THE LIMIT CYCLES OF A CLASS OF PIECEWISE DIFFERENTIAL SYSTEMS

IMANE BENABDALLAH[1], REBIHA BENTERKI[1] AND JAUME LLIBRE[2]

Abstract. In this century many papers have been published on the piecewise differential systems in the plane. The increasing interest for this class of differential systems is motivated by their many applications for modelling several natural phenomena. One of the main difficulties for controlling the dynamics of the planar differential systems consists in determining their periodic orbits and mainly their limit cycles. Hence there are many papers studying the existence or non-existence of limit cycles for the discontinuous and continuous piecewise differential systems. The study of the maximum number of limit cycles is one of the biggest problems in the qualitative theory of planar differential systems. In this paper we provide the maximum number of limit cycles of a class of planar discontinuous piecewise differential systems formed by an arbitrary linear center and an arbitrary quadratic center, separated by the straight line $x = 0$. In general it is a hard problem to find the exact upper bound for the number of limit cycles that a class of differential systems can exhibit. We show that this class of differential systems can have at most 4 limit cycles. Here we also show that there are examples of all types of these differential systems with one, two, three, or four limit cycles.

1. Introduction and statement of the main result

A planar polynomial differential system is a system of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ are polynomial, and the maximum degree of these polynomials is the *degree* of this system.

In the study of differential systems the existence of periodic solutions is very important because they play an important role in many natural phenomena. An isolated periodic orbit in the set of all periodic orbits of a differential system is called a *limit cycle*.

This paper deals with discontinuous piecewise differential systems of the form

$$(\dot{x}, \dot{y}) = F(x, y) = \begin{cases} F^-(x, y) = \Big(F_1^-(x, y), F_2^-(x, y)\Big)^T & \mathbf{y} \in \Sigma^-, \\ F^+(x, y) = \Big(F_1^+(x, y), F_2^+(x, y)\Big)^T & \mathbf{y} \in \Sigma^+, \end{cases} \tag{2}$$

such that the separation line of the plane is $\Sigma = \{(x, y) : x = 0\}$ and

$$\Sigma^- = \{(x, y) : x \le 0\}, \qquad \Sigma^+ = \{(x, y) : x \ge 0\}.$$

In this paper we shall work with discontinuous piecewise differential systems in $\mathbb{R}^2$, and the definition of these differential systems on the separation line of their two pieces in $\mathbb{R}^2$ follows the rules of Filippov [12].

Research on discontinuous piecewise linear differential systems started with the studies of Andronov, Vitt and Khaikin about 1930 in [1]. Recently the dynamics of piecewise differential systems appears frequently in many fields of applied mathematics, mechanics, electronics, economics, neuroscience, etc., see for instance [7, 22, 23].

2020 *Mathematics Subject Classification.* Primary 34C29, 34C25, 47H11.

Key words and phrases. limit cycles, discontinuous piecewise linear differential systems, linear centers, quadratic centers.

The second part of the famous sixteenth Hilbert problem consists in finding an upper bound for the maximum number of limit cycles that the polynomial differential systems in the plane of a given degree can have, see [14, 15, 18]. In the last years many authors have been involved in solving the extension of this problem to some classes of discontinuous piecewise differential systems.

In the literature we find many papers interested in studying piecewise differential linear systems separated by either a straight line or an algebraic curve, such as a conic or a reducible or irreducible cubic curve, see for instance [2, 4, 5, 6, 8, 9, 11, 13, 21].

In [19] it is studied the maximum number of limit cycles of the planar continuous piecewise differential systems formed by an arbitrary linear center and an arbitrary quadratic center, separated by a parabola.

The main goal of this paper is to solve the extension of the second part of the sixteenth Hilbert problem to the class of discontinuous piecewise differential systems formed by an arbitrary linear center and an arbitrary quadratic center separated by the straight line $x = 0$.

Using the first integrals of the linear and quadratic centers we will obtain a set of equations whose solutions provide the upper bound for the maximum number of limit cycles for the class of the discontinuous piecewise differential systems that we study.

Lemma 1. *Every linear center after doing a linear change of variables and a rescaling of the independent variable can be written as*

$$\dot{x} = -\beta x - \frac{(4\beta^2 + \omega^2)}{4\alpha} y + \sigma_1, \quad \dot{y} = \alpha x + \beta y + \delta_1, \text{ with } \omega > 0, \ \alpha > 0, \tag{3}$$

and its first integral is

$$H(x, y) = 8\alpha(\delta_1 x - \sigma_1 y) + 4(\alpha x + \beta y)^2 + y^2\omega^2. \tag{4}$$

For a proof of Lemma 1 see [20].

In the following result we give a normal form of the quadratic centers, for a proof see for instance Theorem 8.15 of [10].

Theorem 2 (Kapteyn-Bautin Theorem)**.** *Any quadratic system candidate to have a center can be written after an affine transformation and a rescaling of the independent variable in the form*

$$\dot{x} = -y - bx^2 - Cxy - dy^2, \quad \dot{y} = x + ax^2 + Axy - ay^2. \tag{5}$$

This system has a center at the origin if and only if one of the following conditions holds

(*i*) $C = a = 0$,
(*ii*) $b + d = 0$,
(*iii*) $C + 2a = A - 2b = 0$,
(*iv*) $C + 2a = A + 3b + 5d = a^2 + bd + 2d^2 = 0$.

The results stated in the next theorem do not depend on which half-plane $x \geq 0$ or $x \leq 0$ are located the linear and the quadratic centers.

Our main results is stated in the next theorem.

Theorem 3. *The maximum number of limit cycles of the discontinuous piecewise differential systems separated by the straight line Σ and formed by an arbitrary linear center and an arbitrary quadratic center satisfying the condition of Kapteyn-Bautin Theorem of*

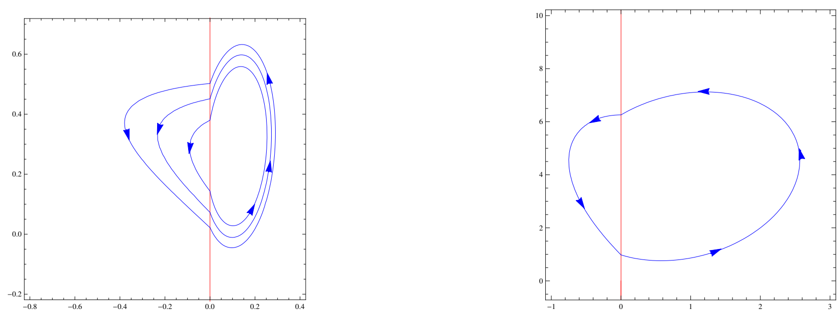
(I) *type $C = a = 0$ is three if $A = -b \neq 0$, and one if eiher $A = 0 \neq b$, or $b = 0 \neq A$, or $A = b = 0$. There are discontinuous piecewise differential systems of these types with three limit cycles see Figure 1(a) and with one limit cycle see Figure 1(b).*

(II) *type* $b + d = 0$ *is three if either* $A + b = 0$ *and* $b \neq 0$*, or* $AbC(A + b)(4b(A + b) + C^2) \neq 0$*, or* $b = C = 0$*, or* $A = 0$ *and* $b \neq 0$*, or* $b = 0$ *and* $A \neq 0$*; two if* $(4b(A + b) + C^2) = 0$*; and one if* $A = b = 0$*. There are discontinuous piecewise differential systems of this type with three limit cycles see Figure 2(a) and with one limit cycle see Figure 2(b).*

(III) *type* $C + 2a = A - 2b = 0$ *is one. There are discontinuous piecewise differential systems of this type with one limit cycle, see Figure 3(a).*

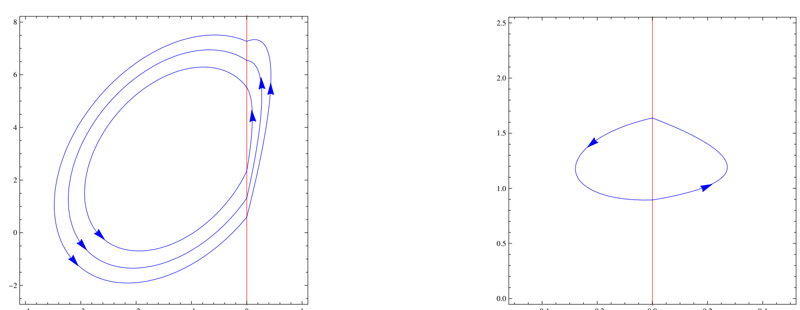
(IV) *type* $C + 2a = A + 3b + 5d = a^2 + bd + 2d^2 = 0$ *is four. There are discontinuous piecewise differential systems of this type with four limit cycles, see Figure 3(b).*

Theorem 3 is proved in section 3.



