb{R}^2$ is a pair of C^r (with $r \geq 1$) differential systems in $\mathbb{R}^2$ separated by a smooth codimension one manifold Σ. The *line of separation* Σ of the piecewise differential system is defined by $\Sigma = h^{-1}(0)$, where $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a differentiable function having 0 as a regular value. Note that Σ is the separating boundary of the regions $\Sigma^+ = \{(x, y) \in \mathbb{R}^2 \,|\, h(x, y) > 0\}$ and $\Sigma^- = \{(x, y) \in \mathbb{R}^2 \,|\, h(x, y) < 0\}$. So the piecewise C^r vector field associated to a piecewise differential system with line of discontinuity Σ is

$$Z(x, y) = \begin{cases} X(x, y), & \text{if } h(x, y) \geq 0, \\ Y(x, y), & \text{if } h(x, y) \leq 0. \end{cases} \tag{1}$$

As usual, system (1) is denoted by $Z = (X, Y, \Sigma)$ or simply by $Z = (X, Y)$, when the separation line Σ is well understood.

When the piecewise differential system, with the vector field $Z = (X, Y)$ given in (1), satisfies $X(x, y) = Y(x, y)$ at all the points (x, y) such that $h(x, y) = 0$ we say that we have a *continuous piecewise differential system*. Otherwise we say that we have a *discontinuous piecewise differential system*.

In order to establish a definition for the trajectories of a discontinuous piecewise differential system Z and investigate its behavior, we need a criterion for the transition of the orbits between Σ^+ and Σ^- across Σ. The contact between the vector field X (or Y) and the line of discontinuity Σ is characterized by the derivative of h in the direction of the vector field X i.e.

$$Xh(p) = \langle \nabla h(p), X(p) \rangle \,,$$

where $\langle ., . \rangle$ is the usual inner product in $\mathbb{R}^2$. The basic results of the discontinuous piecewise differential systems in this context were stated by Filippov [7]. We can divide the line of discontinuity Σ in the following sets:

(a) *Crossing set*: $\Sigma^c : \{p \in \Sigma : Xh(\mathbf{x}) \cdot Yh(\mathbf{x}) > 0\}$.
(b) *Escaping set*: $\Sigma^e : \{p \in \Sigma : Xh(\mathbf{x}) > 0 \text{ and } Yh(\mathbf{x}) < 0\}$.
(c) *Sliding set*: $\Sigma^s : \{p \in \Sigma : Xh(\mathbf{x}) < 0 \text{ and } Yh(\mathbf{x}) > 0\}$.

The *escaping* Σ^e or *sliding* Σ^s regions are respectively defined on points of Σ where both vector fields X and Y simultaneously point outwards or inwards from Σ while the interior of its complement in Σ defines the *crossing region* Σ^c (see Figure 3). The complementary of the union of these regions is the set formed by the *tangency* points between X or Y with Σ.

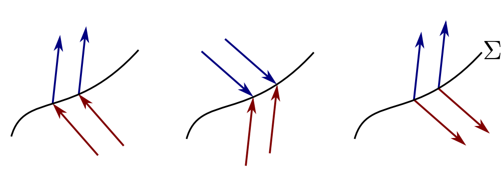


FIGURE 3. Crossing, sliding and escaping regions, respectively.

In order to define a trajectory of a discontinuous piecewise differential system passing through a crossing point, it is enough to concatenate the trajectories of the vector fields X and Y through that point.

A *crossing limit cycle* of a discontinuous piecewise differential system is a limit cycle which intersects the line of discontinuity in isolated crossing points. In the rest of the paper we will say simply limit cycle instead of crossing limit cycle.

1.4. Simultaneous limit cycles intersecting the cubics c_i in two and four points contained in two pieces of the piecewise differential system.

Theorem 2. *Here we study the simultaneous limit cycles intersecting the cubic c_k for $k = 1 \dots 5$ in two or four points which are contained only in two pieces of the discontinuous piecewise linear differential system formed by centers.*

(a) *There are systems in the class C_k which exibit exactly two limit cycles intersecting the cubic curve c_i one in four points and the other in two points. Classes C_1, C_3 and C_4 have one possible configuration see $(\mathcal{C}_1)$, $(\mathcal{C}_3)$ and $(\mathcal{C}_4)$ of Figure 4, respeclively. For the classes C_2 and C_5 we have two possible different configurations see $(\mathcal{C}_2^1)$ and $(\mathcal{C}_2^2)$ of Figure 5, $(\mathcal{C}_5^1)$ and $(\mathcal{C}_5^2)$ of Figure 6.*

(b) *There are systems in the class C_2 and C_5 which can have one limit cycle that intersects the curve c_2 or c_5 in four points and there are also systems in these classes which can have additionally two limit cycles that intersect the curve c_2 or c_5 in two points, see $(\mathcal{C}_2)$ and $(\mathcal{C}_5)$ of Figure 7 for the classes C_2 and C_5, respectively.*

Theorem 2 is proved in section 2.

1.5. Simultaneous limit cycles intersecting the cubics c_2 and c_5 in two and four points contained in three pieces of the piecewise differential system.

Theorem 3. *We study the simultaneous limit cycles that intersect the curves c_2 and c_5 in two or in four points which are contained in three pieces of the discontinuous piecewise linear differential system formed by centers.*

(a) *There are systems in the class C_2 and C_5 which can have one limit cycle that intersects the curve c_2 or c_5 in four points and that additionally they have one limit cycle that intersects the curve c_2 or c_5 in two points. The class C_2 can have two configurations see $(\mathcal{C}_2^1)$ and $(\mathcal{C}_2^2)$ of Figure 8. For the class C_5 we can have four configurations see $(\mathcal{C}_5^1)$ and $(\mathcal{C}_5^2)$ of Figure 9, and $(\mathcal{C}_5^3)$ and $(\mathcal{C}_5^4)$ of Figure 10.*

(b) *There are systems in the class C_2 and C_5 which can have one limit cycle that intersects the curve c_2 or c_5 in four points and that additionally they have two limit cycles that intersect the curve c_2 or c_5 in two points. For the class C_2 we can have two configurations see $(\mathcal{C}_2^1)$ and $(\mathcal{C}_2^2)$ of Figure 11. The class C_5 can have four configurations see $(\mathcal{C}_5^1)$ and $(\mathcal{C}_5^2)$ of Figure 12, and $(\mathcal{C}_5^3)$ and $(\mathcal{C}_5^4)$ of Figure 13.*

(c) *There are systems in the class C_2 and C_5 which can have two limit cycles that intersect the curve c_2 or c_5 in four points and that additionally they have one limit cycle that intersects the curve c_2 or c_5 in two points. For the class C_2 we can have two configurations see $(\mathcal{C}_2^1)$ and $(\mathcal{C}_2^2)$ of Figure 14. The class C_5 can have four configurations see $(\mathcal{C}_5^1)$ and $(\mathcal{C}_5^2)$ of Figure 15, and $(\mathcal{C}_5^3)$ and $(\mathcal{C}_5^4)$ of Figure 16.*

(d) *There are systems in the class C_2 and C_5 which can have one limit cycle that intersects the curve c_2 or c_5 in four points and that additionally they have three limit cycles that intersect the curve c_2 or c_5 in two points. The class C_2 can have two configurations see $(\mathcal{C}_2^1)$ and $(\mathcal{C}_2^2)$ of Figure* 17*. For the class C_5 we give two configurations, see $(\mathcal{C}_5^1)$ and $(\mathcal{C}_5^2)$ of Figure* 18*.*

(e) *There are systems in the class C_2 and C_5 which can have two limit cycles that intersect the curve c_2 or c_5 in four points and that additionally they have two limit cycles that intersect the curve c_2 or c_5 in two points. For the class C_2 see $(\mathcal{C}_2)$ of Figure* 19*. For the class C_5 we give two configurations, see $(\mathcal{C}_5^1)$ of Figure* 19 *and $(\mathcal{C}_5^2)$ of Figure* 20*.*

(f) *There are systems in the class C_2 and C_5 which can have three limit cycles that intersect the curve c_2 or c_5 in four points and that additionally they have one limit cycle that intersects the curve c_2 or c_5 in two points. For the class C_2 see $(\mathcal{C}_2)$ of Figure* 20*. We give two configurations for the class C_5, see $(\mathcal{C}_5^1)$ and $(\mathcal{C}_5^2)$ of Figure* 21*.*

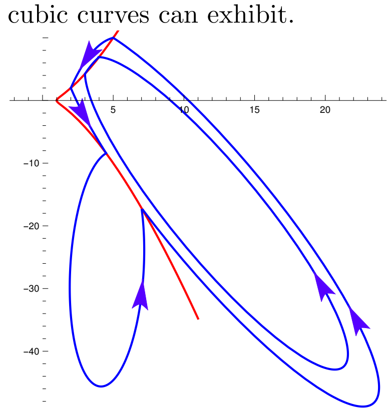
Theorem 3 is proved in section 3.

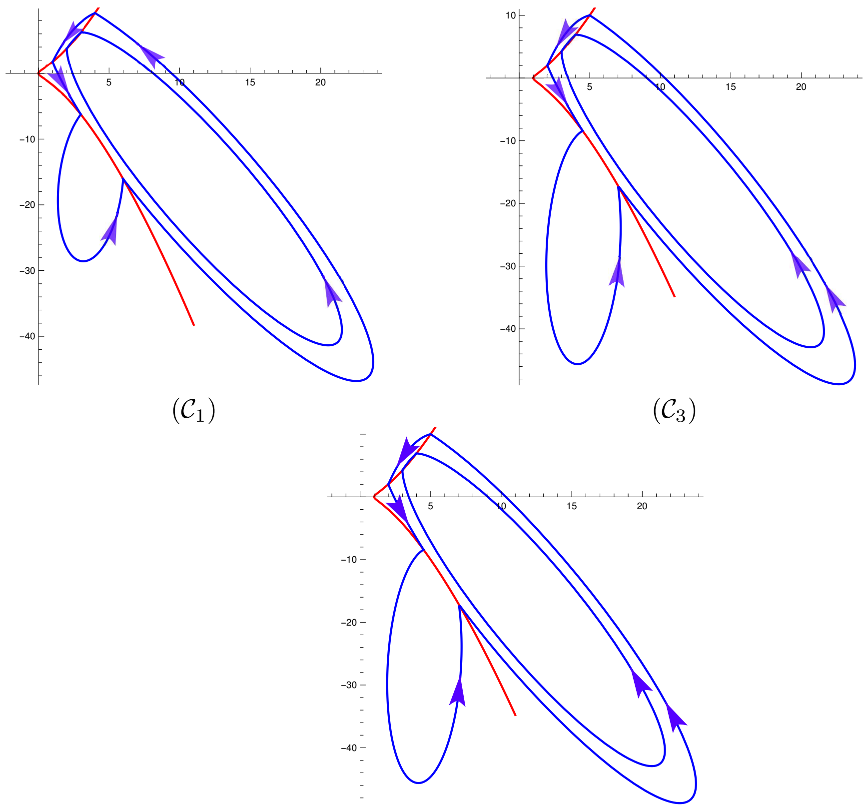
Corollary 4. *The maximum number of known limit cycles that discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubic curves can exhibit is;*

(a) 3 *if we only have limit cycles intersecting in two points the cubic;*
(b) 4 *if we only have limit cycles intersecting in four points the cubic; and*
(c) 4 *if we have have simultaneously limit cycles intersecting in two and four points the cubic.*

Statement (a) of the corollary was proved in [3], statement (b) of the corollary was proved in [4], and finally statement (c) follows immediately from Theorems 2 and 3.

It is an open problem to prove or disprove if four is the maximum number of limit cycles that discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubic curves can exhibit.





$(\mathcal{C}_1)$ $(\mathcal{C}_3)$ $(\mathcal{C}_4)$

FIGURE 4. The two limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_1)$ for (2)–(3), $(\mathcal{C}_3)$ for (5)–(6) and $(\mathcal{C}_4)$ for (7)–(8).

2. Proof of Theorem 2

Proof of statement (*a*) *of Theorem* 2. First we prove the statement for the class C_1. We consider the first linear differential center in the region R_1

$$(2)\quad \dot{x} = -0.36887..x - 0.167751..y + 1.38826.., \quad \dot{y} = x + 0.36887..y - 5.27202..,$$

this system has the first integral

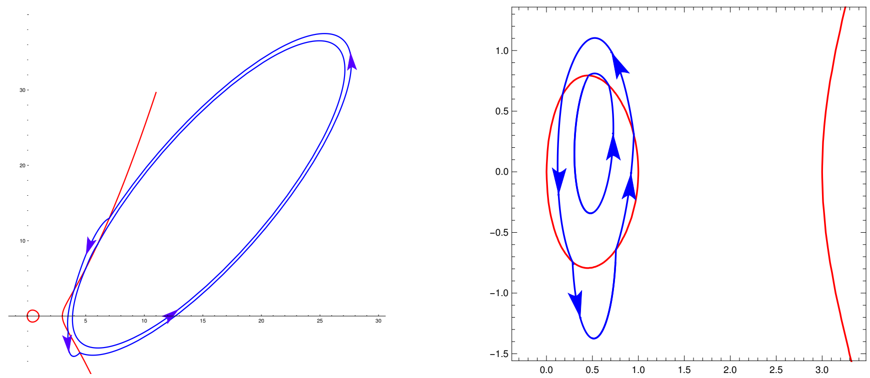
$$H_1(x, y) = 4(x + 0.36887..y)^2 + 8(-5.27202..x - 1.38826..y) + 0.126743..y^2.$$

The second linear differential center in the region R_2 is

$$(3)\quad \dot{x} = 0.0446206..x - 0.0377053..y - 0.789805.., \quad \dot{y} = x - 0.0446206y - 4.44641,$$

its first integral is

$$H_2(x, y) = 4(x - 0.0446206..y)^2 + 8(0.789805..y - 4.44641..x) + 0.142857..y^2.$$



$(\mathcal{C}_2^1)$ $(\mathcal{C}_2^2)$

FIGURE 5. The two limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_2^1)$ for (9)–(10), and $(\mathcal{C}_2^2)$ for (11)–(12)
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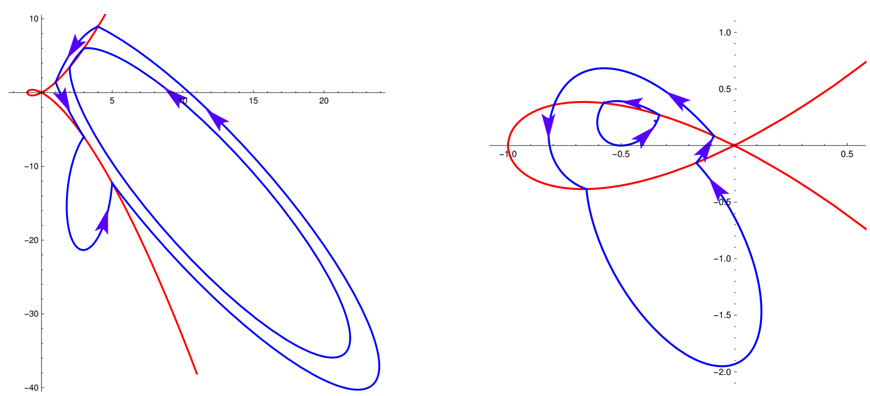


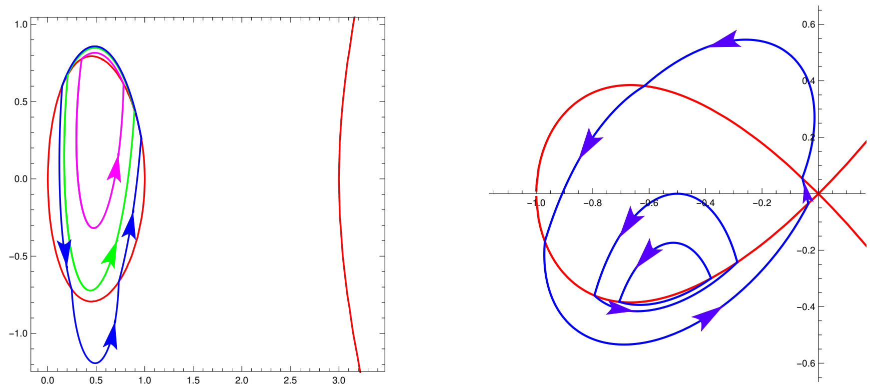
$(\mathcal{C}_5^1)$ $(\mathcal{C}_5^2)$

FIGURE 6. The two limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_5^1)$ for (13)–(14), and $(\mathcal{C}_5^2)$ for (15)–(16).