(a) (b)

FIGURE 1. (a) The three limit cycles of the discontinuous piecewise differential system (22)–(23), and (b) the unique limit cycle of the discontinuous piecewise differential system (24)–(25).

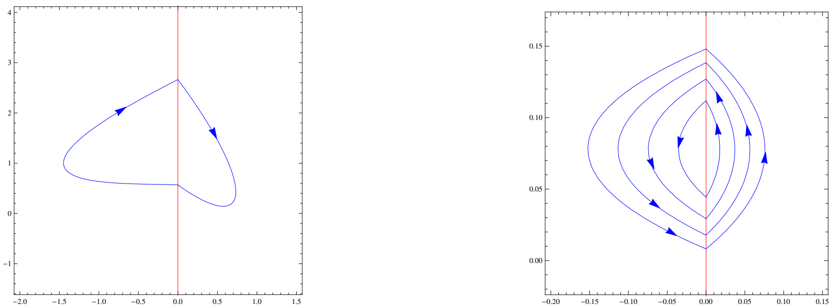


(a) (b)

FIGURE 2. (a) The three limit cycles of the discontinuous piecewise differential system (26)–(27), and (b) the unique limit cycle of the discontinuous piecewise differential system (28)–(29).

2. Quadratic centers after an affine change of variables

In this section we give the expression of an arbitrary quadratic differential center with its corresponding first integral obtained after the general affine change of variables $\{x \to \alpha_1 x + \gamma_1 y + \delta_1, y \to \alpha_2 x + \gamma_2 y + \delta_2\}$ with $\alpha_1\gamma_2 - \alpha_2\gamma_1 \neq 0$.



(a) (b)

FIGURE 3. (a) The unique limit cycle of the discontinuous piecewise differential system (30)–(31), and (b) the four limit cycles of the discontinuous piecewise differential system (32)–(33).

Thus system (5) becomes

(6)
$$\begin{aligned}\dot{x} = & \frac{1}{\alpha_2\gamma_1-\alpha_1\gamma_2}\Big(x^2\left(a\gamma_1(\alpha_1-\alpha_2)(\alpha_1+\alpha_2)+A\alpha_1\alpha_2\gamma_1+\gamma_2\left(\alpha_1^2 b+\alpha_1\alpha_2 C+\alpha_2^2 d\right)\right)+y^2(a\\ & \gamma_1^3+\gamma_1^2\gamma_2(A+b)\gamma_1\gamma_2^2(C-a)+\gamma_2^3 d)+\delta_1(a\gamma_1\delta_1+b\gamma_2\delta_1+\gamma_1)+\delta_2(A\gamma_1\delta_1\\ & +\gamma_2+\gamma_2C\delta_1)+\delta_2^2(\gamma_2 d-a\gamma_1)y(\gamma_1\gamma_2(-2a\delta_2+A\delta_1+2b\delta_1+C\delta_2)+\gamma_1^2(2a\delta_1\\ & +A\delta_2+1)+\gamma_2^2(C\delta_1+2d\delta_2+1))+x(\alpha_1(2a\gamma_1^2y+\gamma_1+2a\gamma_1\delta_1+\gamma_1\gamma_2y+(A\\ & +2b)+A\gamma_1\delta_2+2b\gamma_2\delta_1+\gamma_2C(\delta_2+\gamma_2y))+\alpha_2(A\gamma_1(\delta_1\gamma_1y)+\gamma_2-2a\gamma_1(\delta_2\\ & +\gamma_2y)+\gamma_2+C(\delta_1+\gamma_1y)+2\gamma_2d(\delta_2+\gamma_2y)))\Big),\\ \dot{y} = & \frac{1}{\alpha_1\gamma_2-\alpha_2\gamma_1}\Big(x^2\left(a\alpha_1^3+\alpha_1\alpha_2^2(C-a)+\alpha_1^2\alpha_2(A+b)+\alpha_2^3 d\right)+y^2(a\alpha_1(\gamma_1-\gamma_2)(\gamma_1+\gamma_2)\\ & +\gamma_2(A\alpha_1\gamma_1+\alpha_2\gamma_1C+\alpha_2\gamma_2 d)+\alpha_2 b\gamma_1^2)+\delta_1(a\alpha_1\delta_1+\alpha_1+\alpha_2 b\delta_1)+\delta_2^2\\ & (\alpha_2 d-a\alpha_1)+\delta_2(A\alpha_1\delta_1+\alpha_2+\alpha_2C\delta_1)+y(\alpha_1(2a\gamma_1\delta_1-2a\gamma_2\delta_2+A\gamma_1\delta_2\\ & +A\gamma_2\delta_1+\gamma_1)+\alpha_2(2b\gamma_1\delta_1+\gamma_2+\gamma_1C\delta_2+\gamma_2C\delta_1+2\gamma_2 d\delta_2))+x(\alpha_1\alpha_2\\ & (-(2a-C)(\delta_2+\gamma_2y)+A(\delta_1+\gamma_1y)+2b(\delta_1+\gamma_1y))+\alpha_1^2(2a(\delta_1+\gamma_1y)\\ & +A(\delta_2+\gamma_2y)+1)+\alpha_2^2(C(\delta_1+\gamma_1y)+2d(\delta_2+\gamma_2y)+1))\Big).\end{aligned}$$

For its corresponding first integral we distinguish the following cases.

I. The quadratic system (6) **satisfying condition** (i) **of Theorem 2.** The corresponding first integral of this differential system if $A=-b\neq 0$ becomes

(7)
$$H_1^{(1)}(x,y) = (A(\delta_2+\alpha_2x+\gamma_2y)+1)^{2d}e^{Z_1(x,y)},$$

where

$$Z_1(x,y)=\frac{1}{(A(\delta_2+\alpha_2x+\gamma_2y)+1)^2}\Big(A\Big(A^2(\delta_1+\alpha_1x+\gamma_1y)^2-2A(\delta_2+\alpha_2x+\gamma_2y)\\ +4d(\delta_2+\alpha_2x+\gamma_2y)-1\Big)+3d\Big).$$

If $A=0\neq b$ it becomes

$$H_2^{(1)}(x,y) = e^{2b(\delta_2+\alpha_2 x+\gamma_2 y)}\Big(2b^3(\delta_1+\alpha_1 x+\gamma_1 y)^2+2b^2 d(\delta_2+\alpha_2 x+\gamma_2 y)^2+2b(b-d)(\delta_2+\alpha_2 x+\gamma_2 y)-b+d\Big). \tag{8}$$

If $b = 0 \neq A$ it becomes

$$H_3^{(1)}(x,y) = e^{A\left(A^2(\delta_1+\alpha_1 x+\gamma_1 y)^2+Ad(\delta_2+\alpha_2 x+\gamma_2 y)^2+2(A-d)(\delta_2+\alpha_2 x+\gamma_2 y)\right)}(A(\delta_2+\alpha_2 x+\gamma_2 y)+1)^{2d-2A}. \tag{9}$$

If $A = b = 0$ it becomes

$$H_4^{(1)}(x,y) = 2d(\delta_2+\alpha_2 x+\gamma_2 y)^3+3\left((\delta_1+\alpha_1 x+\gamma_1 y)^2+(\delta_2+\alpha_2 x+\gamma_2 y)^2\right). \tag{10}$$

II. The quadratic system (6) **satisfying condition (ii) of Theorem 2.** The first integral of the differential system (6) if $A = -b \neq 0$ and $a = 0 \neq C$ becomes

$$H_1^{(2)}(x,y) = e^{Z(x,y)}(1-b(\delta_2+\alpha_2 x+\gamma_2 y))^{-b^2-C^2}(-b(\delta_2+\alpha_2 x+\gamma_2 y)+C(\delta_1+\alpha_1 x+\gamma_1 y)+1)^{b^2}, \tag{11}$$

where

$$Z(x,y) = \frac{bC}{b(\delta_2+\alpha_2 x+\gamma_2 y)-1}\Big(b(\delta_1+\alpha_1 x+\gamma_1 y)+C(\delta_2+\alpha_2 x+\gamma_2 y)\Big).$$

If $AbC(A+b)\Delta \neq 0$ and $a = 0$ with $\Delta = 4b(A+b)+C^2 < 0$ and $L = \sqrt{-\Delta}$ it becomes

$$H_2^{(2)}(x,y) = \left(-\frac{\left(C^2+L^2\right)(\delta_2+\alpha_2 x+\gamma_2 y)}{4b}-b(\delta_2+\alpha_2 x+\gamma_2 y)+1\right)^{-\frac{8b}{4b^2+C^2+L^2}} e^{\frac{-2CM}{bL}}, \tag{12}$$

where $M = \arctan\left(\dfrac{2b(\delta_2+\alpha_2 x+\gamma_2 y)-C(\delta_1+\alpha_1 x+\gamma_1 y)-2}{L(\delta_1+\alpha_1 x+\gamma_1 y)}\right)$.

If $AbC(A+b)\Delta \neq 0$ and $a = 0$ with $\Delta = 4b(A+b)+C^2 > 0$ and $r = \sqrt{\Delta}$ it becomes

$$H_3^{(2)}(x,y) = (A(\delta_2+\alpha_2 x+\gamma_2 y)+1)^{1/A}\left(\frac{1}{2}(C-r)(\delta_1+\alpha_1 x+\gamma_1 y)-b(\delta_2+\alpha_2 x+\gamma_2 y)+1\right)^{\frac{r-C}{2rb}}\left(\frac{1}{2}(C+r)(\delta_1+\alpha_1 x+\gamma_1 y)-b(\delta_2+\alpha_2 x+\gamma_2 y)+1\right)^{\frac{r+C}{2rb}}. \tag{13}$$

If $b = C = 0$ it becomes

$$H_4^{(2)}(x,y) = e^{Z(x,y)}(a(\delta_1+\alpha_1 x+\gamma_1 y)+1)^{-2\sqrt{4a^2+A^2}}\Bigg(\frac{1}{(a(\delta_1+\alpha_1 x+\gamma_1 y)+1)^2}((\delta_2+\alpha_2 x+\gamma_2 y)\big(a^2(-(\delta_2+\alpha_2 x+\gamma_2 y))+aA(\delta_1+\alpha_1 x+\gamma_1 y)+A\big)+\big(a(\delta_1+\alpha_1 x+\gamma_1 y)+1)^2\big)\Bigg)^{-\sqrt{4a^2+A^2}}, \tag{14}$$

where

$$Z(x,y) = 2a\sqrt{4a^2+A^2}(\delta_1+\alpha_1 x+\gamma_1 y)-2A\tanh^{-1}\left(\frac{-2a^2(\delta_2+\alpha_2 x+\gamma_2 y)+aA(\delta_1+\alpha_1 x+\gamma_1 y)+A}{\sqrt{4a^2+A^2}(a(\delta_1+\alpha_1 x+\gamma_1 y)+1)}\right).$$

If $A = a = 0$, $C \neq 0$ and $b \neq 0$ it becomes

$$H_5^{(2)}(x,y) = e^{\delta_2+\alpha_2 x+\gamma_2 y}\left(\frac{1}{2}(C-r)(\delta_1+\alpha_1 x+\gamma_1 y)-b(\delta_2+\alpha_2 x+\gamma_2 y)+1\right)^{\frac{r+C}{2rb}}\left(\frac{1}{2}(C+r)(\delta_1+\alpha_1 x+\gamma_1 y)-b(\delta_2+\alpha_2 x+\gamma_2 y)+1\right)^{\frac{r-C}{2rb}}, \tag{15}$$

where $r=\sqrt{4b^2+C^2}$.

If $b=a=0$, $C\neq 0$ and $A\neq 0$ it becomes

$$H_6^{(2)}(x,y)= e^{-2AC(A(\delta_1+\alpha_1x+\gamma_1y)+C(\delta_2+\alpha_2x+\gamma_2y))}(C(\delta_1+\alpha_1x+\gamma_1y)+1)^{2A^2}(A(\delta_2+\alpha_2x+\gamma_2y)+1)^{2C^2}. \tag{16}$$

If $\Delta=4b(A+b)+C^2=0$ and $a=0\neq C$ it becomes

$$H_7^{(2)}(x,y)= \frac{1}{2}e^{Z(x,y)+1}\left(-\frac{C^2(\delta_2+\alpha_2x+\gamma_2y)}{4b}-b(\delta_2+\alpha_2x+\gamma_2y)+1\right)^{-\frac{4b^2}{4b^2+C^2}}(-2b(\delta_2+\alpha_2x+\gamma_2y)+C(\delta_1+\alpha_1x+\gamma_1y)+2), \tag{17}$$

where $Z(x,y)=\dfrac{C(\delta_1+\alpha_1x+\gamma_1y)}{2b(\delta_2+\alpha_2x+\gamma_2y)-C(\delta_1+\alpha_1x+\gamma_1y)-2}$.

If $A=b=0$ and $a=0\neq C$ it becomes

$$H_8^{(2)}(x,y)= (C(\delta_1+\alpha_1x+\gamma_1y)+1)^2e^{-C\left(C(\delta_2+\alpha_2x+\gamma_2y)^2+2(\delta_1+\alpha_1x+\gamma_1y)\right)}. \tag{18}$$

III. The quadratic system (6) **satisfying condition (iii) of Theorem 2.** Has the first integral

$$H_1^{(3)}(x,y)= \frac{1}{6}(2a(\delta_1+\alpha_1x+\gamma_1y)^3+6b(\delta_1+\alpha_1x+\gamma_1y)^2(\delta_2+\alpha_2x+\gamma_2y)+3(\delta_1+\alpha_1x+\gamma_1y)^2-6a(\delta_1+\alpha_1x+\gamma_1y)(\delta_2+\alpha_2x+\gamma_2y)^2+2d(\delta_2+\alpha_2x+\gamma_2y)^3+3(\delta_2+\alpha_2x+\gamma_2y)^2). \tag{19}$$

IV. The quadratic system (6) **satisfying condition (iv) of Theorem 2.** Has the first integral

$$H_1^{(4)}(x,y)= \Big((a^2+d^2)(d(\delta_2+\alpha_2x+\gamma_2y)-a(\delta_1+\alpha_1x+\gamma_1y))^3-3ad(a^2+d^2)(\delta_1+\alpha_1x+\gamma_1y)(\delta_2+\alpha_2x+\gamma_2y)+3d^2(a^2+d^2)(\delta_2+\alpha_2x+\gamma_2y)^2+3d(a^2+d^2)(\delta_2+\alpha_2x+\gamma_2y)+d^2\Big)^2\Big/\Big((a^2+d^2)(a(\delta_1+\alpha_1x+\gamma_1y)-d(\delta_2+\alpha_2x+\gamma_2y))^2+2d(a^2+d^2)(\delta_2+\alpha_2x+\gamma_2y)+d^2\Big)^3. \tag{20}$$

3. Proof of Theorem 3

Now we should give the proof of Theorem 3, where we provide the maximum number of limit cycles which can have the discontinuous piecewise differential systems separated by the straight line Σ, and formed by an arbitrary linear center and an arbitrary quadratic centers.

In one half-plane we consider the linear differential center (3) with its first integral $H(x,y)$ given in (4). In the other half-plane we consider system (6) satisfying one of the four condition of Theorem 2, with its corresponding first integral $H_k^{(j)}(x,y)$ with $k=1,...,8$ and $j=1,...,4$.

In order that the discontinuous piecewise differential system (3)–(6) has a limit cycle that intersects the straight line Σ at the points $(0,y_1)$ and $(0,y_2)$ with $y_1<y_2$, these points must satisfy the following system

$$\begin{aligned} e_1 &= H(0,y_1)-H(0,y_2)=(y_1-y_2)\left(\left(4\beta^2+\omega^2\right)(y_1+y_2)-8\alpha\sigma_1\right)\right)=0,\\ e_2 &= H_k^{(j)}(0,y_1)-H_k^{(j)}(0,y_2)=h_k^{(j)}(y_1,y_2)=0. \end{aligned} \tag{21}$$

From $e_1 = 0$, we obtain $y_1 = \dfrac{8\alpha\sigma_1}{4\beta^2+\omega^2} - y_2$ and by substituting it in $e_2 = 0$ we obtain the equation $F(y_2) = 0$ in the variable y_2, which differs according with the first integrals of system (6).