For the piecewise linear differential system (2)–(3) the real solutions of the system of equations

$$
\begin{array}{ll}
 & H_1(\alpha_1,\beta_1)-H_1(\gamma_1,\delta_1)=0,\\
 & H_1(\alpha_2,\beta_2)-H_1(\gamma_2,\delta_2)=0,\\
 & H_1(f,g)-H_1(h,k)=0,\\
(4) & H_2(\gamma_2,\delta_2)-H_2(\alpha_2,\beta_2)=0,\\
 & H_2(\gamma_1,\delta_1)-H_2(h,k)=0,\\
 & H_2(\alpha_1,\beta_1)-H_2(f,g)=0,\\
 & c_i(\alpha_s,\beta_s)=0, \quad c_i(\gamma_s,\delta_s)=0, \quad s=1,2,\\
 & c_i(f,g)=0, \quad c_i(h,k)=0.
\end{array}
$$

when $i=1$ are $(\alpha_1,\beta_1,\gamma_1,\delta_1,f,g,h,k)=(1,\sqrt{3},4,2\sqrt{21},3,-\sqrt{39},6,-\sqrt{258})$ and $(\alpha_2,\beta_2,\gamma_2,\delta_2)=(2,\sqrt{14},3,\sqrt{39})$. Hence this piecewise differential system has exactly two limit cycles, the $(\mathcal{C}_1)$ of Figure 4.



($\mathcal{C}_2$) ($\mathcal{C}_5$)

FIGURE 7. The three limit cycles of the discontinuous piecewise linear differential system ($\mathcal{C}_2$) for (17)–(18), and ($\mathcal{C}_5$) for (20)–(21).

We prove the statement for the class C_3. We consider the first linear differential center in the region R_1

$$(5)\quad \dot{x} = -0.414893..x - 0.21524..y + 1.58113.., \quad \dot{y} = x + 0.414893..y - 4.92165..,$$

with its first integral

$$H_1(x,y) = 4(x + 0.414893..y)^2 + 8(-4.92165..x - 1.58113..y) + 0.172414..y^2.$$

The second linear differential center in the region R_2 is

$$(6)\quad \dot{x} = 0.0354525..x - 0.0325069..y - 0.53107.., \quad \dot{y} = x - 0.0354525..y - 3.71414..,$$

this differential system has the first integral

$$H_2(x,y) = 4(x - 0.0354525..y)^2 + 8(0.53107..y - 3.71414..x) + 0.125..y^2.$$

For this piecewise linear differential centers the real solutions of system (4) when $i = 3$ are $(\alpha_1, \beta_1, \gamma_1, \delta_1, f, g, h, k) = (1, 1, 4, 8, 2.7.., -4.43655.., 5, -5\sqrt{5})$ and $(\alpha_2, \beta_2, \gamma_2, \delta_2) = (2, 2\sqrt{2}, 3, 3\sqrt{3})$. Hence the discontinuous piecewise linear differential system (5)–(6) has exactly two limit cycles, the ($\mathcal{C}_3$) of Figure 4.

We prove the statement for the class C_4. We consider the first linear differential center in the region R_1

$$(7)\quad \dot{x} = -0.335761..x - 0.13919..y + 1.6245.., \quad \dot{y} = x + 0.335761y - 6.25651,$$

this system has the first integral

$$H_1(x,y) = 4(x + 0.335761..y)^2 + 8(-6.25651..x - 1.6245..y) + 0.10582..y^2.$$

In the region R_2 we consider the linear differential center

$$(8)\quad \dot{x} = 0.0223501..x - 0.0197303..y - 0.63202.., \quad \dot{y} = x - 0.0223501..y - 5.15837..,$$

its first integral is

$$H_2(x,y) = 4(x - 0.0223501..y)^2 + 8(0.63202..y - 5.15837..x) + 0.0769231..y^2.$$

For this piecewise differential system the real solutions of system (4), when $i = 4$ are $(\alpha_1, \beta_1, \gamma_1, \delta_1, f, g, h, k) = (1, 2, 5, 10, 4.5, -8.41873.., 7, -7\sqrt{6})$ and $(\alpha_2, \beta_2, \gamma_2, \delta_2) =$

$(3, 3\sqrt{2}, 4, 4\sqrt{3})$. Hence the discontinuous piecewise linear differential system (7)–(8) has two limit cycles, see $(\mathcal{C}_4)$ of Figure 4.

For the first configuration of the class C_2 we consider in the region R_1 the linear differential center

(9) $\dot{x} = 0.463879..x - 0.325803..y - 1.92759.., \quad \dot{y} = x - 0.463879..y - 8.00533..,$

its first integral is

$$H_1(x, y) = 4(x - 0.463879..y)^2 + 8(1.92759..y - 8.00533..x) + 0.442478..y^2.$$

In the region R_2 we consider the center

(10) $\dot{x} = 0.173662..x - 0.0539679..y - 0.773986.., \quad \dot{y} = x - 0.173662..y - 4.90342..,$

with the first integral

$$H_2(x, y) = 4(x - 0.173662..y)^2 + 8(0.773986..y - 4.90342..x) + 0.0952381..y^2.$$

For this piecewise differential system the real solutions of system (4) when $i = 2$ are $(\alpha_1, \beta_1, \gamma_1, \delta_1, f, g, h, k) = (4, 2\sqrt{3}, 7, 2\sqrt{42}, 3.5, -2.09165, 4.5, -4.86056)$ and $(\alpha_2, \beta_2, \gamma_2, \delta_2) = (5, 2\sqrt{10}, 6, 2/21)$. So the discontinuous piecewise linear differential system (9)–(10) has exactly two limit cycles, the $(\mathcal{C}_2^1)$ of Figure 5.

For the second configuration of the class C_2 we consider in the region R_2 the linear differential center

(11)
$\dot{x} = 0.00451658..x - 0.15002..y - 0.0544755.., \quad \dot{y} = x - 0.00451658..y - 0.520808..,$

its first integral is

$$H_1(x, y) = 4x^2 - 0.0361326..xy - 4.16646..x + 0.360082..y^2 + 0.435804..y.$$

In the region R_3 we consider the center

$$\dot{x} = \frac{x}{16} - \frac{229y}{1280}, \quad \dot{y} = x - \frac{y}{16} - \frac{1}{2}, \tag{12}$$

which has the first integral

$$H_2(x, y) = 4x^2 - \frac{xy}{2} - 4x + \frac{809y^2}{1600}.$$

For this piecewise differential system the real solutions of system (4) when $i = 2$ are $(\alpha_1, \beta_1, \gamma_1, \delta_1, f, g, h, k) = (0.950831.., 0.309517.., 0.175649.., 0.639497.., 0.755729.., -0.64366.., 0.283963.., -0.743133..)$ and $(\alpha_2, \beta_2, \gamma_2, \delta_2) = (0.679472.., 0.710906.., 0.459819.., 0.79432..)$. Then the discontinuous piecewise linear differential system (11)–(12) has exactly two limit cycles, the $(\mathcal{C}_2^2)$ of Figure 5. Hence the statement holds for this class.

Now we prove the statement for the first configuration of the class C_5. In the region R_3 we consider the linear differential center

(13) $\dot{x} = -0.413021..x - 0.223778..y + 1.58022.., \quad \dot{y} = x + 0.413021..y - 5.75093..,$

its first integral is

$$H_1(x, y) = 4(x + 0.413021..y)^2 + 8(-5.75093..x - 1.58022..y) + 0.212766..y^2.$$

In the region R_1 we consider the linear center

(14) $\dot{x} = 0.0543018..x - 0.038663..y - 0.691405.., \quad \dot{y} = x - 0.0543018y - 4.11607,$

with the first integral

$$H_2(x,y) = 4(x - 0.0543018..y)^2 + 8(0.691405..y - 4.11607x) + 0.142857..y^2.$$

The real solutions of system (4) for this piecewise diferential system are $(\alpha_1, \beta_1, \gamma_1, \delta_1, f, g, h, k) = (1, \sqrt{2}, 4, 4\sqrt{5}, 3, -6, 5, -5\sqrt{6})$ and $(\alpha_2, \beta_2, \gamma_2, \delta_2) = (2, 2\sqrt{3}, 3, 6)$. Hence the discontinuous piecewise differential system (13)–(14) has exactly two limit cycles, see $(\mathcal{C}_5^1)$ of Figure 6.

Now we prove the existence of the second configuration for the class C_5. In the region R_3 we consider the linear center

(15)
$$\dot{x} = -0.197926..x - 0.129175..y - 0.165481.., \quad \dot{y} = x + 0.197926..y + 0.440113..,$$

its first integral is

$$H_1(x,y) = 4(x + 0.197926..y)^2 + 8(0.440113..x + 0.165481..y) + 0.36..y^2.$$

The linear differential center in the region R_1 is

$$\dot{x} = \frac{x}{8} - \frac{17y}{64} + \frac{13}{100}, \quad \dot{y} = x - \frac{y}{8} + \frac{1}{2}, \tag{16}$$

this differential system has the first integral

$$H_2(x,y) = 4\left(x - \frac{y}{8}\right)^2 + 8\left(\frac{x}{2} - \frac{13y}{100}\right) + y^2.$$

The real solutions of system (4) for this piecewise differential system are $(\alpha_1, \beta_1, \gamma_1, \delta_1, f, g, h, k) = (-0.0896583.., 0.0855446.., -0.79597.., 0.359537.., -0.167711.., -0.153002.., -0.653327.., -0.384672..)$, and $(\alpha_2, \beta_2, \gamma_2, \delta_2) = (-0.331148.., 0.270824.., -0.577381.., 0.37535..)$. Then the piecewise differential system (15)–(16) has exactly two limit cycles, the $(\mathcal{C}_5^2)$ of Figure 6. This completes the proof of statement (a). □

Proof of statement (b) *of Theorem* 2. First we prove the statement for the class C_2. In the region R_2 we consider the linear differential center

(17)
$$\dot{x} = -0.0023633..x - 0.204451..y - 0.0976703.., \quad \dot{y} = x + 0.0023633..y - 0.48701..,$$

its first integral is

$$H_1(x,y) = 4x^2 + 0.0189064..xy - 3.89608..x + 0.66879..y^2 + 0.781362..y.$$

The second linear differential center in the region R_3 is

$$\dot{x} = \frac{x}{12} - \frac{113y}{720}, \quad \dot{y} = x - \frac{y}{12} - \frac{1}{2}, \tag{18}$$

this differential system has the first integral

$$H_2(x,y) = 4x^2 - \frac{2xy}{3} - 4x + \frac{349y^2}{900}.$$

For the piecewise differential system (17)–(18) the real solutions of the system of equations

$$
\begin{array}{l}
H_1(\alpha,\beta)-H_1(f_1,g_1)=0,\\
H_1(\gamma,\delta)-H_1(h_1,k_1)=0,\\
H_1(f_l,g_l)-H_2(h_l,k_l)=0, \quad l=2,3\\
H_2(\gamma,\delta)-H_2(\alpha,\beta)=0,\\
H_2(f_s,g_s)-H_2(h_s,k_s)=0, \quad s=1,2,3\\
c_i(\alpha,\beta)=c_i(\gamma,\delta)=0,\\
c_i(f_s,g_s)=c_i(h_s,k_s)=0.
\end{array} \tag{19}
$$

when $i=2$ are $(\alpha,\beta,\gamma,\delta,f_1,g_1,h_1,k_1)=(0.96332..,0.268263..,0.149696..,$ $0.602335,0.732696..,-0.666377..,0.248059..,-0.716045..)$, $(f_2,g_2,h_2,k_2)=$ $(0.892948..,0.448795..,0.20665..,0.676726..)$ and $(f_3,g_3,h_3,k_3)=(0.781319..,$ $0.615697..,0.349844..,0.776393..)$. Hence the piecewise differential system (17)–(18) has exactly three limit cycles, the $(\mathcal{C}_2)$ of Figure 7.

Now we prove the statement for the class C_5. In the region R_3 we consider the linear differential center

$$\dot{x}=0.399509..x-0.786467..y+0.2184.., \quad \dot{y}=x-0.399509..y+0.475968.., \tag{20}$$

its first integral is

$$H_1(x,y)=4(x-0.399509..y)^2+8(0.475968..x-0.2184..y)+2.50744..y^2.$$

In the region R_3 we consider the linear differential center

$$\dot{x}=\frac{x}{12}-\frac{37y}{144}-\frac{1}{10}, \quad \dot{y}=x-\frac{y}{12}+\frac{1}{2}, \tag{21}$$

it has the first integral

$$H_2(x,y)=4\left(x-\frac{y}{12}\right)^2+8\left(\frac{x}{2}+\frac{y}{10}\right)+y^2.$$

The real solutions of system (19) for this piecewise differential system, when $i=5$ are $(\alpha,\beta,\gamma,\delta,f_1,g_1,h_1,k_1)=(-0.0574586..,0.0557834..,-0.616663..,0.381802..,$ $-0.0360021..,-0.0353481..-0.969557..,-0.169168..)$, $(f_2,g_2,h_2,k_2)=(-0.287727..,$ $-0.242831..,-0.793931..,-0.360404..)$ and $(f_3,g_3,h_3,k_3)=(-0.380796..,-0.299647..,$ $-0.705582..,-0.382851..)$. Then the piecewise differential system (20)–(21) has exactly three limit cycles, the $(\mathcal{C}_5)$ of Figure 7.

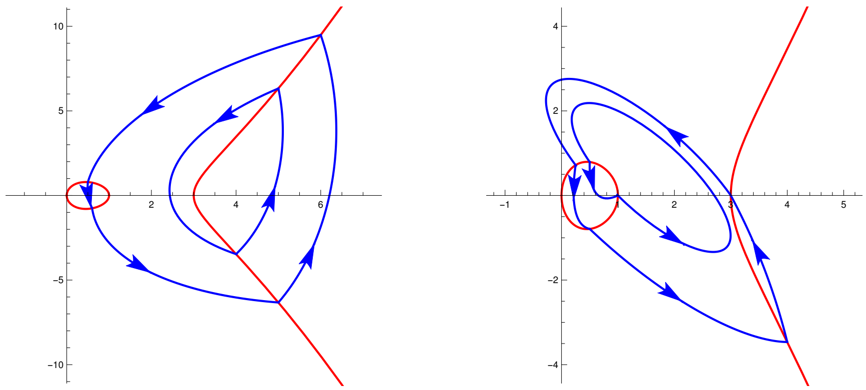
This completes the proof of Theorem 2. $\square$

$(\mathcal{C}_5^3)$

3. Proof of Theorem 3

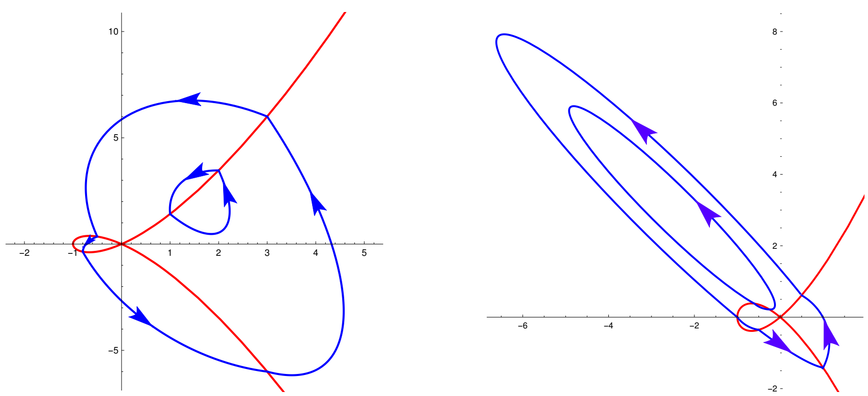
Proof of statement (*a*) *of Theorem* 3. We prove the statement for the first possible configuration of the class C_2. We consider the first linear differential center in the region R_1

$$\dot{x}=\frac{23}{156}\sqrt{\sqrt{\frac{10}{3}}+\frac{17}{6}}-\frac{y}{12}, \quad \dot{y}=x+\frac{1}{468}(115\sqrt{30}-1824), \tag{22}$$



$(\mathcal{C}_2^1)$ $(\mathcal{C}_2^2)$

FIGURE 8. The two limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_2^1)$ for (22)–(24), and $(\mathcal{C}_2^2)$ for (26)–(28).



$(\mathcal{C}_5^1)$ $(\mathcal{C}_5^2)$

FIGURE 9. The two limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_5^1)$ for (30)–(32), and $(\mathcal{C}_5^2)$ for (33)–(35).

this system has the first integral

$$H_1(x,y) = \frac{1}{6\sqrt{3}-9\sqrt{10}}(4x(6\sqrt{3}x - 9\sqrt{10}x - 91\sqrt{3} + 79\sqrt{10}) + y(2\sqrt{3}y - 3\sqrt{10} + 46)).$$

In the second region R_2 we consider the linear differential center

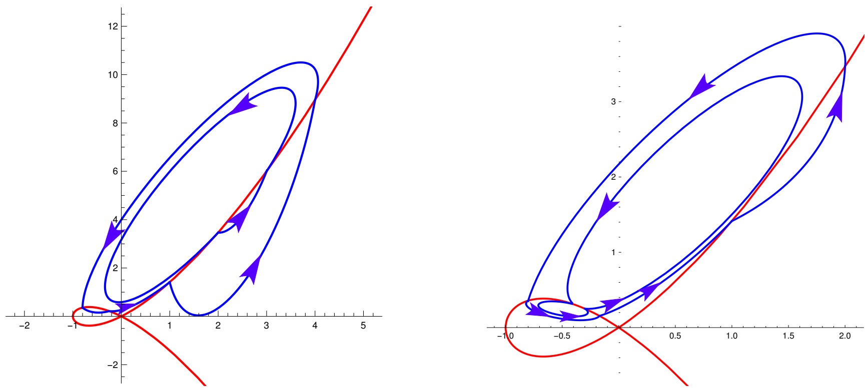
$$\begin{aligned} \dot{x} &= 0.257906..x - 0.816516..y - 0.369385.., \\ \dot{y} &= x - 0.257906..y - 7.81763.., \end{aligned} \tag{23}$$

its first integral is given by

$$H_2(x,y) = 4(x - 0.257906..y)^2 + 8(0.369385..y - 7.81763..x) + 3y^2.$$

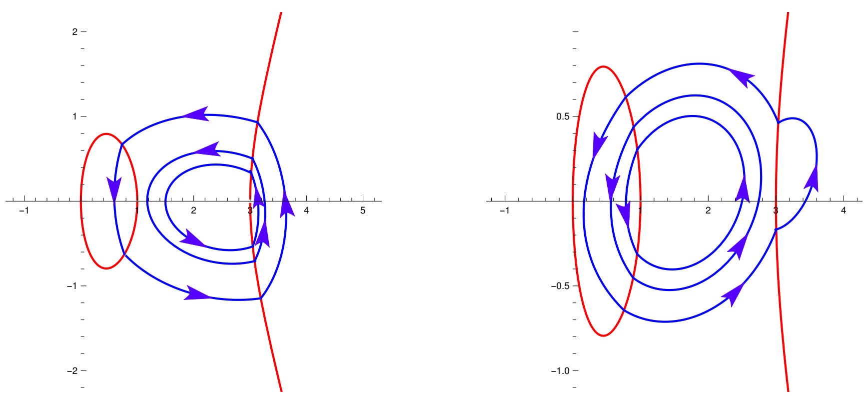
In the third region R_3 we consider the linear differential center

$$\dot{x} = \frac{x}{6} - \frac{19y}{36} + \frac{1}{6}, \quad \dot{y} = x - \frac{y}{6} - \frac{3029}{7200} - \frac{253}{20\sqrt{10}}, \tag{24}$$



($\mathcal{C}_5^3$) ($\mathcal{C}_5^4$)

FIGURE 10. The two limit cycles of the discontinuous piecewise linear differential system ($\mathcal{C}_5^3$) for (36)–(38), and ($\mathcal{C}_5^4$) for (39)–(41).



($\mathcal{C}_2^1$) ($\mathcal{C}_2^2$)

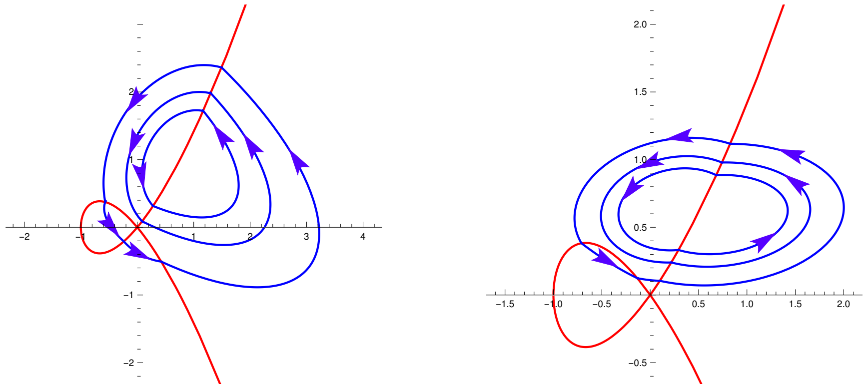
FIGURE 11. The two limit cycles of the discontinuous piecewise linear differential system ($\mathcal{C}_2^1$) for (42)–(44), and ($\mathcal{C}_2^2$) for (45)–(47).

this differential system has the first integral

$$H_3(x,y) = 4x^2 - \frac{1}{900}x(1200y + 9108\sqrt{10} + 3029) + \frac{1}{9}y(19y - 12).$$

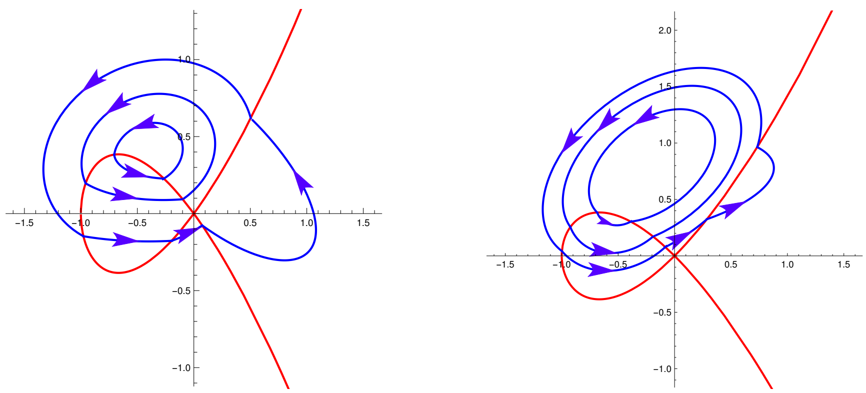
For the piecewise linear differential system (22)–(24) the real solutions of the system of equations

$$\begin{aligned}
&H_1(\alpha_i,\beta_i) - H_1(\gamma_i,\delta_i) = 0,\\
&H_2(\alpha_s,\beta_s) - H_2(\gamma_s,\delta_s) = 0,\\
&H_2(f,g) - H_2(\alpha_1,\beta_1) = 0,\\
&H_2(\gamma_1,\delta_1) - H_2(h,k) = 0,\\
&H_3(h,k) - H_3(f,g) = 0,\\
&c_l(\alpha_i,\beta_i) = c_l(\gamma_i,\delta_i) = 0,\\
&c_l(f,g) = c_l(h,k) = 0.
\end{aligned} \tag{25}$$



$(\mathcal{C}_5^1)$ $(\mathcal{C}_5^2)$

FIGURE 12. The two limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_5^1)$ for (48)–(50), and $(\mathcal{C}_5^2)$ for (51)–(53).



$(\mathcal{C}_5^3)$ $(\mathcal{C}_5^4)$

FIGURE 13. The two limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_5^3)$ for (54)–(56), and $(\mathcal{C}_5^4)$ for (57)–(59).

when $l = 2$, $i = 1, 2$ and $s = 2$ are $(\alpha_1, \beta_1, \gamma_1, \delta_1, f, g, h, k) = (6, 3\sqrt{10}, 5, -2\sqrt{10}, 1/2, \sqrt{5}/(2\sqrt{2}), 3/5, -(6\sqrt{2})/(5\sqrt{5}))$ and $(\alpha_2, \beta_2, \gamma_2, \delta_2) = (5, 2\sqrt{10}, 4, -2\sqrt{3})$. Then the piecewise differential system (22)–(24) has exactly two limit cycles, the $(\mathcal{C}_2^1)$ of Figure 8.

For the second configuration of the class C_2 in the region R_1 we consider the linear center

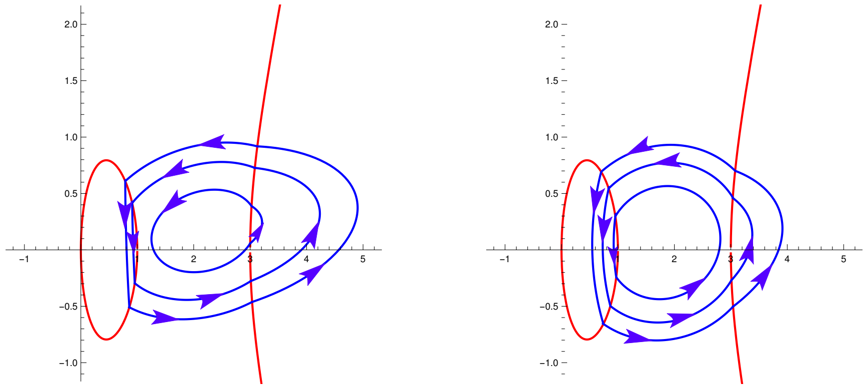
$$\dot{x} = -\frac{x}{6} - \frac{5y}{18} + \frac{1}{36}\left(24 - 37\sqrt{3}\right), \quad \dot{y} = x + \frac{y}{6} + 1, \tag{26}$$

which has the first integral

$$H_1(x, y) = 4\left(x + \frac{y}{6}\right)^2 + 8\left(x - \frac{1}{36}\left(24 - 37\sqrt{3}\right)y\right) + y^2.$$

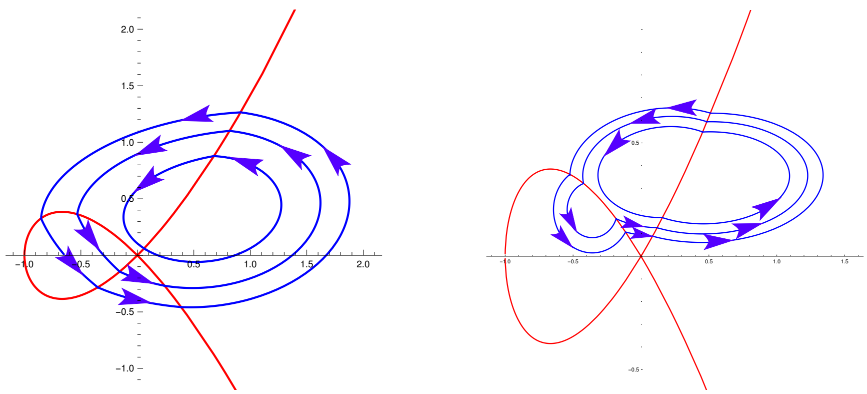
In the region R_2 we have the differential center

$$\dot{x} = -0.623758..x - 0.639075..y + 1.26279.., \quad \dot{y} = x + 0.623758..y - 1.8541.., \tag{27}$$



$(\mathcal{C}_2^1)$ $(\mathcal{C}_2^2)$

FIGURE 14. The three limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_2^1)$ for (60)–(62), and $(\mathcal{C}_2^2)$ for (64)–(66).