Proof of statement (I) *of Theorem* 3. Now we prove the statement (I) for the discontinuous piecewise differential system formed by the linear differential center (3) and the quadratic differential center (6) of type $C = a = 0$, and we distinguish the following cases:

Case 1. If $A = -b \neq 0$ then $k = 1$ and $j = 1$ in system (21), the first integral of (6) is $H_1^{(1)}(x, y)$ given in (7), so the solutions of $F(y_2) = 0$ are equivalent to the solutions of the non-algebraic equation $f_1(y_2) = g_1(y_2)$ where

$$f_1(y_2) = \left(\frac{L_1 + L_2\ y_2}{L_3 - L_2\ y_2}\right)^r \text{ and } g_1(y_2) = e^{\dfrac{k_0 + k_1\ y_2 + k_2\ y_2^2 + k_3\ y_2^3}{(L_1 + L_2\ y_2)^2(L_3 - L_2\ y_2)^2}},$$

where

$$\begin{aligned} k_0 = \ & \frac{1}{(4\beta^2+\omega^2)^2} 16A\sigma_1\alpha(A^3(\beta^2(8\gamma_1\delta_1\delta_2 - 4\gamma_2\delta_1^2 + 4\gamma_2\delta_2^2) + \omega^2(2\gamma_1\delta_1\delta_2 - \gamma_2\delta_1^2 + \gamma_2\delta_2^2) \\ & +8\alpha\sigma_1\delta_2(\gamma_1^2+\gamma_2^2)) - \gamma_2 d(4\beta^2+\omega^2) - 3A\gamma_2 d\left(\delta_2\left(4\beta^2+\omega^2\right) + 4\alpha\gamma_2\sigma_1\right) - A^4(\gamma_2\delta_1 \\ & -\gamma_1\delta_2)\Big(\delta_1\delta_2\left(4\beta^2+\omega^2\right) + 4\alpha\sigma_1(\gamma_1\delta_2+\gamma_2\delta_1)\Big) + A^2\Big(\left(4\beta^2+\omega^2\right)(\gamma_1\delta_1 + \gamma_2\delta_2(1 \\ & -2d\delta_2)) + 4\alpha\sigma_1\left(\gamma_1^2+\gamma_2^2 - 4\gamma_2^2 d\delta_2\right)\Big), \\ k_1 = \ & 4A\big((A\delta_2+1)(d(2A\gamma_2\delta_2+\gamma_2) - A^2(\delta_2(A\gamma_1\delta_1+\gamma_2) + \delta_1(\gamma_1 - A\gamma_2\delta_1))) + \frac{32\alpha^2A^2\gamma_2\sigma_1^2}{(4\beta^2+\omega^2)^2} \\ & (A^2\gamma_1(\gamma_1\delta_2 - \gamma_2\delta_1) + A(\gamma_1^2+\gamma_2^2) - 2\gamma_2^2 d) - \frac{4\alpha A\sigma_1}{4\beta^2+\omega^2}(A^3(\gamma_1^2\delta_2^2 - \gamma_2^2\delta_1^2) + 2A^2\delta_2(\gamma_1^2 \\ & +\gamma_2^2) + A(\gamma_1^2 + \gamma_2^2(1-4d\delta_2)) - 3d\gamma_2^2)\big), \\ k_2 = \ & -\frac{48\alpha A^3\gamma_2\sigma_1}{4\beta^2+\omega^2}\left(A^2\gamma_1(\gamma_1\delta_2 - \gamma_2\delta_1) + A\left(\gamma_1^2+\gamma_2^2\right) - 2\gamma_2^2 d\right), \\ k_3 = \ & 4A^3\gamma_2\left(A^2\gamma_1(\gamma_1\delta_2 - \gamma_2\delta_1) + A\left(\gamma_1^2+\gamma_2^2\right) - 2\gamma_2^2 d\right), \\ L_1 = \ & A\delta_2 + 1,\ L_2 = A\gamma_2,\ L_3 = \frac{8\alpha A\gamma_2\sigma_1}{4\beta^2+\omega^2} + A\delta_2 + 1,\ r = 2d. \end{aligned}$$

We note that $(f_1)'(y_2)$ and $(g_1)'(y_2)$ are the derivatives of the functions $f_1(y_2)$ and $g_1(y_2)$, respectively. Where

$$(f_1)'(y_2) = \frac{\eta(L_1 + L_2\ y_2)^{r-1}}{(L_3 - L_2\ y_2)^{r+1}},$$

and

$$(g_1)'(y_2) = e^{\dfrac{k_0 + k_1\ y_2 + k_2\ y_2^2 + k_3\ y_2^3}{(L_1 + L_2\ y_2)^2(L_3 - L_2\ y_2)^2}} \frac{P(y_2)}{(L_1 + L_2\ y_2)^3(L_1 - L_2\ y_2)^3},$$

with $\eta = rL_2(L_3 + L_1)$, and

$$\begin{aligned} P(y_2) = \ & k_3L_2^2\ y_2^4 + L_2\big(2k_2L_2 + k_3(L_3 - L_1)\big)\ y_2^3 + 3\left(k_1L_2^2 + k_3L_1L_3\right)\ y_2^2 + \big(4k_0L_2^2 + k_1L_2(L_1 - L_3) \\ & +2k_2L_1L_3\big)\ y_2 + 2k_0L_1L_2 - 2k_0L_2L_3 + k_1L_1L_3. \end{aligned}$$

We denoted by (C_{f_1}) and (C_{g_1}) the graphics of the functions $f_1(y_2)$ and $g_1(y_2)$, respectively.

According with the sign of $(f_1)'(y_2)$ which depends on r and with the sign of the parameter $\eta \in \mathbb{R}$, we obtain that all the possible graphics (C_{f_1}) of the function $f_1(y_2)$ are as follows.

If r is an even integer or the rational $r = p/(2q+1)$ with p is an even integer and q is an arbitrary integer, then the sign of $\big(f_1\big)'(y_2)$ depends on the sign of η $(L_1 + L_2\ y_2)$ $(L_3 - L_2\ y_2)$. Therefore the graphic (C_{f_1}) is given in Figure 4(a) if $\eta > 0$, or 4(b) if $\eta < 0$.

If r is an odd integer or the rational $r = p/(2q+1)$ with p is an odd integer and q is an arbitrary integer, then the sign of $\big(f_1\big)'(y_2)$ depends only on the sign of η. Therefore the graphic (C_{f_1}) is given in Figure 4(c) if $\eta < 0$, or Figure 4(d) if $\eta > 0$.

If r is irrational or the rational $r = p/(2q)$ where p is an odd integer and q is an arbitrary integer, then the sign of $\big(f_1\big)'(y_2)$ depends on the sign of η. Consequently the graphics (C_{f_1}) are the same than in the case that r is an odd integer but in the domain of definition of $f_1(y_2)$.

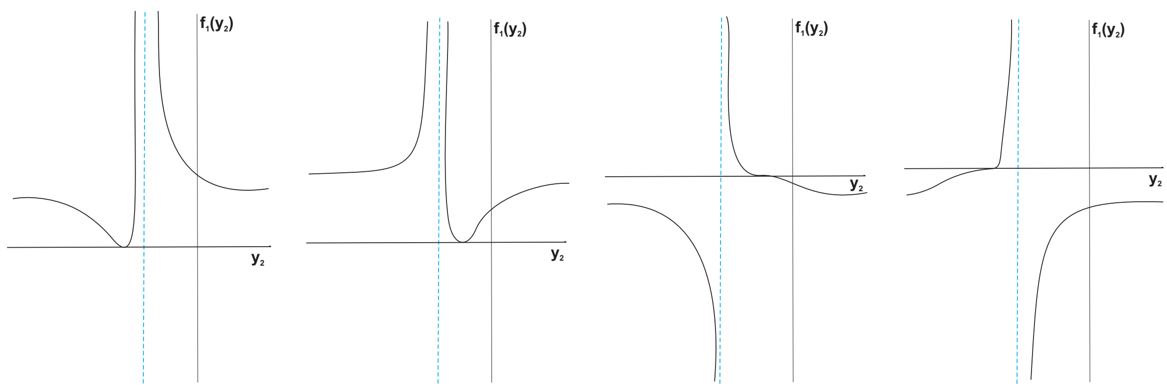


FIGURE 4. The graphics of the function $f_1(y_2)$. The dashed straight line is the vertical asymptote straight line.

According with the sign of $\big(g_1\big)'(y_2)$ and with the different kind of the roots r_i with $i \in \{1, \ldots, 4\}$ of the polynomial $P(y_2)$, and by considering the case when $L_3 \neq -L_1$, we shall obtain the different possible topologically distinct graphics (C_{g_1}).

If $P(y_2)$ has four simple real roots, then the positions of these roots with respect to the two vertical asymptotes straight lines $y_{21} = -\dfrac{L_1}{L_2}$ and $y_{22} = -\dfrac{L_3}{L_2}$ play a main role in the variation of the graphics (C_{g_1}). So all the possible topologically distinct graphics (C_{g_1}) are given in Figure 5(a) if $y_{21} < r_1 < y_{22} < r_2 < r_3 < r_4$, or Figure 5$(b)$ if $r_1 < r_2 < r_3 < y_{21} < r_4 < y_{22}$, or Figure 5$(c)$ if $r_1 < y_{21} < r_2 < y_{22} < r_3 < r_4$, or Figure 5$(d)$ if $r_1 < r_2 < y_{21} < r_3 < y_{22} < r_4$, or Figure 5$(e)$ if $y_{21} < r_1 < r_2 < y_{22} < r_3 < r_4$, or Figure 5$(f)$ if $r_1 < r_2 < y_{21} < r_3 < r_4 < y_{22}$, or Figure 5$(g)$ if $r_1 < y_{21} < r_2 < r_3 < y_{22} < r_4$, or Figure 5$(h)$ if $r_1 < y_{21} < r_2 < r_3 < r_4 < y_{22}$, or Figure 5$(i)$ if $y_{21} < r_1 < r_2 < r_3 < y_{22} < r_4$.

If $P(y_2)$ has one triple and one simple real root, or two complex and two simple real roots, the graphics (C_{g_1}) are given in Figure 5(j) if $y_{21} < r_1 < r_2 < y_{22}$, or Figure 5$(k)$ if $r_1 < y_{21} < r_2 < y_{22}$, or Figure 5$(l)$ if $y_{21} < r_1 < y_{22} < r_2$.

If $P(y_2) = 0$ has one double real and two complex roots, the graphics (C_{g_1}) are given in Figure 6(a).

If $P(y_2) = 0$ has two double real roots, the graphics (C_{g_1}) are given in Figure 6(b) if $r_1 < y_{21} < r_2 < y_{22}$, or Figure 6$(c)$ if $y_{21} < r_1 < y_{22} < r_2$.

If $P(y_2) = 0$ has four complex roots, see Figure 6(d).

If $P(y_2) = 0$ has one double real r_0 and two simple real roots r_1 and r_2, then if $r_0 < r_1 < y_{21} < r_2 < y_{22}$ see Figure 6(e), or if $y_{21} < r_1 < y_{22} < r_0 < r_2$ see Figure 6(f), or if $r_1 < y_{21} < r_0 < r_2 < y_{22}$ see Figure

6(g), or if $y_{21} < r_1 < r_0 < y_{22} < r_2$ see Figure 6(h), or if $r_1 < y_{21} < r_0 < y_{22} < r_2$ see Figure 6(i), or if $y_{21} < r_1 < r_2 < y_{22} < r_0$ see Figure 6(j), or if $r_0 < y_{21} < r_1 < r_2 < y_{22}$ see Figure 6(k), or if $y_{21} < r_0 < y_{22} < r_1 < r_2$ see Figure 6(l), or if $r_1 < r_2 < y_{21} < r_0 < y_{22}$ see Figure 7(a), or if $r_0 < y_{21} < r_1 < y_{22} < r_2$ see Figure 7(b).

If $P(y_2) = 0$ has one real root of order four this root must equal one of the two asymtotes y_{21} or y_{22}, then the graphics (C_{g_1}) are given in Figure 7(c), or Figure 7(d).

Now if $L_3 = -L_1$ we obtain that $P(y_2) = 0$ is a cubic equation, therefore the graphics (C_{g_1}) are as follows.

If $P(y_2) = 0$ has one triple real root or one simple and two complex roots, the graphics (C_{g_1}) are given in Figure 7(c) or 7(d).

If $P(y_2) = 0$ has one double real and one simple real root, the graphics (C_{g_1}) are given in Figure 7(e) or 7(f) if $r_1 = r_2 < y_{21} < r_3$, or $r_1 < y_{21} < r_2 = r_3$, respectively.

If $P(y_2) = 0$ has three real roots, the graphics (C_{g_1}) are given in Figure 7(g) if $y_{21} < r_1 < r_2 < r_3$, or Figure 7(h) if $r_1 < r_2 < y_{21} < r_3$, or Figure 7(i) if $r_1 < y_{21} < r_2 < r_3$, or Figure 7(j) if $r_1 < r_2 < r_3 < y_{21}$.

We will only give the graphics (C_{g_1}) when the derivative $\big(g_1\big)'(y_2)$ starts with a negative sign because when the derivative start with a positive sign their graphics are topologically equivalent to the previous ones.

For the function $f_1(y_2)$ we remark that the sign of the derivative changes at most three times when r an even integer or $r = p/(2q+1)$ with p an even integer and q is an arbitrary integer which guarantees that (C_{f_1}) can have at most one local extrem in (a) or (b) of Figure 4, and on the other hand it is obvious that the graphics (C_{g_1}) can have at most four local extremes in (a), or (b), or (c), or (d), or (e), or (f), or (g), or (h), or (i) or (j) of Figure 5, and since the function $g_1(y_2)$ is positive and it has the horizontal asymptote straight line $g_1(y_2) = 1$, then we guarantee that the maximum number of intersection points between the graphics (C_{f_1}) and (C_{g_1}) can be precisely between (a) or (b) of Figure 4 and (a), or (b), or (c), or (d), or (e), or (f), or (g), or (h), or (i) or (j) of Figure 5. It is clear that the graphics (C_{f_1}) and (C_{g_1}) intersect at most in seven points, see for example Figure 8. Hence, $F(y_2) = 0$ has at most seven real solutions. We can show easily that if (y_1, y_2) is a solution of (21), then (y_2, y_1) is also a solution of this system. Consequently, the maximum number of limit cycles of the discontinuous piecewise differential system (3)–(6) in this case is at most three.

In what follows we construct an example with exactly seven intersection points between the graphics (C_{f_1}) and (C_{g_1}) by considering $\{L_1, L_2, L_3, k_0, k_1, k_2, k_3, r\} \to \{-2, 1, -3.5, 2, 2.5, -1, -6.9, 2\}$, these points are shown in Figure 8.

To complete the proof of this case we provide an example with three limit cycles.

Three limit cycles for a discontinuous piecewise differential system (3)–(6) of type $C = a = 0$ with $A = -b \neq 0$.

In the half-plane Σ^- we consider the quadratic center

$$
\begin{aligned}
\dot{x} = &\ -0.41137..x^2 + x(-4.7083..y - 0.8342..) + y(-12.0159..y - 0.866409..) + 1.05188..,\\
\dot{y} = &\ 0.053171..x^2 + x(0.67274..y + 0.0188591..) + y(1.9488..y + 0.11029..) - 0.720841..,
\end{aligned} \tag{22}
$$

its corresponding first integral is

$$
\begin{aligned}
H_2^{(1)}(x,y) = &\ -0.001233..(-x - 10.5578..y + 9.06)^2\Big(x^2 + (9.8313..y + 6.51292..)x + (21.4047..y\\
&+14.3032..)y + 2.38945..\Big).
\end{aligned}
$$