$(\mathcal{C}_5^1)$ $(\mathcal{C}_5^2)$

FIGURE 15. The three limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_5^1)$ for (68)–(70), and $(\mathcal{C}_5^2)$ for (71)–(73).

it has the first integral

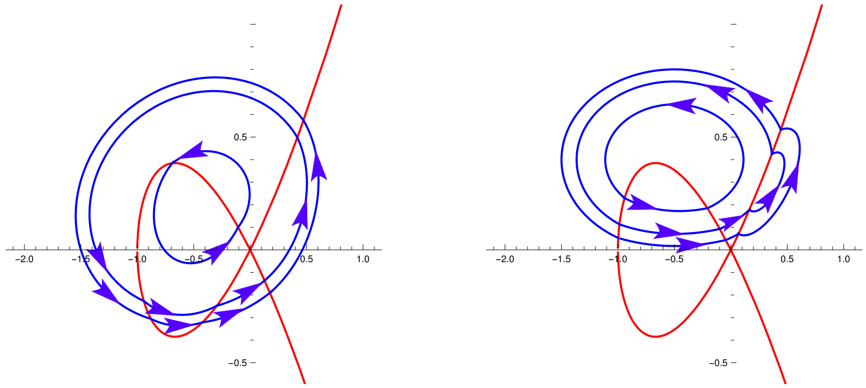
$$H_2(x,y) = 4(x + 0.623758..y)^2 + 8(-1.8541..x - 1.26279..y) + y^2.$$

The third linear center in the region R_3 is given by

$$\begin{aligned} \dot{x} &= \frac{x}{3} - \frac{11y}{72} + \frac{189 - 32\left(2\sqrt{10}+\sqrt{33}\right)}{384\left(\sqrt{10}+\sqrt{33}\right)}, \\ \dot{y} &= x - \frac{y}{3} + \frac{-567\sqrt{330} - 96\sqrt{330\left(43 - 2\sqrt{330}\right)} - 29014}{52992}, \end{aligned} \tag{28}$$

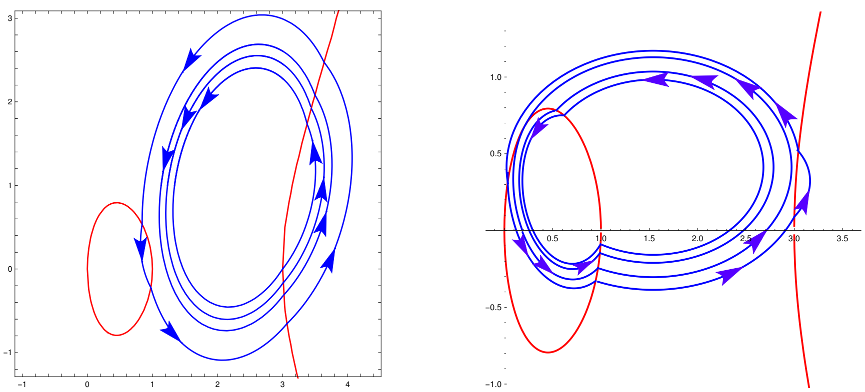
its first integral is

$$\begin{aligned} H_3(x,y) = \;& 4\left(x - \frac{y}{3}\right)^2 + \frac{y^2}{6} + \frac{1}{288\left(\sqrt{10}+\sqrt{33}\right)} 6\left(64\sqrt{10} + 32\sqrt{33} - 189\right) y \\ & - \left(2075\sqrt{10} + 1508\sqrt{33} + 96\sqrt{330}\right) x. \end{aligned}$$



$(\mathcal{C}_5^3)$ $(\mathcal{C}_5^4)$

FIGURE 16. The three limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_5^3)$ for (74)–(76), and $(\mathcal{C}_5^4)$ for (77)–(79).



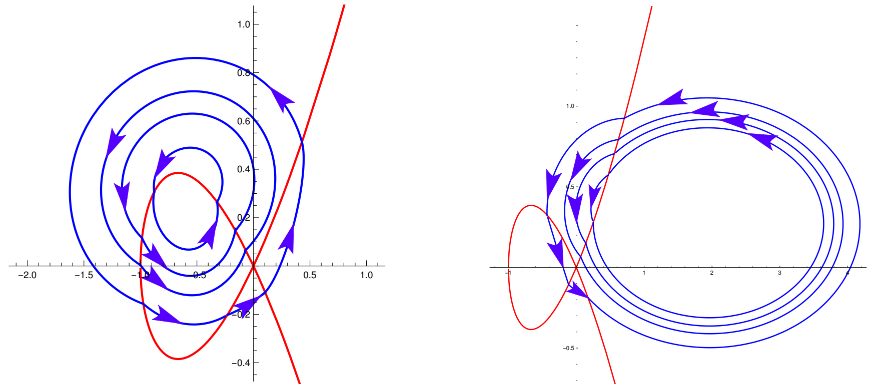
$(\mathcal{C}_2^1)$ $(\mathcal{C}_2^2)$

FIGURE 17. The four limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_2^1)$ for (80)–(82), and $(\mathcal{C}_2^2)$ for (83)–(85).

For the piecewise linear differential system (26)–(28) the real solutions of the system of equations

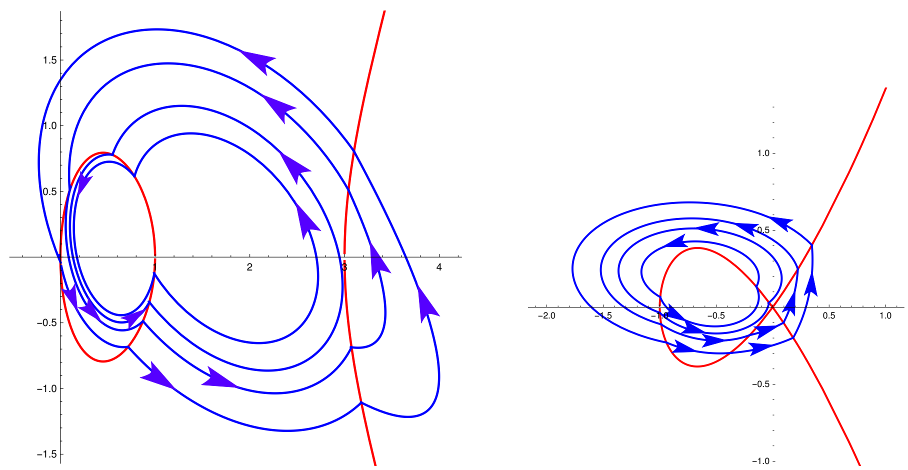
$$
\begin{aligned}
&H_1(\alpha,\beta) - H_1(\gamma,\delta) = 0,\\
&H_2(f_1,g_1) - H_2(\alpha,\beta) = 0,\\
&H_2(\gamma,\delta) - H_2(h_1,k_1) = 0,\\
&H_2(h_s,k_s) - H_2(f_s,g_s) = 0,\\
&H_3(h_i,k_i) - H_3(f_i,g_i) = 0,\\
&c_l(\alpha,\beta) = c_l(\gamma,\delta) = 0,\\
&c_l(f_i,g_i) = c_l(h_i,k_i) = 0.
\end{aligned}
\tag{29}
$$

when $l = 2$, $i = 1, 2$ and $s = 1$ are $(\alpha, \beta, \gamma, \delta, f_1, g_1, h_1, k_1) = (3, 0, 4, -2\sqrt{3}, 1/4, \sqrt{33}/8, 1/2, -\sqrt{5}/(2\sqrt{2}))$ and $(f_2, g_2, h_2, k_2)) = (1/2, \sqrt{5}/(2\sqrt{2}), 1, 0)$. Hence the piecewise differential system (26)–(28) has exactly two limit cycles, the $(\mathcal{C}_2^2)$ of Figure 8.



$(\mathcal{C}_5^1)$ $(\mathcal{C}_5^2)$

FIGURE 18. The four limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_5^1)$ for (86)–(88), and $(\mathcal{C}_5^2)$ for (89)–(91).



$(\mathcal{C}_2)$ $(\mathcal{C}_5^1)$

FIGURE 19. The four limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_2)$ for (92)–(94), and $(\mathcal{C}_5^1)$ for (96)–(98).

We prove statement (a) for the first configuration of the class C_5. We consider the linear differential center in the region R_1

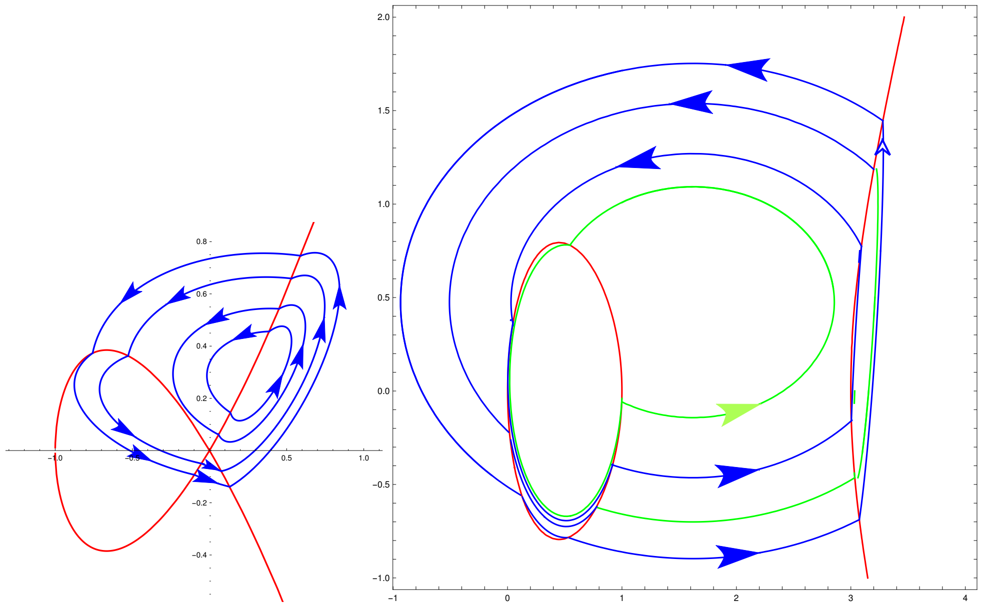
$$\dot{x} = \frac{1}{8}(-2x - y + 6), \quad \dot{y} = x + \frac{1}{8}\left(2y - 4\sqrt{2} + 4\sqrt{3} - 17\right), \tag{30}$$

this system has the first integral

$$H_1(x, y) = 4x^2 + x\left(2y - 4\sqrt{2} + 4\sqrt{3} - 17\right) + \frac{1}{2}(y - 12)y.$$

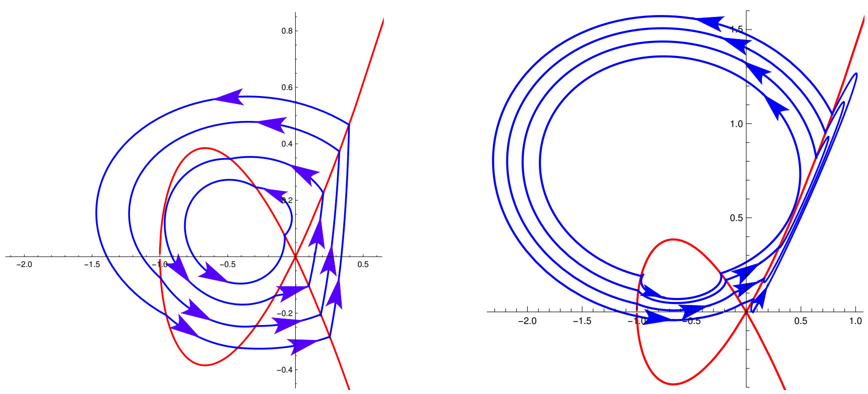
In the second region R_2 we consider the linear differential center

$$\dot{x} = -0.163799..x - 0.27683..y + 0.613613.., \quad \dot{y} = x + 0.163799..y - 2.5295.., \tag{31}$$



$(\mathcal{C}_5^2)$ $(\mathcal{C}_2)$

FIGURE 20. The four limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_5^2)$ for (99)–(101), and $(\mathcal{C}_2)$ for (103)–(105).



$(\mathcal{C}_5^1)$ $(\mathcal{C}_5^2)$

FIGURE 21. The four limit cycles of the discontinuous piecewise linear differential system $(\mathcal{C}_5^1)$ for (106)–(108), and $(\mathcal{C}_5^2)$ for (109)–(111).

its first integral is

$$H_2(x,y) = 4(x + 0.163799..y)^2 + 8(-2.5295..x - 0.613613..y) + y^2.$$

In the third region R_3 we consider the linear differential center

$$\dot{x} = -\frac{7x}{10} - \frac{99y}{100} - 1, \quad \dot{y} = x + \frac{7y}{10} - \frac{5}{12}\left(2\sqrt{2} - \frac{39297}{25000} - \frac{7}{5\sqrt{2}} + \frac{352}{125\sqrt{5}}\right), \tag{32}$$

this differential system has the first integral

$$H_3(x,y) = 4\left(x + \frac{7y}{10}\right)^2 + 8\left(y - \frac{5}{12}\left(2\sqrt{2} - \frac{39297}{25000} - \frac{7}{5\sqrt{2}} + \frac{352}{125\sqrt{5}}\right)x\right) + 2y^2.$$

For this piecewise differential system the real solutions of the system of equations (25) when $l = 5$ are $(\alpha_1, \beta_1, \gamma_1, \delta_1, f, g, h, k) = (3, 6, 3, -6, -1/2, 1/(2\sqrt{2}), -4/5, -4/(5\sqrt{5}))$, and $(\alpha_2, \beta_2, \gamma_2, \delta_2) = (2, 2\sqrt{3}, 1, \sqrt{2})$, which provide exactly two limit cycles, the $(\mathcal{C}_5^1)$ of Figure 9.