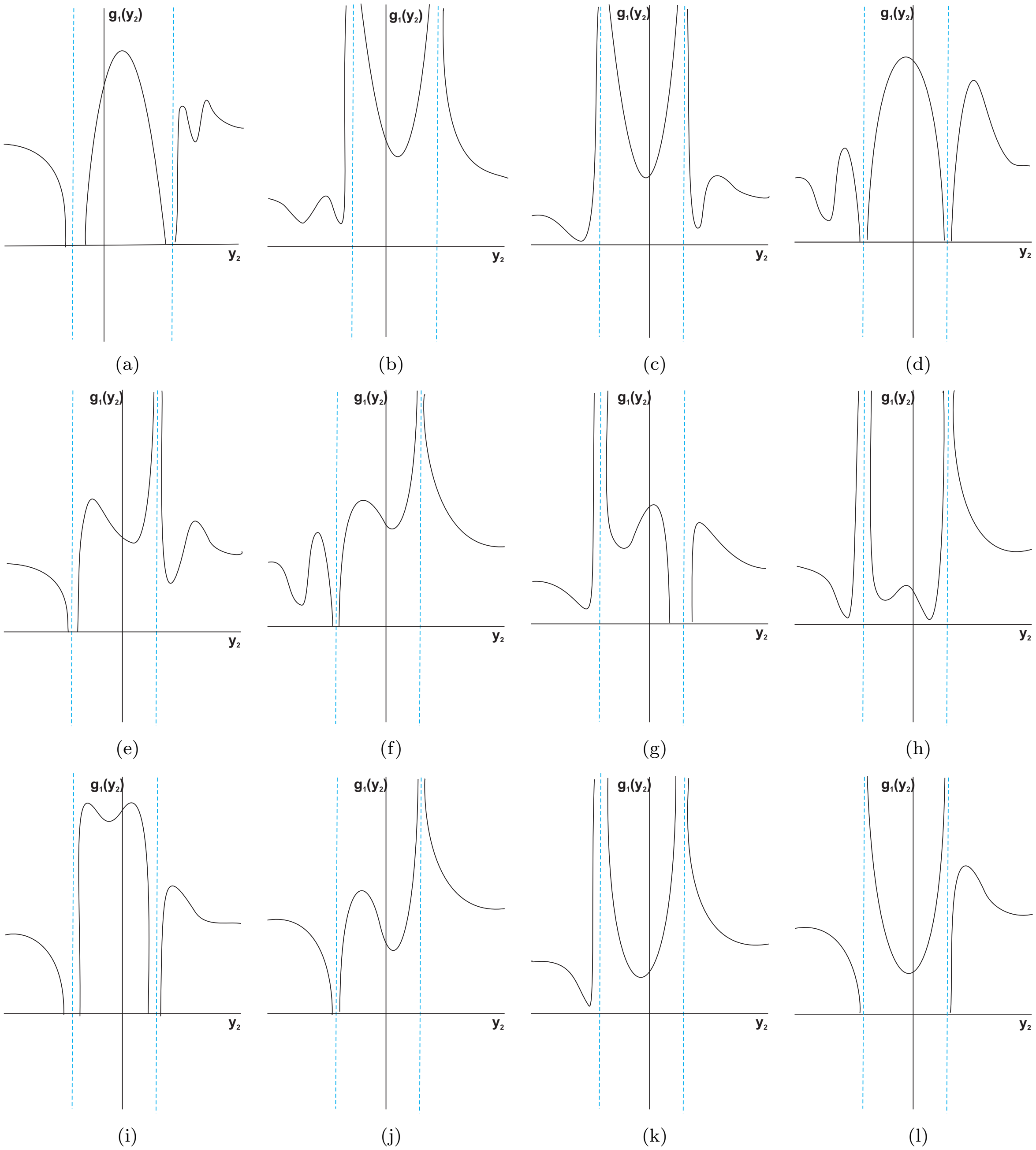


FIGURE 5. The graphics of the function $g_1(y_2)$.

In the half-plane Σ^+ we consider the linear differential center

$$\dot{x} = \frac{1}{5}x - \frac{229}{300}y + \frac{1}{5}, \quad \dot{y} = 3x - \frac{1}{5}y - \frac{3}{10}, \tag{23}$$

with the first integral

$$H(x,y) = 4\Big(3x - \frac{2}{10}y\Big)^2 + 24\Big(-\frac{3}{10}x - \frac{2}{10}y\Big) + 9y^2.$$

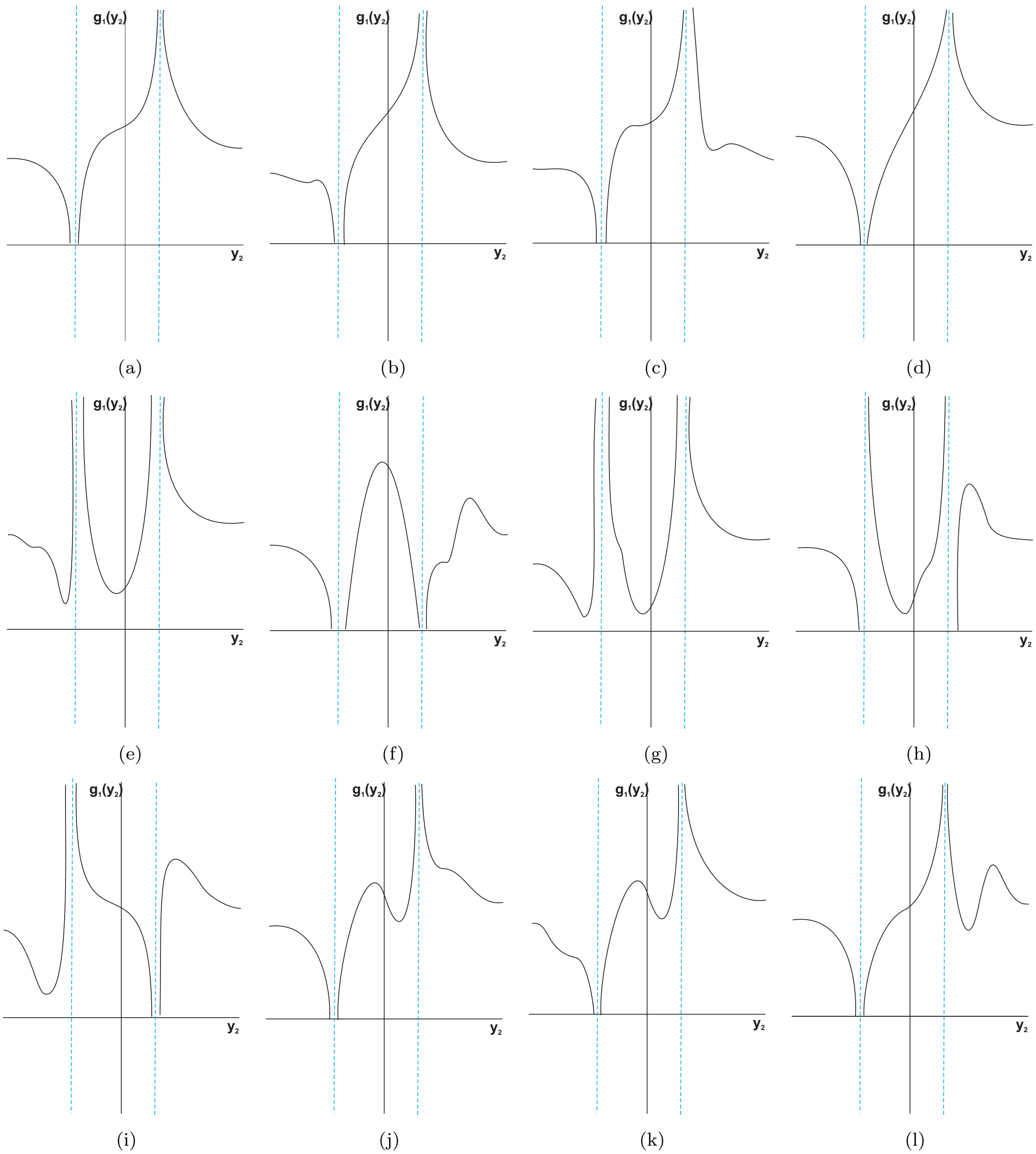


FIGURE 6. The graphics of the function $g_1(y_2)$.

In this case system (21) has the three solutions $(y_1, y_2) = (0.0217348.., 0.502283..)$, $(y_3, y_4) = (0.0725424.., 0.451475..)$ and $(y_5, y_6) = (0.143419.., 0.380598..)$ which provide the three limit cycles for the discontinuous piecewise differential system (22)–(23) shown in Figure 1(a).

Case 2. If $A = 0 \neq b$ then $k = 2$ and $j = 1$ in system (21), then the first integral of system (6) is $H_2^{(1)}(x, y)$ given in (8), and the solutions y_2 of $F(y_2) = 0$ are equivalent to the solutions of the