For the second configuration of the class C_5 in the region R_1 we consider the linear center

$$\dot{x} = -\frac{3y}{4} - \frac{347}{80\sqrt{2}\left(\sqrt{3}+4\right)}, \quad \dot{y} = x + \frac{1}{5}, \tag{33}$$

its first integral is

$$H_1(x,y) = 4x^2 + \frac{8x}{5} + 3y^2 + \frac{347y}{10\sqrt{2}\left(\sqrt{3}+4\right)}.$$

In the region R_2 we consider the following linear center

$$\dot{x} = -0.804636..x - 0.70994..y + 0.145892.., \quad \dot{y} = x + 0.804636..y + 0.056572.., \tag{34}$$

its first integral is

$$H_2(x,y) = 4(x + 0.804636..y)^2 + 8(0.056572..x - 0.145892..y) + 0.25y^2.$$

The third linear center in the region R_3 is

$$\begin{aligned} \dot{x} &= \frac{x}{4} - \frac{7y}{16} - \frac{5600\sqrt{2} + 2000\sqrt{3} - 4608\sqrt{10} + 18781}{1280\left(-35\sqrt{2} - 25\sqrt{3} + 24\sqrt{10}\right)}, \\ \dot{y} &= x - \frac{y}{4} - \frac{43519\sqrt{2} + 44500\sqrt{3} + 1536\sqrt{5} + 2000\sqrt{6} - 42720\sqrt{10}}{2560\left(-35\sqrt{2} - 25\sqrt{3} + 24\sqrt{10}\right)}, \end{aligned} \tag{35}$$

which has the first integral

$$\begin{aligned} H_3(x,y) = \ & \frac{1}{320(-35\sqrt{2} - 25\sqrt{3} + 24\sqrt{10})} 1280(-35\sqrt{2} - 25\sqrt{3} + 24\sqrt{10})x^2 \\ & -x(640(-35\sqrt{2} - 25\sqrt{3} + 24\sqrt{10})y + 43519\sqrt{2} + 44500\sqrt{3} + 96(16 \\ & -445\sqrt{2})\sqrt{5} + 2000\sqrt{6}) + 2y(280(-35\sqrt{2} - 25\sqrt{3} + 24\sqrt{10})y \\ & +5600\sqrt{2} + 2000\sqrt{3} - 4608\sqrt{10} + 18781). \end{aligned}$$

For this piecewise differential system the real solutions of the system of equations (29) when $l = 5$ are $(\alpha, \beta, \gamma, \delta, f_1, g_1, h_1, k_1) = (1/2, \sqrt{3}/(2\sqrt{2}), 1, -\sqrt{2}, -1, 0, -1/2, -1/(2\sqrt{2}))$ and $(f_2, g_2, h_2, k_2) = (-3/5, (3\sqrt{2})/(5\sqrt{5}), -1/4, \sqrt{3}/8)$, which provide exactly two limit cycles, the $(\mathcal{C}_5^2)$ of Figure 9.

Now we prove the statement for the third configuration of the class C_5. In the region R_1 we consider the linear center

$$
\begin{aligned}
\dot{x} &= \frac{x}{8} - \frac{3y}{64} + \frac{-8\sqrt{2} + 96\sqrt{3} + 128\sqrt{5} - 441}{64\left(\sqrt{2} - 6\sqrt{3} - 4\sqrt{5} + 18\right)}, \\
\dot{y} &= x - \frac{y}{8} - \frac{50\sqrt{2} - 597\sqrt{3} - 488\sqrt{5} + 8\sqrt{6} + 64\sqrt{15} + 1791}{32\left(\sqrt{2} - 6\sqrt{3} - 4\sqrt{5} + 18\right)},
\end{aligned} \tag{36}
$$

its first integral is

$$
\begin{aligned}
H_1(x, y) = \; & \frac{1}{16(\sqrt{2} - 6\sqrt{3} - 4\sqrt{5} + 18)} 64(\sqrt{2} - 6\sqrt{3} - 4\sqrt{5} + 18)x^2 - 4x(4(\sqrt{2} \\
& -6\sqrt{3} - 4\sqrt{5} + 18)y + 50\sqrt{2} - 597\sqrt{3} - 488\sqrt{5} + 8\sqrt{6} + 64\sqrt{15} \\
& +1791) + y(3(\sqrt{2} - 6\sqrt{3} - 4\sqrt{5} + 18)y + 2(8\sqrt{2} - 96\sqrt{3} - 128\sqrt{5} \\
& +441)).
\end{aligned}
$$

In the region R_2 we consider the following linear center

$$
\dot{x} = 0.37865..x - 0.196567..y + 0.369324.., \quad \dot{y} = x - 0.37865..y + 0.270094.., \tag{37}
$$

its first integral is

$$
H_2(x, y) = 4.(x - 0.37865..y)^2 + 8(0.270094..x - 0.369324..y) + 0.212766..y^2.
$$

Now we consider the third linear center in the region R_3

$$
\dot{x} = \frac{x}{5} - \frac{29y}{100} + \frac{1}{2}, \quad \dot{y} = x - \frac{y}{5} - \frac{5}{22}\left(-\frac{9\sqrt{3}}{20} - \frac{480821}{200000} + \frac{272}{125\sqrt{5}}\right), \tag{38}
$$

its first integral is

$$
H_3(x, y) = 4\left(x - \frac{y}{5}\right)^2 + 8\left(-\frac{5}{22}\left(-\frac{9\sqrt{3}}{20} - \frac{480821}{200000} + \frac{272}{125\sqrt{5}}\right)x - \frac{y}{2}\right) + y^2.
$$

In this case the real solutions of system (25) when $l = 5$ are $(\alpha_1, \beta_1, \gamma_1, \delta_1, f, g, h, k) = (1, \sqrt{2}, 4, 4\sqrt{5}, -1/4, \sqrt{3}/8, -4/5, 4/(5\sqrt{5}))$ and $(\alpha_2, \beta_2, \gamma_2, \delta_2) = (2, 2\sqrt{3}, 3, 6)$. So the piecewise differential system (36)–(38) has exactly two limit cycles, the $(\mathcal{C}_5^3)$ of Figure 10.

For the fourth configuration of the class C_5 in the region R_1 we consider the linear center

$$
\dot{x} = -\frac{3y}{4} - \frac{109}{20\left(\sqrt{2} - 2\sqrt{3}\right)}, \quad \dot{y} = x + \frac{1}{5}, \tag{39}
$$

its first integral is

$$
H_1(x, y) = 4x^2 + \frac{8x}{5} + 3y^2 - \frac{109}{25}\left(\sqrt{2} + 2\sqrt{3}\right)y.
$$

In the second region R_2 we consider the following linear center

$$
\dot{x} = 0.552823..x - 0.450613..y + 0.485408.., \quad \dot{y} = x - 0.552823..y + 0.407605.., \tag{40}
$$

its first integral is

$$
H_2(x, y) = 4(x - 0.552823..y)^2 + 8(0.407605x - 0.485408..y) + 0.58..y^2.
$$

The third linear center in the region R_3 is

$$
\dot{x} = -x - \frac{19y}{4} + 0.686294.., \quad \dot{y} = x + y + 0.243569.., \tag{41}
$$

its first integral is

$$H_3(x,y)=4x^2+8x(y+0.243569..)+19(y-0.288966..)y.$$

In this case the real solutions of system (29) when $l=5$ are $(\alpha,\beta,\gamma,\delta,f_1,g_1,h_1,k_1)=(1,\sqrt{2},2,2\sqrt{3},-0.17..,0.154877..,-4/5,4/5\sqrt{5})$ and $(f_2,g_2,h_2,k_2)=(-0.3,0.250998..,-0.4,0.309839..),\sqrt{2})/(5\sqrt{5}),-1/4,\sqrt{3}/8)$, which provide exactly two limit cycles, the $(\mathcal{C}_5^4)$ of Figure 10. This completes the proof of statement (a) of Theorem 3. □

Proof of statement (*b*) *of Theorem* 3. We prove the statement for the first configuration of the class C_2. In the region R_1 we consider the linear differential center

$$\begin{aligned}\dot{x}&=-0.0646393..x-1.00418..y+0.0706613..,\\ \dot{y}&=x+0.0646393..y-2.26028..,\end{aligned}\tag{42}$$

this system has the first integral

$$H_1(x,y)=4(x+0.0646393..y)^2+8(-2.26028..x-0.0706613..y)+4y^2.$$

In the region R_2 we consider the linear differential center

$$\dot{x}=-\frac{x}{4}-\frac{65y}{16}+\frac{1}{3},\quad \dot{y}=x+\frac{y}{4}-\frac{5}{2},\tag{43}$$

which has the first integral

$$H_2(x,y)=4\left(x+\frac{y}{4}\right)^2+8\left(\frac{1}{2}(-5)x-\frac{y}{3}\right)+16y^2.$$

In the third region R_3 we consider the differential center

$$\dot{x}=0.0628346..-y,\quad \dot{y}=x-2,\tag{44}$$

this differential system has the first integral

$$H_3(x,y)=4x^2+8(-2x-0.0628346..y)+4y^2.$$

For the piecewise differential system (42)–(44) the real solutions of the system of equations (25) when $l=2$, $i=1,2,3$ and $s=2,3$ are $(\alpha_1,\beta_1,\gamma_1\delta_1,f,g,h,k)=(3.12988..,0.930497..,3.18837..,-1.14645..,0.729843..,0.669038..,0.772505..,-0.625669..)$, $(\alpha_2,\beta_2,\gamma_2,\delta_2)=(3.04078..,0.503058..,3.07822..,-0.707378..)$ and $(\alpha_3,\beta_3,\gamma_3,\delta_3)=(3.01822..,0.333121..,3.04581..,-0.534252..),\sqrt{2})/(5\sqrt{5}),-1/4,\sqrt{3}/8)$, which provide exactly three limit cycles, the $(\mathcal{C}_2^1)$ of Figure 11.

For the fourth configuration of the class C_5 we consider the linear center

$$\dot{x}=\frac{x}{2}-\frac{5y}{2}-1.14278,\quad \dot{y}=x-\frac{y}{2}-3,\tag{45}$$

in the region R_1 with its first integral

$$H_1(x,y)=4\left(x-\frac{y}{2}\right)^2+8(1.14278y-3x)+9y^2.$$

In the region R_2 we have the linear center

$$\dot{x}=\frac{x}{3}-\frac{37y}{9}-\frac{1}{3},\quad \dot{y}=x-\frac{y}{3}-\frac{8}{5},\tag{46}$$

with its first integral

$$H_2(x,y)=4\left(x-\frac{y}{3}\right)^2+8\left(\frac{y}{3}-\frac{8x}{5}\right)+16y^2.$$

Now we consider the third linear center in the region R_3

(47) $\dot{x} = 0.875275..x - 9.76611..y - 0.875404.., \quad \dot{y} = x - 0.875275..y - 3.66604..,$

it has the first integral

$$H_3(x,y) = 4(x - 0.875275..y)^2 + 8(0.875404..y - 3.66604..x) + 36y^2.$$

For the piecewise differential system (45)–(47) the real solutions of the system of equations (29) when $l = 2$, $i = 1,2,3$ and $s = 2,3$ are $(\alpha, \beta, \gamma, \delta, f_1, g_1, h_1, k_1) = (3.03463.., 0.462412.., 3.00461.., -0.166713.., 0.782368.., 0.614485.., 0.755912.., -0.643471..)$, $(f_2, g_2, h_2, k_2) = (0.898459.., 0.437863.., 0.891227.., -0.452137..)$ and $(f_3, g_3, h_3, k_3) = (0.952472.., 0.304451.., 0.950276.., -0.311213..)$. Then the piecewise differential system (45)–(47) has exactly three limit cycles, the $(\mathcal{C}_2^2)$ of Figure 11.

Now we prove the existence of the first configuration of three limit cycles for the class C_5. We consider the linear differential center

(48) $\dot{x} = -0.954848..x - 1.91173..y + 2.89911.., \quad \dot{y} = x + 0.954848..y - 1.28151..,$

in the region R_1 with its first integral

$$H_1(x,y) = 4(x + 0.954848..y)^2 + 8(-1.28151..x - 2.89911..y) + 4y^2.$$

In the region R_2 we consider the differential center

$$\dot{x} = \frac{x}{5} - \frac{26y}{25} + \frac{4}{5}, \quad \dot{y} = x - \frac{y}{5} - \frac{7}{10}, \tag{49}$$

with its first integral

$$H_2(x,y) = 4\left(x - \frac{y}{5}\right)^2 + 8\left(\frac{1}{10}(-7)x - \frac{4y}{5}\right) + 4y^2.$$

In the region R_3 we consider the following linear differential center

$$\dot{x} = -x - \frac{13y}{4} + 0.329095, \quad \dot{y} = x + y - \frac{1}{2}, \tag{50}$$

this differential system has the first integral

$$H_3(x,y) = 4(x+y)^2 + 8\left(-\frac{x}{2} - 0.329095..y\right) + 9y^2.$$

For the piecewise differential system (48)–(50) the real solutions of the system of equations (25) when $l = 5$, $i = 1,2,3$ and $s = 2,3$ are $(\alpha_1, \beta_1, \gamma_1, \delta_1, f, g, h, k) = (1.16805.., 1.71987.., 0.279315.., 0.315925.., -0.567895.., 0.373304.., -0.238818.., -0.208359..)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2) = (1.30402, 1.97936, 0.0856017.., 0.0891903..)$ and $(\alpha_3, \beta_3, \gamma_3, \delta_3) = (1.49483.., 2.3611.., 0.424632.., -0.506832..)$, which provide exactly three limit cyles, the $(\mathcal{C}_5^1)$ of Figure 12.