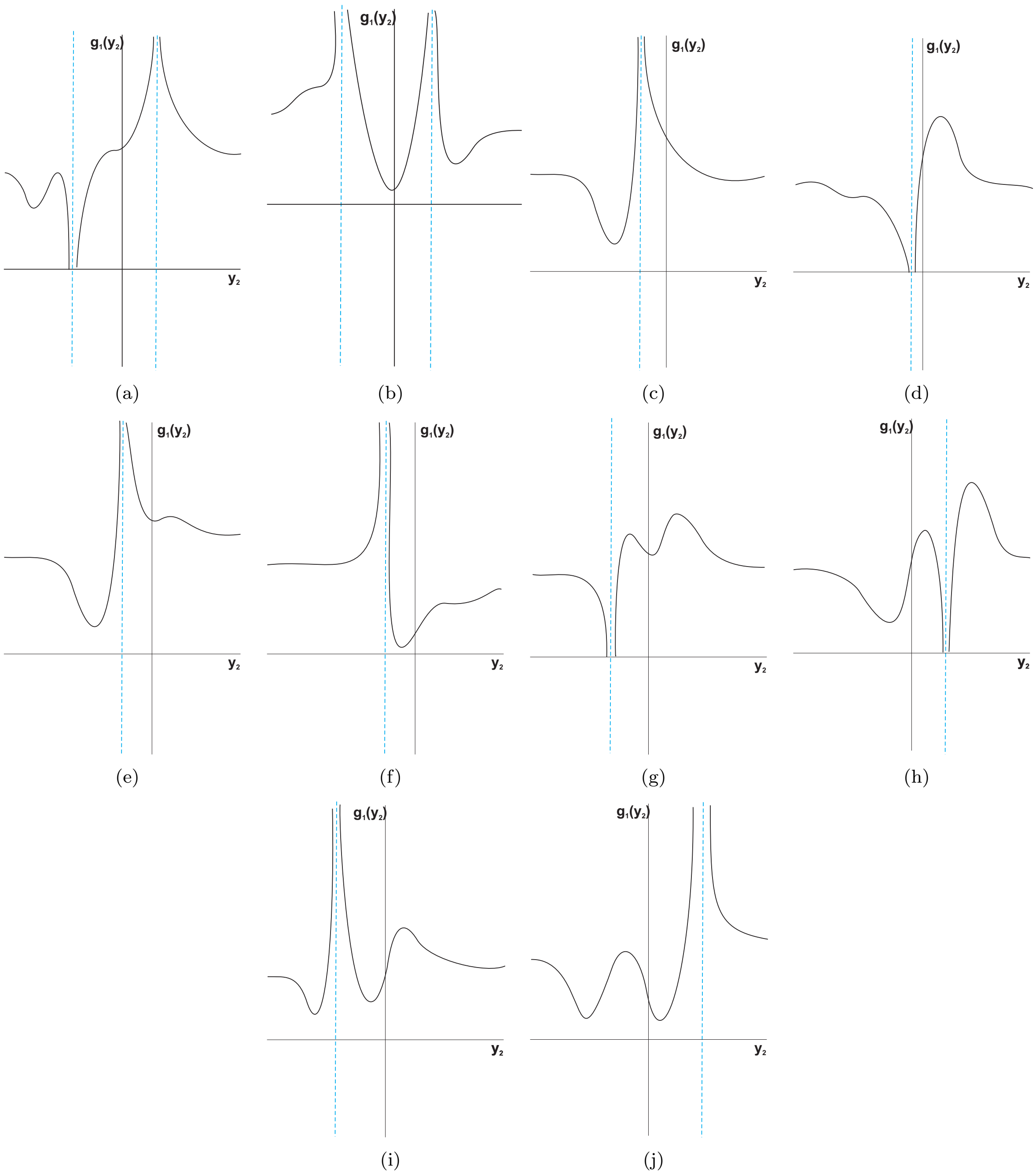


FIGURE 7. The graphics of the function $g_1(y_2)$.

non-algebraic equation $f_2(y_2) = g_2(y_2)$, where

$$f_2(y_2) = e^{(l_0+l_1 y_2)} \quad \text{and} \quad g_2(y_2) = \frac{k_0 + k_1\ y_2 + k_2\ y_2^2}{z_0 + z_1\ y_2 + k_2\ y_2^2},$$

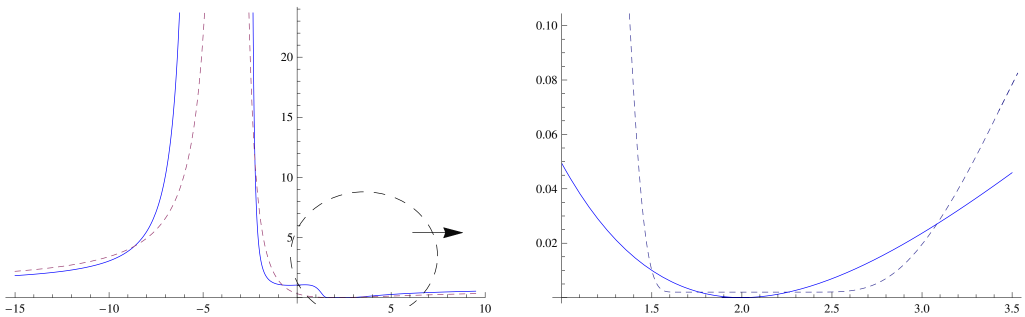


FIGURE 8. The seven intersection poitns between the two functions $f_1(y_2)$ drawn in continuous line and $g_1(y_2)$ drawn in dashed line.

with

$$\begin{aligned} l_0 = & -\frac{16\alpha b\gamma_2\sigma_1}{4\beta^2+\omega^2},\ l_1 = 4b\gamma_2,\ k_0 = b\left(2b^2\delta_1^2 + 2b\delta_2(d\delta_2+1) - 2d\delta_2 - 1\right) + d, \\ k_1 = & \ 2b\left(2b^2\gamma_1\delta_1 + b(\gamma_2 + 2\gamma_2 d\delta_2) - \gamma_2 d\right),\ k_2 = 2b^2\left(b\gamma_1^2 + \gamma_2^2 d\right), \\ z_0 = & \ 2b\Big(\frac{8\alpha\sigma_1}{4\beta^2+\omega^2}\left(2b^2\gamma_1\delta_1 + b(\gamma_2 + 2\gamma_2 d\delta_2) - \gamma_2 d\right) + b^2\delta_1^2 + \frac{64\alpha^2 b\sigma_1^2}{(4\beta^2+\omega^2)^2}\left(b\gamma_1^2 + \gamma_2^2 d\right) + bd\delta_2^2 + b\delta_2 \\ & -d\delta_2\Big) - b + d,\ z_1 = 2b\Big(-2b^2\gamma_1\delta_1 - b\gamma_2 - 2b\gamma_2 d\delta_2 - \frac{16\alpha b\sigma_1}{4\beta^2+\omega^2}\left(b\gamma_1^2 + \gamma_2^2 d\right) + \gamma_2 d\Big). \end{aligned}$$

We denoted by (C_{f_2}) and (C_{g_2}) the graphics of $f_2(y_2)$ and $g_2(y_2)$, respectively.

The possible graphics of $f_2(y_2)$ are shown either in Figure 10(a) if $l_1 > 0$, or in Figure 10(b) if $l_1 < 0$.

For the function $g_2(y_2)$ its derivative is

$$(g_2)'(y_2) = \frac{P_1(y_2)}{\Big(P_2(y_2)\Big)^2},$$

with

$$P_1(y_2) = \quad (k_2z_1 - k_1k_2)y_2^2 + (2k_2z_0 - 2k_0k_2)y_2 - k_0z_1 + k_1z_0 \quad and \quad P_2(y_2) = k_2y_2^2 + z_1y_2 + z_0.$$

We see that the discriminant of the numerator of $g_2(y_2) = 0$ is equal to the discriminant of $P_2(y_2) = 0$ which is $\Delta_0 = k_1^2 - 4k_0k_2 = z_1^2 - 4z_0k_2$, and $\Delta = (2k_2z_0 - 2k_0k_2)^2 - 4(k_2z_1 - k_1k_2)(-k_0z_1 + k_1z_0)$ is the discriminant of $P_1(y_2) = 0$, then according to the sign of the determinants Δ_0 and Δ the graphics of $g_1(y_2)$ are given in Figure 9(a) or Figure 9(b) if $\Delta_0 > 0$ and $\Delta < 0$, Figure 9(c) or Figure 9(d) if $\Delta_0 > 0$ and $\Delta > 0$, Figure 9(e) or Figure 9(f) if $\Delta_0 > 0$ and $\Delta = 0$ or $\Delta_0 = 0$ and $\Delta > 0$, and Figure 9(g) or Figure 9(h) if $\Delta_0 < 0$ and $\Delta < 0$.

It is clear that the graphics (C_{g_2}) can have the maximum number of local extremes in (c), or (d), or (g) or (h) of Figure 9, then we know that the maximum number of intersection points between the graphics (C_{f_2}) and (C_{g_2}) can be precisely between (a) or (b) of Figure 10 and (c), or (d), or (g) or (h) of Figure 9. It is obvious that the graphics (C_{f_2}) and (C_{g_2}) intersect at most in three points see for example Figure 11. Due to the symmetry of the solutions of system (21) we know that the maximum number of limit cycles in this case is at most one.

In the following we build an example with exactly three intersection points between the graphics (C_{f_2}) and (C_{g_2}) by taking $\{l_0, l_1, k_0, k_1, k_2, z_0, z_1\} \to \{0.35, 0.2, 3.9, -2, -1, 4, -2\}$, these points are shown in Figure 11.