For the second configuration we consider the linear differential center

(51) $\dot{x} = 0.248983..x - 6.31199..y + 3.57874.., \quad \dot{y} = x - 0.248983..y - 0.538596..,$

in the region R_1 with its first integral

$$H_1(x,y) = 4(x - 0.248983..y)^2 + 8(-0.538596..x - 3.57874..y) + 25y^2.$$

In the region R_2 we consider the linear center

$$\dot{x} = \frac{x}{4} - \frac{65y}{16} + \frac{5}{2}, \quad \dot{y} = x - \frac{y}{4} - \frac{1}{8}, \tag{52}$$

with its first integral

$$H_2(x,y) = 4\left(x - \frac{y}{4}\right)^2 + 8\left(-\frac{x}{8} - \frac{5y}{2}\right) + 16y^2.$$

In the region R_3 we consider the following linear differential center

$$\dot{x} = -x - 2y + 1.57994, \quad \dot{y} = x + y - \frac{1}{2}, \tag{53}$$

this differential system has the first integral

$$H_3(x,y) = 4(x+y)^2 + 8\left(-\frac{x}{2} - 1.57994y\right) + 4y^2.$$

For this piecewise differential system the real solutions of the system of equations (25) when $l = 5$, $i = 1,2,3$ and $s = 2,3$ are $(\alpha_1, \beta_1, \gamma_1, \delta_1, f, g, h, k) = (0.826701.., 1.11733.., 0.100008.., 0.10489.., -0.710248.., 0.382317.., -0.133816.., 0.124541..)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2) = (0.74172.., 0.978879.., 0.217731.., 0.240268..)$ and $(\alpha_3, \beta_3, \gamma_3, \delta_3) = (0.682347.., 0.885041.., 0.292618.., 0.332687..)$, which provide exactly three limit cycles, the $(\mathcal{C}_5^2)$ of Figure12.

Now we prove the existence of the third configuration. We consider the differential center

$$\dot{x} = -x - 2y + 0.867702, \quad \dot{y} = x + y - \frac{1}{2}, \tag{54}$$

in the region R_1 with its corresponding first integral

$$H_1(x,y) = 4(x+y)^2 + 8\left(-\frac{x}{2} - 0.867702y\right) + 4y^2.$$

In the region R_2 we consider the linear center

$$\dot{x} = \frac{x}{4} - \frac{37y}{16} + 1, \quad \dot{y} = x - \frac{y}{4} + \frac{1}{2}, \tag{55}$$

with its first integral

$$H_2(x,y) = 4\left(x - \frac{y}{4}\right)^2 + 8\left(\frac{x}{2} - y\right) + 9y^2.$$

In the region R_3 we consider the following linear differential center

$$\dot{x} = 0.475023x - 9.22565y + 3.93923, \quad \dot{y} = x - 0.475023y + 0.313639, \tag{56}$$

this differential system has the first integral

$$H_3(x,y) = 4(x - 0.475023..y)^2 + 8(0.313639..x - 3.93923..y) + 36y^2.$$

For this piecewise differential system the real solutions of the system of equations (29) when $l = 5$, $i = 1,2,3$ and $s = 2,3$ are $(\alpha, \beta, \gamma, \delta, f_1, g_1, h_1, k_1) = (0.504181.., 0.618352.., 0.0741887.., -0.0768915.., -0.704592.., 0.382956.., -0.265331.., 0.227423..)$, $(f_2, g_2, h_2, k_2) = (-0.957879.., 0.19659.., -0.0980231.., 0.0930949..)$ and $(f_3, g_3, h_3, k_3) = (-0.977383.., -0.146987.., -0.197311.., -0.176777..)$,, which provide exactly three limit cycles, the $(\mathcal{C}_5^3)$ of Figure 13.

Finally for the fourth configuration of the class C_5 we consider the linear differential center

$$\dot{x} = x - \frac{29y}{4} + 4.76533, \quad \dot{y} = x - y + 1, \tag{57}$$

in the region R_1 with its first integral

$$H_1(x,y) = 4(x-y)^2 + 8(x - 4.76533y) + 25y^2.$$

In the region R_2 we consider the linear center

$$\dot{x} = \frac{x}{2} - \frac{5y}{4} + \frac{11}{10}, \quad \dot{y} = x - \frac{y}{2} + \frac{3}{5}, \tag{58}$$

with its first integral

$$H_2(x,y) = 4\left(x - \frac{y}{2}\right)^2 + 8\left(\frac{3x}{5} - \frac{11y}{10}\right) + 4y^2.$$

We consider the linear differential center

$$\dot{x} = 0.212791x - 1.04528y + 0.620591, \quad \dot{y} = x - 0.212791y + 0.570351, \tag{59}$$

in the region R_3 with its first integral

$$H_3(x,y) = 4(x - 0.212791..y)^2 + 8(0.570351..x - 0.620591..y) + 4y^2.$$

For this piecewise differential system the real solutions of the system of equations (29) when $l = 5$, $i = 1,2,3$ and $s = 2,3$ are $(\alpha, \beta, \gamma, \delta, f_1, g_1, h_1, k_1) = (0.732306.., 0.96384.., -0.998207.., 0.042272.., -0.0818663.., 0.0784437.., -0.0818663.., 0.0784437..)$, $(f_2, g_2, h_2, k_2) = (-0.923415.., 0.255547.., -0.190494.., 0.171392..)$ and $(f_3, g_3, h_3, k_3) = (-0.685802.., 0.384415.., -0.388156.., 0.303617..)$. Hence the piecewise differential system (57)–(59) has exactly three limit cycles, the $(\mathcal{C}_5^4)$ of Figure 13.

This completes the proof of statement (b) of Theorem 3. □

Proof of statement (c) *of Theorem* 3. For the first configuration of the class C_2 we consider the differential center

$$\dot{x} = 1.02441..x - 13.2994..y - 0.0637063.., \quad \dot{y} = x - 1.02441..y - 1.96403.., \tag{60}$$

in the region R_1 with its first integral

$$H_1(x,y) = 4(x - 1.02441..y)^2 + 8(0.0637063..y - 1.96403..x) + 49y^2.$$

In the region R_2 we consider the linear differential center

$$\dot{x} = \frac{x}{2} - \frac{13y}{2}, \quad \dot{y} = x - \frac{y}{2} - \frac{21}{10}, \tag{61}$$

its first integral is

$$H_2(x,y) = 4\left(x - \frac{y}{2}\right)^2 - \frac{84x}{5} + 25y^2.$$

In the region R_3 we consider the differential center

$$\dot{x} = 0.699079 \ - y, \quad \dot{y} = x - 10.8215, \tag{62}$$

its first integral is

$$H_3(x,y) = 4x^2 + 8(-10.8215..x - 0.699079..y) + 4y^2.$$

For the piecewise linear differential system (60)–(62) the real solutions of the system of equations

$$
\begin{aligned}
&H_1(\alpha_i,\beta_i)-H_1(\gamma_i,\delta_i)=0,\\
&H_2(\alpha_3,\beta_3)-H_2(\gamma_3,\delta_3)=0,\\
&H_2(\gamma_s,\delta_s)-H_2(h_s,g_s)=0,\\
&H_2(\alpha_s,\beta_s)-H_2(f_s,g_s)=0,\\
&H_3(f_s,g_s)-H_3(h_s,g_s)=0,\\
&c_l(\alpha_i,\beta_i)=c_l(\gamma_i,\delta_i)=0,\\
&c_l(f_s,g_s)=c_l(h_s,k_s)=0.
\end{aligned}
\tag{63}
$$

when $l=2$, $i=1,2,3$ and $s=1,2$ are $(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1)=(3.12669..,0.91783..,$ $3.03453..,-0.461703..,0.78665..,0.609484..,0.859216..,-0.508878..)$, $(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,$ $g_2,h_2,k_2)=(3.08236..,0.727064..,3.01257..,-0.276091..,0.909377..,0.415077..,0.955309..,$ $-0.29546..)$ and $(\alpha_3,\beta_3,\gamma_3,\delta_3)=(3.02459..,0.388066..,3.00055..,0.0574778..)$. Then the piecewise differential system (60)–(62) has exactly three limit cycles, the $(\mathcal{C}_2^1)$ of Figure 14.

For the second configuration of the class C_2 we consider the linear center

$$\dot{x}=x-10y-2.01329..,\quad \dot{y}=x-y-1.23779..,\tag{64}$$

in the region R_1 with its first integral

$$H_1(x,y)=4(x-y)^2+8(2.01329..y-1.23779..x)+36y^2.$$

In the region R_2 we consider the following linear center

$$\dot{x}=\frac{x}{7}-\frac{197y}{49},\quad \dot{y}=x-\frac{y}{7}-\frac{9}{5},\tag{65}$$

its first integral is

$$H_2(x,y)=4\left(x-\frac{y}{7}\right)^2-\frac{72x}{5}+16y^2.$$

Now we consider the third linear center in the region R_3

$$\dot{x}=0.0132443..x-1.00018..y+0.0431369..,\quad \dot{y}=x-0.0132443..y-1.91288..,\tag{66}$$

its first integral is

$$H_3(x,y)=4(x-0.0132443..y)^2+8(-1.91288..x-0.0431369..y)+4y^2.$$

For the piecewise linear differential system (64)–(66) the real solutions of the system of equations

$$
\begin{aligned}
&H_1(\alpha_s,\beta_s)-H_1(\gamma_s,\delta_s)=0,\\
&H_2(f_3,g_3)-H_2(h_3,k_3)=0,\\
&H_2(\gamma_s,\delta_s)-H_2(h_s,g_s)=0,\\
&H_2(\alpha_s,\beta_s)-H_2(f_s,g_s)=0,\\
&H_3(f_i,g_i)-H_3(h_i,g_i)=0,\\
&c_l(\alpha_i,\beta_i)=c_l(\gamma_i,\delta_i)=0,\\
&c_l(f_s,g_s)=c_l(h_s,k_s)=0.
\end{aligned}
\tag{67}
$$

when $l=2$, $i=1,2,3$ and $s=1,2$ are $(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1)=(3.0769..,0.701031..,$ $3.04066..,-0.502302..,0.703362..,0.692228..,0.742906..,-0.65658..)$, $(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,$ $g_2,h_2,k_2)=(0.836248..,0.544333..,3.01191..,-0.268626..,3.03495..,0.464584..,0.866123..,$ $-0.497426..)$ and $(f_3,g_3,h_3,k_3)=(0.954587..,0.297776..,0.970358..,-0.241617..)$. Then the piecewise differential system (64)–(66) has exactly three limit cycles, see $(\mathcal{C}_2^2)$ of Figure 14.

Now we prove the statement for the first configuration of the class C_5. We consider the linear differential center

$$(68)\quad \dot{x} = 0.115964..x - 2.26345..y + 0.862862.., \quad \dot{y} = x - 0.115964..y - 0.521857..,$$

in the region R_1 with its first integral

$$H_1(x,y) = 4(x - 0.115964..y)^2 + 8(-0.521857..x - 0.862862..y) + 9y^2.$$

In the region R_2 we consider the linear center

$$(69)\quad \dot{x} = \frac{x}{4} - \frac{65y}{16} + \frac{7}{5}, \quad \dot{y} = x - \frac{y}{4} - \frac{4}{5},$$

with its first integral

$$H_2(x,y) = 4\left(x - \frac{y}{4}\right)^2 + 8\left(-\frac{9x}{5} - \frac{7y}{5}\right) + 16y^2.$$

In the region R_3 we consider the following linear differential center

$$(70)\quad \dot{x} = -0.3x - 6.34..y + 4.12242.., \quad \dot{y} = x + 0.3y - 4.45837..,$$

this differential system has the first integral

$$H_3(x,y) = 4(x + 0.3y)^2 + 8(-4.45837..x - 4.12242..y) + 25y^2.$$

For this piecewise differential system the real solutions of the system of equations (63) when $l = 5$, $i = 1,2,3$ and $s = 1,2$ are $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.677708.., 0.87781.., 0.0595682.., 0.0613168.., -0.53197.., 0.363935.., -0.156495.., -0.143729..)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.816561.., 1.10056.., 0.241118.., -0.268619.., -0.85279.., 0.327198.., -0.350653.., -0.282564..)$ and $(\alpha_3, \beta_3, \gamma_3, \delta_3) = (0.913978.., 1.26446.., 0.388329.., -0.457558..)$. Then the piecewis differential system (68)–(70) has exactly three limit cycles, the $(\mathcal{C}_5^1)$ of Figure 15.

For the second configuration we consider the linear differential center

$$(71)\quad \dot{x} = 0.122682..x - 9.01505..y + 3.06092.., \quad \dot{y} = x - 0.122682..y - 0.441974..,$$

in the region R_1 with its first integral

$$H_1(x,y) = 4(x - 0.122682..y)^2 + 8(-0.441974..x - 3.06092..y) + 36y^2.$$

In the region R_2 we consider the linear center

$$(72)\quad \dot{x} = \frac{x}{6} - \frac{113y}{18} + \frac{23}{10}, \quad \dot{y} = x - \frac{y}{6} - \frac{1}{8},$$

with its first integral

$$H_2(x,y) = 4\left(x - \frac{y}{6}\right)^2 + 8\left(-\frac{x}{8} - \frac{23y}{10}\right) + 25y^2.$$

In the region R_3 we consider the following linear differential center

$$(73)\quad \dot{x} = 0.462659 - \frac{9y}{4}, \quad \dot{y} = x + 0.358257,$$

this differential system has the first integral

$$H_3(x,y) = 4x^2 + 8(0.358257x - 0.462659y) + 9y^2.$$

For this piecewise differential system the real solutions of the system of equations (63) when $l = 5$, $i = 1,2,3$ and $s = 1,2$ are $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.513192.., 0.631288.., 0.0851022.., 0.0886494.., -0.523164.., 0.361262.., -0.113678.., 0.107022..)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.486714.., 0.593455.., 0.118954.., 0.12583..,$

$-0.422925.., 0.321277.., -0.183158.., 0.165537..)$ and $(\alpha_3, \beta_3, \gamma_3, \delta_3) = (0.454598.., 0.548276.., 0.158304.., 0.170374..)$. Then the piecewise differential system (71)–(73) has exactly three limit cycles, see $(\mathcal{C}_5^2)$ of Figure 15.

For the third configuration of the class C_5 we consider the linear differential center

$$\dot{x} = \frac{x}{4} - \frac{37y}{16} + 0.557263.., \quad \dot{y} = x - \frac{y}{4} + 0.224583.., \tag{74}$$

in the region R_1 with its first integral

$$H_1(x, y) = 4\left(x - \frac{y}{4}\right)^2 + 8(0.224583..x - 0.557263..y) + 9y^2.$$

In the region R_2 we consider the linear center

$$\dot{x} = \frac{x}{4} - \frac{65y}{16} + 1, \quad \dot{y} = x - \frac{y}{4} + \frac{1}{2}, \tag{75}$$

with its first integral

$$H_2(x, y) = 4\left(x - \frac{y}{4}\right)^2 + 8\left(\frac{x}{2} - y\right) + 16y^2.$$

In the region R_3 we consider the linear differential center

$$\dot{x} = 0.327353..x - 2.35716..y + 0.638815.., \quad \dot{y} = x - 0.327353..y + 0.508354.., \tag{76}$$

this differential system has the first integral

$$H_3(x, y) = 4(x - 0.327353..y)^2 + 8(0.508354..x - 0.638815..y) + 9y^2.$$

For this piecewise differential system the real solutions of the system of equations (67) when $l = 5$, $i = 1, 2, 3$ and $s = 1, 2$ are $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.418072.., 0.497852.., 0.127989.., -0.135933.., -0.690255.., 0.38416.., -0.10652.., 0.100687..)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.472678.., 0.573613.., 0.175388.., -0.190147.., -0.294939.., -0.247654.., -0.942264.., -0.226411..)$ and $(f_3, g_3, h_3, k_3) = (-0.42299.., -0.321308.., -0.879188.., -0.305589..)$. Then the piecewise differential system (74)–(76) has exactly three limit cycles, the $(\mathcal{C}_5^3)$ of Figure 16.

Finally we prove the statement for the fourth configuration of the class C_5. We consider the linear differential center

$$\dot{x} = \frac{4x}{5} - \frac{41y}{25} + 0.247984.., \quad \dot{y} = x - \frac{4y}{5} - 0.066548.., \tag{77}$$

in the region R_1 with its first integral

$$H_1(x, y) = 4\left(x - \frac{4y}{5}\right)^2 + 8(-0.066548..x - 0.247984..y) + 4y^2.$$

In the region R_2 we consider the linear center

$$\dot{x} = \frac{5}{2} - \frac{25y}{4}, \quad \dot{y} = x + \frac{1}{2}, \tag{78}$$

with its first integral

$$H_2(x, y) = 4x^2 + 8\left(\frac{x}{2} - \frac{5y}{2}\right) + 25y^2.$$

In the region R_3 we consider the following linear differential center

$$\dot{x} = 0.255935..x - 9.0655..y + 4.10027.., \quad \dot{y} = x - 0.255935..y + 0.499658.., \tag{79}$$

this differential system has the first integral

$$H_3(x,y) = 4(x - 0.255935..y)^2 + 8(0.499658..x - 4.10027..y) + 36y^2.$$

For this piecewise differential system the real solutions of the system of equations (67) when $l = 5$, $i = 1, 2, 3$ and $s = 1, 2$ are $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.443382.., 0.532683.., 0.0686511.., 0.0709685.., -0.997181.., 0.0529413.., -0.0448512.., 0.0438338..)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.363795, 0.424846.., 0.164996.., 0.18088.., -0.986773.., 0.113489.., -0.0980884.., 0.0931536..)$ and $(f_3, g_3, h_3, k_3) = (-0.940295.., 0.229758.., -0.207579.., 0.184783..)$. Then the piecewise differential system (77)–(79) has exactly three limit cycles, the $(\mathcal{C}_5^4)$ of Figure 16.

This completes the proof of statement (c) of Theorem 3. □

Proof of statement (d) of Theorem 3. For the first configuration of the class C_2 we consider the differential center in the region R_1

$$\dot{x} = \frac{x}{5} - \frac{61y}{80} + \frac{1}{5}, \quad \dot{y} = x - \frac{y}{5} - \frac{11}{5}, \tag{80}$$

this system has the first integral

$$H_1(x,y) = 4\left(x - \frac{y}{5}\right)^2 + 8\left(-\frac{11x}{5} - \frac{y}{5}\right) + \frac{289y^2}{100}.$$

In the second region R_2 we consider the linear differential center

$$\dot{x} = 0.190222.. + 0.147015..x - 0.553968..y, \quad \dot{y} = -2.2358.. + x - 0.147015..y, \tag{81}$$

its first integral is

$$H_2(x,y) = 8(-2.2358..x - 0.190222..y) + 4(x - 0.147015..y)^2 + 2.12942..y^2.$$

In the third region R_3 we consider the linear differential center

$$\dot{x} = \frac{3x}{5} - \frac{261y}{100} + \frac{3}{10}, \quad \dot{y} = x - \frac{3y}{5} - 3.29905, \tag{82}$$

it has the first integral

$$H_3(x,y) = 4\left(x - \frac{3y}{5}\right)^2 + 8\left(-3.29905x - \frac{3y}{10}\right) + 9y^2.$$

For this piecewise differential system the real solutions of the system of equations (25) when $l = 2$, $i = 1, 2, 3, 4$ and $s = 2, 3, 4$ are $(\alpha_1, \beta_1, \gamma_1, \delta_1, f, g, h, k) = (3.63717.., 2.47216.., 3.06727.., -0.653125.., 0.842035.., 0.535756.., 0.975, -0.22217..)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2) = (3.37534.., 1.73474.., 3, 0)$, $(\alpha_3, \beta_3, \gamma_3, \delta_3) = (3.43988.., 1.92143.., 3.00474.., -0.168945..)$ and $(\alpha_4, \beta_4, \gamma_4, \delta_4) = (3.49629.., 2.08122.., 3.01595.., -0.311427..)$. Then the piecewise differential system (80)–(82) has exactly four limit cycles, the $(\mathcal{C}_2^1)$ of Figure 17.

For the second configuration of the class C_2 we consider the linear center

$$\dot{x} = \frac{x}{4} - \frac{101y}{16} + 1.25061.., \quad \dot{y} = x - \frac{y}{4} - 2, \tag{83}$$

in the region R_1 with its first integral

$$H_1(x,y) = 4(x - \frac{y}{4})^2 + 8(-2x - 1.25061..y) + 25y^2.$$

In the second region R_2 we consider the following linear center

$$\dot{x} = \frac{33}{20} - 4y, \quad \dot{y} = x - \frac{77}{50}, \tag{84}$$

its first integral is

$$H_2(x, y) = 4x^2 + 8\left(-\frac{77x}{50} - \frac{33y}{20}\right) + 16y^2.$$

Now we consider the third linear center in the region R_3

(85)
$$\dot{x} = -0.113659..x - 0.905475..y + 0.315143.., \quad \dot{y} = x + 0.113659..y - 0.676964..,$$

its first integral is

$$H_3(x, y) = 4(x + 0.113659y)^2 + 8(-0.676964x - 0.315143y) + 3.57023y^2.$$

For this piecewise differential system the real solutions of the system of equations (29) when $l = 2$, $i = 1, 2, 3, 4$ and $s = 2, 3, 4$ are $(\alpha, \beta, \gamma, \delta, f_1, g_1, h_1, k_1) = (3.04294.., 0.51666.., 3.00134.., 0.0897351.., 0.0349525.., 0.316249.., 0.944348.., -0.328686..)$, $(f_2, g_2, h_2, k_2) = (0.219327.., 0.690011.., 0.969621.., -0.244556..)$, $(f_3, g_3, h_3, k_3) = (0.537538.., 0.782397.., 0.989244.., -0.146269..)$ and $(f_4, g_4, h_4, k_4) = (0.61897.., 0.749371.., 0.996025.., -0.0890765..)$. Then the piecewise differential system (83)–(85) has exactly four limit cycles, the $(\mathcal{C}_2^2)$ of Figure 17.

We prove the statement for the first configuration of the class C_5. We consider the linear differential center

$$\dot{x} = \frac{x}{2} - \frac{y}{2}, \quad \dot{y} = x - \frac{y}{2} - 0.103914.., \tag{86}$$

in the region R_1 with its first integral

$$H_1(x, y) = 4\left(x - \frac{y}{2}\right)^2 - 0.831311..x + y^2.$$

In the region R_2 we consider the linear center

$$\dot{x} = \frac{x}{8} - \frac{257y}{64} + \frac{143}{100}, \quad \dot{y} = x - \frac{y}{8} + \frac{31}{50}, \tag{87}$$

with its first integral

$$H_2(x, y) = 4\left(x - \frac{y}{8}\right)^2 + 8\left(\frac{31x}{50} - \frac{143y}{100}\right) + 16y^2.$$

In the region R_3 we consider the following linear differential center

$$\dot{x} = -0.151423..x - 2.50168..y + 0.529126.., \quad \dot{y} = x + 0.151423..y + 0.563689.., \tag{88}$$

this differential system has the first integral

$$H_3(x, y) = 4(x + 0.151423..y)^2 + 8(0.563689..x - 0.529126..y) + 9.91502..y^2.$$

For this piecewise differential systeem the real solutions of the system of equations (29) when $l = 5$, $i = 1, 2, 3, 4$ and $s = 2, 3, 4$ are $(\alpha, \beta, \gamma, \delta, f_1, g_1, h_1, k_1) = (0.428889.., 0.512678.., 0.103449.., -0.108668.., -0.877924.., 0.306741.., -0.321728.., 0.264966..)$, $(f_2, g_2, h_2, k_2) = (-0.986092.., 0.116294.., -0.160009.., 0.14665..)$, $(f_3, g_3, h_3, k_3) = (-0.999985.., 0.0038499.., -0.0683007.., 0.065927..)$ and $(f_4, g_4, h_4, k_4) = (-0.974086.., -0.156808.., -0.231262.., -0.202765..)$. Then the piecewise differential system (86)–(88) has exactly four limit cycles, the $(\mathcal{C}_5^1)$ of Figure 18.

For the second configuration in the region R_1 we consider the linear differential center

$$(89)\quad \dot{x} = -0.0261907..x - 8.3125..y + 2.35413.., \quad \dot{y} = x + 0.0261907..y - 1.95466..,$$

it has the first integral

$$H_1(x,y) = 4(x + 0.0261907..y)^2 + 8(-1.95466..x - 2.35413..y) + 33.2473..y^2.$$

In the region R_2 we consider the linear center

$$(90)\quad \dot{x} = \frac{x}{8} - \frac{257y}{64} + \frac{143}{100}, \quad \dot{y} = x - \frac{y}{8} - \frac{31}{50},$$

with its first integral

$$H_2(x,y) = 4\left(x - \frac{y}{8}\right)^2 + 8\left(-\frac{31x}{50} - \frac{143y}{100}\right) + 16y^2.$$

In the region R_3 we consider the following linear differential center

$$(91)\quad \dot{x} = 1 - 0.6x - 0.61y, \quad \dot{y} = -1.28181.. + x + 0.6y,$$

this differential system has the first integral

$$H_3(x,y) = 4(x + 0.6y)^2 + 8(-1.28181..x - y) + y^2.$$

For the piecewise differential system (89)–(91) the real solutions of the system of equations (29) when $l = 5$, $i = 1,2,3,4$ and $s = 2,3,4$ are $(\alpha_1, \beta_1, \gamma_1, \delta_1, f, g, h, k) = ((0.47254.., 0.573419.., 0.249609.., 0.279028.. - 0.431069.., 0.325145.., -0.113056.., -0.106474..)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2) = (0.564055.., 0.70542.., 0.141024.., 0.15064..)$, $(\alpha_3, \beta_3, \gamma_3, \delta_3) = (0.622667.., 0.793177.., 0.0646561.., 0.0667136..)$ and $(\alpha_4, \beta_4, \gamma_4, \delta_4) = (0.707817.., 0.925.., 0.172916.., -0.18727..)$. Then the piecewise differential system (89)–(91) has four limit cycles, the $(\mathcal{C}_5^2)$ of Figure 18.

This completes the proof of statement (d) of Theorem 3. □

Proof of statement (e) *of Theorem* 3. For the class C_2 we consider the differential center in the region R_1

$$(92)\quad \dot{x} = -x - \frac{29y}{25} + 2.94178.., \quad \dot{y} = x + y - 2.45325..,$$

this system has the first integral

$$H_1(x,y) = 4(x+y)^2 + 8(-2.45325x - 2.94178y) + \frac{16y^2}{25}.$$

In the second region R_2 we consider the linear differential center

$$(93)\quad \dot{x} = -\frac{9x}{20} - \frac{61y}{40} + 1, \quad \dot{y} = x + \frac{9y}{20} - \frac{9}{5},$$

its first integral is

$$H_2(x,y) = 4\left(x + \frac{9y}{20}\right)^2 + 8\left(-\frac{9x}{5} - y\right) + \frac{529y^2}{100}.$$

In the third region R_3 we consider the linear differential center

$$(94)\quad \dot{x} = -0.10723..x - 0.559468..y + 0.141448.., \quad \dot{y} = x + 0.10723..y - 0.59413..,$$

which has the first integral

$$H_3(x,y) = 4(x + 0.10723..y)^2 + 8(-0.59413..x - 0.141448..y) + 2.19188..y^2.$$

For the piecewise differential system (92)–(94) the real solutions of the system of equations

$$
\begin{aligned}
&H_1(\alpha_s,\beta_s)-H_1(\gamma_s,\delta_s)=0,\\
&H_2(f_s,g_s)-H_2(\alpha_s,\beta_s)=0,\\
&H_2(\gamma_s,\delta_s)-H_2(h_s,k_s)=0,\\
&H_2(h_n,k_n)-H_2(f_n,g_n)=0,\\
&H_3(h_i,k_i)-H_3(f_i,g_i)=0,\\
&c_l(\alpha_s,\beta_s)=c_l(\gamma_s,\delta_s)=0,\\
&c_l(f_i,g_i)=c_l(h_i,k_i)=0.
\end{aligned}
\tag{95}
$$

when $l=2$, $i=1\dots4$, $n=3,4$ and $s=1,2$ are $(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1)=$ $(3.04187..,0.509953..,3.07388..,-0.686287..,0.782419..,0.614426..,0.992143..,$ $-0.125105..)$, $(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2)=(3.0997..,0.805554..,3.17726..,-1.10734..,$ $0.550134..,0.778659..,0.939394..,-0.342515..)$, $(f_3,g_3,h_3,k_3)=(0.0992579..,0.509257..,$ $0.872091..,-0.4872..)$ and $(f_4,g_4,h_4,k_4)=(0.00046361..,-0.0372823..,0.714642..,$ $-0.682678..)$. Then the piecewise differential system (92)–(94) has exactly four limit cycles, the $(\mathcal{C}_2)$ of Figure 19.

For the first configuration of the class C_5 we consider the linear center

$$
\dot{x}=0.307738..-y,\quad \dot{y}=x+0.480645..,\tag{96}
$$

in the region R_1. Its first integral is

$$
H_1(x,y)=4x^2+8(0.480645..x-0.307738..y)+4y^2.
$$

In the region R_3 we consider the following linear center

$$
\dot{x}=-\frac{x}{5}-\frac{29y}{5}+\frac{11}{10},\quad \dot{y}=x+\frac{y}{5}+\frac{3}{5},\tag{97}
$$

its first integral is

$$
H_2(x,y)=4\left(x+\frac{y}{5}\right)^2+8\left(\frac{3x}{5}-\frac{11y}{10}\right)+\frac{576y^2}{25}.
$$

Now we consider the third linear center in the region R_2

$$
\dot{x}=-0.251554..x-4.18795..y+0.244301..,\quad \dot{y}=x+0.251554..y+0.54238..,\tag{98}
$$

its first integral is

$$
H_3(x,y)=4(x+0.251554..y)^2+8(0.54238..x-0.244301..y)+16.4987..y^2.
$$

For the piecewise differential system (96)–(98) the real solutions of the system of equations (95) when $l=5$, $i=1\dots4$, $n=3,4$ and $s=1,2$ are $(\alpha_1,\beta_1,\gamma_1,\delta_1,f_1,g_1,h_1,k_1)=$ $(0.22318..,0.246831..,0.0959506..,-0.100448..,-0.995586..,$ $0.0661471..,-0.144069..,0.133287..)$, $(\alpha_2,\beta_2,\gamma_2,\delta_2,f_2,g_2,h_2,k_2)=(0.347472..,0.403348..,$ $0.182072..,-0.197955..,-0.998702..,-0.0359796..,-0.0315597..,0.0310577..)$, $(f_3,g_3,h_3,$ $k_3)=(-0.987333..,-0.111122..,-0.202436..,-0.180788..)$, and $(f_4,g_4,h_4,k_4)=$ $(-0.940774..,-0.22895..,-0.376017..,-0.297026..)$. Then the piecewise differential system (96)–(98) has exactly four limit cycles, the $(\mathcal{C}_5^1)$ of Figure 19.

We prove the statement for the second configuration of the class C_5. We consider the linear differential center

$$
\dot{x}=0.759097x-1.22317..y+0.109357..,\quad \dot{y}=x-0.759097..y-0.107506..,\tag{99}
$$

in the region R_1 with its first integral

$$H_1(x,y) = 4(x - 0.759097..y)^2 + 8(-0.107506..x - 0.109357..y) + 2.58776..y^2.$$

In the region R_3 we consider the following linear center

$$\dot{x} = -\frac{x}{5} - \frac{629y}{100} + \frac{9}{5}, \quad \dot{y} = x + \frac{y}{5} - \frac{1}{2}, \tag{100}$$

its first integral is

$$H_2(x,y) = 4\left(x + \frac{y}{5}\right)^2 + 8\left(-\frac{x}{2} - \frac{9y}{5}\right) + 25y^2.$$

Now we consider the third linear center in the region R_2

$$\dot{x} = 1.63633.. - x - 10y, \quad \dot{y} = -0.181207.. + x + y, \tag{101}$$

its first integral is

$$H_3(x,y) = 4(x+y)^2 + 8(-0.181207..x - 1.63633y) + 36y^2.$$

For the piecewise differential system (99)–(101) the real solutions of the system of equations

$$\begin{array}{l} H_1(\alpha_i,\beta_i) - H_1(\gamma_i,\delta_i) = 0, \\ H_2(f_s,g_s) - H_2(\alpha_s,\beta_s) = 0, \\ H_2(\gamma_s,\delta_s) - H_2(h_s,k_s) = 0, \\ H_2(\alpha_n,\beta_n) - H_2(\gamma_n,\delta_n) = 0, \\ H_3(h_i,k_i) - H_3(f_i,g_i) = 0, \\ c_l(\alpha_i,\beta_i) = c_l(\gamma_i,\delta_i) = 0, \\ c_l(f_s,g_s) = c_l(h_s,k_s) = 0. \end{array} \tag{102}$$

when $l = 5$, $i = 1\ldots 4$, $n = 3, 4$ and $s = 1, 2$ are $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.387278.., 0.456147.., 0.136272.., 0.145261.., -0.527085.., 0.36247.., -0.0511948.., -0.0498671..)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.450783.., 0.542961.., 0.0591371.., 0.0608605.., -0.757207.., 0.373106.., -0.0978544.., -0.0929434..)$, $(\alpha_3, \beta_3, \gamma_3, \delta_3) = (0.532278.., 0.658882.., 0.0759308.., -0.0787608..)$ and $(\alpha_4, \beta_4, \gamma_4, \delta_4) = (0.591395.., 0.746048.., 0.130519. - 0.138776..)$. Then the piecewise differential system (99)–(101) has exactly four limit cycles, the $(\mathcal{C}_5^2)$ of Figure 20.

This completes the proof of statement (e) of Theorem 3. □

Proof of statement (f) *of Theorem* 3. For the class C_2 we consider the differential center in the region R_1

$$\dot{x} = 0.0425516..x - 0.00821064..y - 0.12984.., \quad \dot{y} = x - 0.0425516..y - 3.0847.., \tag{103}$$

this system has the first integral

$$H_1(x,y) = 4x^2 - 0.792208..xy - 24.7562..x + 0.0648246..y^2 + 2.4331..y.$$

In the second region R_2 we consider the linear differential center

$$\dot{x} = -\frac{7x}{25} - \frac{9221y}{2500} + \frac{47}{25}, \quad \dot{y} = x + 7\,\text{y25} - \frac{17}{10}, \tag{104}$$

its first integral is

$$H_2(x,y) = 4x^2 - 12.96x + 16y^2 - 15.2y.$$

In the third region R_3 we consider the linear differential center

(105)
$$\dot{x} = 0.0328614..x - 0.577764..y + 0.0613607.., \quad \dot{y} = x - 0.0328614..y - 0.561582..,$$

this differential system has the first integral

$$H_3(x,y) = 4x^2 + 0.0155494..xy - 4.09113..x + 1.84261..y^2 - 0.213258..y.$$

For the piecewise differential system (103)–(105) the real solutions of the system of equations (95) when $l = 2$, $i = 1 \dots 4$, $n = 4$ and $s = 1, 2, 3$ are $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (3.09263.., 0.774271.., 3.00423.., -0.159588..0.545491.., 0.780095.., 0.998262.., -0.0589339..)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (3.20022.., 1.18735.., 3.03483.., -0.46377.., 0.0465301.., 0.361982.., 0.917981.., -0.395929..)$, $(\alpha_3, \beta_3, \gamma_3, \delta_3, f_3, g_3, h_3, k_3) = (3.2797.., 1.44612.., 3.07427.., -0.688186.., 0.0172143.., -0.224639.., 0.775737.., -0.622055..)$ and $(f_4, g_4, h_4, k_4) = (0.125055.., -0.560862.., 0.531161.., -0.7841..)$.Then the piecewise differential system (103)–(105) has exactly four limit cycles, the $(\mathcal{C}_2)$ of Figure 20.

For the first configuration of the class C_5 we consider the linear center

(106)
$$\dot{x} = -0.543006..x - 0.31048..y + 0.435475.., \quad \dot{y} = x + 0.543006..y + 0.84298..,$$

in the region R_1. Its first integral is

$$H_1(x,y) = 4(x + 0.543006..y)^2 + 8(0.84298..x - 0.435475..y) + \frac{y^2}{16}.$$

In the region R_3 we consider the following linear center

$$\dot{x} = -\frac{x}{10} - \frac{73y}{10} + \frac{99}{100}, \quad \dot{y} = x + \frac{y}{10} + \frac{3}{10}, \tag{107}$$

its first integral is

$$H_2(x,y) = 4\left(x + \frac{y}{10}\right)^2 + 8\left(\frac{3x}{10} - \frac{99y}{100}\right) + \frac{729y^2}{25}.$$

Now we consider the following linear center in the region R_2

(108)
$$\dot{x} = -0.224603..x - 4.01713..y + 0.255369.., \quad \dot{y} = x + 0.224603..y + 0.427013..,$$

its first integral is

$$H_3(x,y) = 4(x + 0.224603y)^2 + 8(0.427013x - 0.255369y) + 15.8667y^2.$$

For the piecewise differential system (106)–(108) the real solutions of the system of equations (95) when $l = 5$, $i = 1 \dots 4$, $n = 4$ and $s = 1, 2, 3$ are $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.205616.., 0.225767.., 0.0987227.., -0.103481.., -0.292089.., 0.245756.., -0.0785997.., 0.0754475..)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.324003.., 0.372815.., 0.187602.., -0.204443.., -0.481088.., 0.346554.., -0.149974.., -0.138271..)$, $(\alpha_3, \beta_3, \gamma_3, \delta_3, f_3, g_3, h_3, k_3) = (0.395613.., 0.467362.., 0.253269.., -0.283534.., -0.994136.., -0.0761275.., -0.291586.., -0.24542..)$ and $(f_4, g_4, h_4, k_4) = (-0.952604.., -0.207387.., -0.4252.., -0.322368..)$. Then the piecewise differential system (106)–(108) has exactly four limit cycles, the $(\mathcal{C}_5^1)$ of Figure 21.

Finaly we prove the statement for the second configuration of the class C_5. We consider the linear differential center

(109)
$$\dot{x} = 0.757252..x - 0.587119..y - 0.0263454.., \quad \dot{y} = x - 0.757252..y - 0.0462943..,$$

in the region R_1. Its first integral is

$$H_1(x,y) = 4(x - 0.757252..y)^2 + 8(0.0263454..y - 0.0462943..x) + 0.054756..y^2.$$

In the region R_3 we consider the following linear center

$$\dot{x} = -\frac{x}{10} - \frac{401y}{100} + \frac{299}{100}, \quad \dot{y} = x + \frac{y}{10} + \frac{31}{50}, \tag{110}$$

its first integral is

$$H_2(x,y) = 4\left(x + \frac{y}{10}\right)^2 + 8\left(\frac{31x}{50} - \frac{299y}{100}\right) + 16y^2.$$

Now we consider the following linear center in the region R_2

$$\dot{x} = 0.365039..x - 12.5909..y + 2.33152.., \quad \dot{y} = x - 0.365039..y + 0.662151.., \tag{111}$$

its first integral is

$$H_3(x,y) = 4(x - 0.365039..y)^2 + 8(0.662151..x - 2.33152..y) + 49.8306..y^2.$$

For the piecewise differential system (109)–(111) the real solutions of the system of equations (95) when $l = 5$, $i = 1 \ldots 4$, $n = 4$ and $s = 1, 2, 3$ are $(\alpha_1, \beta_1, \gamma_1, \delta_1, f_1, g_1, h_1, k_1) = (0.639843.., 0.819359.., 0.302944.., 0.3458.., -0.960551.., 0.190783.., -0.233631.., 0.204526..)$, $(\alpha_2, \beta_2, \gamma_2, \delta_2, f_2, g_2, h_2, k_2) = (0.724239.., 0.951001.., 0.16497.., 0.178058.., -0.986741.., 0.113619.., -0.187539.., 0.169041..)$, $(\alpha_3, \beta_3, \gamma_3, \delta_3, f_3, g_3, h_3, k_3) = (0.785915.., 1.05028.., 0.065594.., 0.0677111.., -0.99814.., 0.043048.., -0.109294.., 0.103149..)$ and $(f_4, g_4, h_4, k_4) = (-0.999498.., -0.0223905.., -0.0435657.., 0.0426061..)$. Then the piecewise differential system (109)–(111) has exactly four limit cycles, the $(\mathcal{C}_5^2)$ of Figure 21.

This completes the proof of statement (f) of Theorem 3. □

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References

[1] A.A. Andronov, A.A. Vitt and S.E. Khaikin, *Theory of Oscillators*, Pergamon Press, Oxford. 1996.

[2] S. Banerjee and G. Verghese, *Nonlinear phenomena in power electronics.* Attractors, bifurcations chaos and nonlinear control, Wiley-IEEE Press, New York, 2001.

[3] R. Benterki and J. LLibre, *The limit cycles of discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubic curves I*, Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis **28** (2021) 153-192.

[4] R. Benterki and J. LLibre, *The limit cycles of discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubic curves II*, Differ. Equ. Dyn. Syst. (2021). https://doi.org/10.1007/s12591-021-00564-w.

[5] R. Bix, *Conics and cubics*, Undergraduat Texts in Mathematics, Second Edition, Springer, 2006.

[6] M. di Bernardo, C.J. Budd, A. Champneys and P. Kowalczyk, *Piecewise-Smooth Dynamical Systems: Theory and Applications*, Applied Mathematical Sciences Series 163, Springer, London, 2008.

[7] A.F. Filippov, *Differential Equations with Discontinuous Right-Hand Sides*, Kluwer Academic, Dordrecht, 1988.

[8] M. Han and W. Zhang, *On hopf bifurcation in non-smooth planar systems*, J. Differential Equations **248** (2010), 2399–2416.

[9] D. Hilbert, *Mathematische Probleme*, Lecture, Second Internat. Congr. Math. (Paris, 1900), *Nachr. Ges. Wiss. G"ottingen Math. Phys. KL.* (1900), 253–297; English transl., *Bull. Amer. Math. Soc.* **8** (1902), 437–479; *Bull. (New Series) Amer. Math. Soc.* **37** (2000), 407–436.

[10] S.M. Huan and X.S. Yang, *On the number of limit cycles in general planar piecewise linear systems*, Disc. Cont. Dyn. Syst. **32** (2012), 2147–2164.

[11] Yu. Ilyashenko, *Centennial history of Hilbert's 16th problem*, *Bull. (New Series) Amer. Math. Soc.* **39** (2002), 301–354.

[12] J. Jimenez, J. Llibre and J.C. Medrado, *Crossing limit cycles for a class of piecewise linear differential centers separated by a conic*, Electronic J. of Differential Equations **2020**, No. 4, 36 pp.

[13] R. I. Leine and H. Nijmeijer, *Dynamics and bifurcations of non-smooth mechanical systems*, Lecture Notes in Applied and Computational Mechanics, Vol. 18, Springer-Verlag, Berlin, 2004.

[14] J. Li, *Hilbert's 16th problem and bifurcations of planar polynomial vector fields*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **13** (2003), 47–106.

[15] D. Liberzon, *Switching in systems and control, Systems and Control: Foundations and Applications*, Birkhäuser, Boston, 2003.

[16] J. Llibre and E. Ponce, *Three nested limit cycles in discontinuous piecewise linear differential systems with two zones*, Dyn. Contin. Discr. Impul.st., Ser. B **19** (2012), 325–335.

[17] S. Shui, X. Zhang and J. Li, *The qualitative analysis of a class of plana Filippov systems*, Nonlinear Anal. **73**(5) (2010), 1277–1288.

[18] D.J.W. Simpson, *Bifurcations in Piecewise-Smooth Continuous Systems*, World Scientific Series on Nonlinear Science A, vol. 69. World Scientific, Singapore; 2010.

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