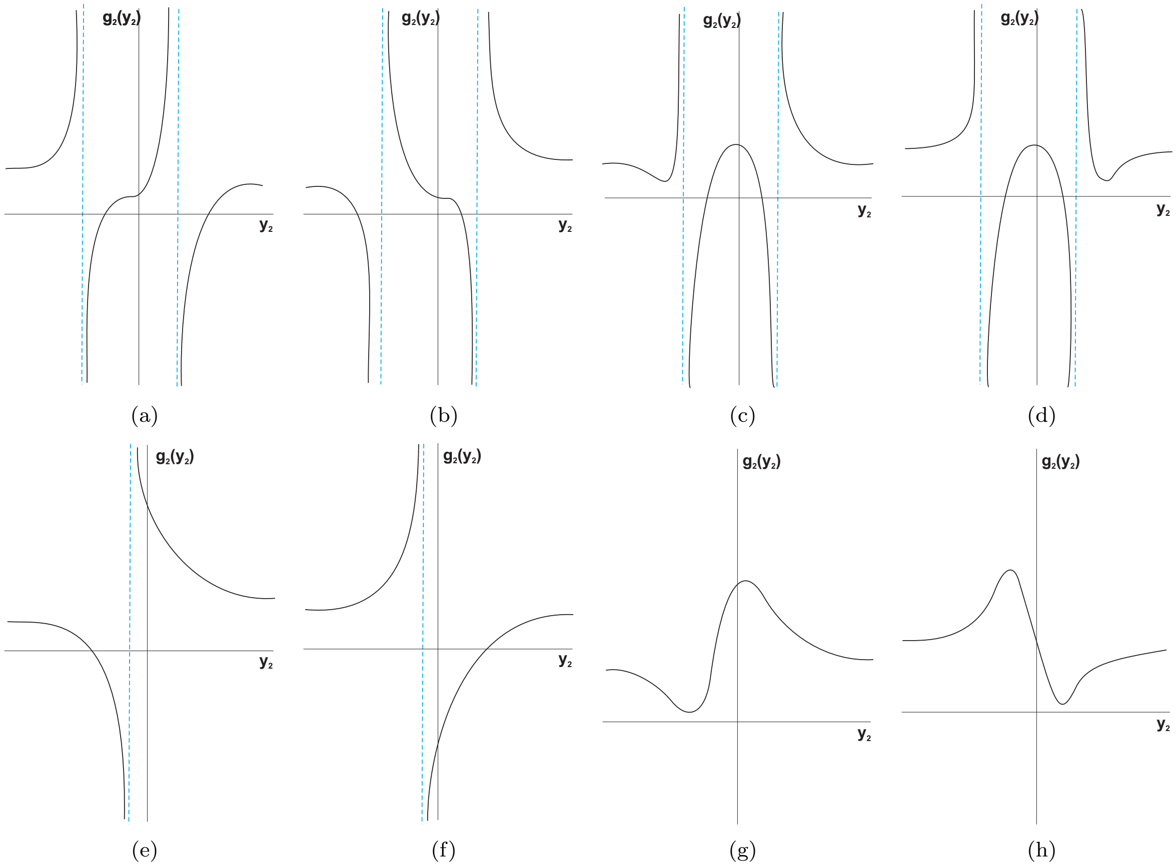


FIGURE 9. The graphic of the function $g_2(y_2)$.

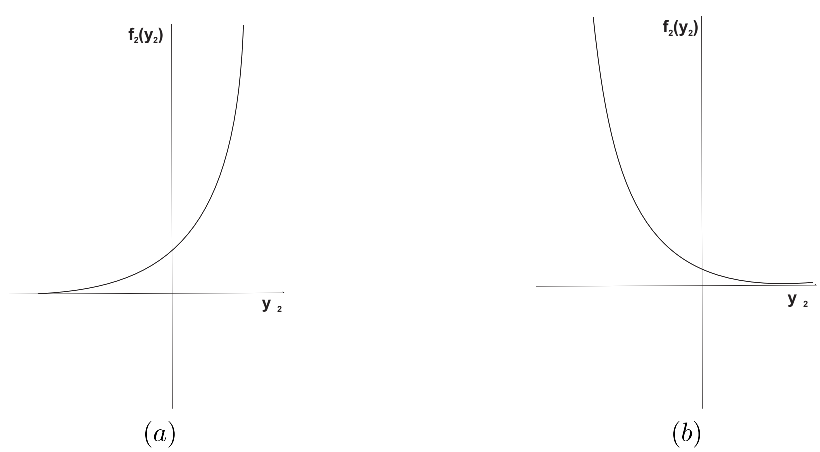


FIGURE 10. The graphic of the function $f_2(y_2)$.

Case 3. If $b = 0 \neq A$ then $k = 3$ and $j = 1$ in system (21), the first integral of system (6) is $H_3^{(1)}(x, y)$ given in (9), and the solutions of $F(y_2) = 0$ are equivalent to the ones of the equation $f_3(y_2) = g_3(y_2)$.

We have $f_3(y_2) = f_1(y_2)$, with $r = 2d - 2A$, therefore the graphics of $f_3(y_2)$ are given in Figure 4.

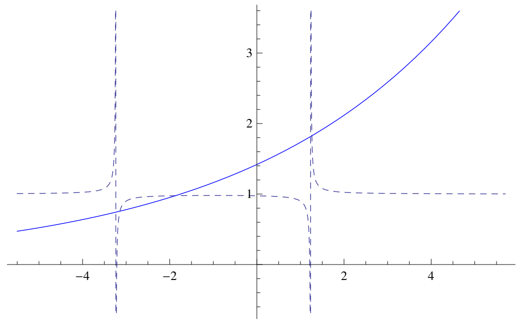


FIGURE 11. The three intersection poitns between the two functions $f_2(y_2)$ drawn in continous line and $g_2(y_2)$ drawn in dashed line.

We have also $g_3(y_2) = f_2(y_2)$, then we know that the graphics of $g_3(y_2)$ are given in Figure 10.

The parameters of the function $g_3(y_2)$ are:

$$\begin{aligned} l_0 = & \frac{1}{(4\beta^2+\omega^2)^2}\Big(16\alpha A\sigma_1(A^2\gamma_1(\delta_1(4\beta^2+\omega^2)+4\alpha\gamma_1\sigma_1)+A\gamma_2(4\beta^2(d\delta_2+1)+4\alpha\gamma_2\sigma_1 d+d\delta_2\omega^2 \\ & +\omega^2)-\gamma_2 d(4\beta^2+\omega^2))\Big), \\ l_1 = & -\frac{1}{4\beta^2+\omega^2}\Big(4A(A^2\gamma_1(\delta_1(4\beta^2+\omega^2)+4\alpha\gamma_1\sigma_1)+A\gamma_2(4\beta^2(d\delta_2+1)+4\alpha\gamma_2\sigma_1 d+d\delta_2\omega^2+\omega^2) \\ & -\gamma_2 d(4\beta^2+\omega^2))\Big). \end{aligned}$$

Due to the fact that $f_3(y_2) = f_1(y_2)$ there is at most one local extrem at zero in (a) or (b) of Figure 4, we know also that $g_3(y_2) = f_2(y_2)$, then it is clear that the maximum number of intersections points between the graphics (C_{g_3}) and (C_{f_3}) can be precisely between (a) or (b) of Figure 4 and Figure 10. Consequently the graphics of functions $f_3(y_2)$ and $g_3(y_2)$ intersect at most in three points see for example Figure 12. Due to the symmetry of the solutions of system (21) we know that the maximum number of limit cycles of the discontinuous piecewise differential system (3)–(6) is at most one.

By considering $\{l_0, l_1, L_1, L_2, L_3, r\} \to \{-0.2, 1.75, 4, -7, -0.4, -2\}$ we constuct an example with exactly three intersection points between the graphics of the two functions $f_3(y_2)$ and $g_3(y_2)$, these points are illustrated in Figure 12.

Case 4. If $A = b = 0$ then $k = 4$ and $j = 1$ in system (21), the first integral in this case is $H_4^{(1)}(x, y)$ given in (10), and

$$\begin{aligned} F(y_2) = \; & 2d\Big(\frac{8\alpha\gamma_2\sigma_1}{4\beta^2+\omega^2}+\delta_2-\gamma_2 y_2\Big)^3-2d(\delta_2+\gamma_2 y_2)^3+3\Big(\Big(\frac{8\alpha\gamma_1\sigma_1}{4\beta^2+\omega^2}+\delta_1-\gamma_1 y_2\Big)^2 \\ & +\Big(\frac{8\alpha\gamma_2\sigma_1}{4\beta^2+\omega^2}+\delta_2-\gamma_2 y_2\Big)^2\Big)-3\left((\delta_1+\gamma_1 y_2)^2+(\delta_2+\gamma_2 y_2)^2\right). \end{aligned}$$

Since $F(y_2) = 0$ is a cubic equation in the variable y_2 the maximum number of real solutions of system (21) is at most three. Eventually, the upper bound of the maximum number of limit cycles for this case is at most one.

To complete the proof of this case we build an example with one limit cycle of the discontinuous piecewise differential system (3)–(6) of type $C = a = 0$ with $A = b = 0$.

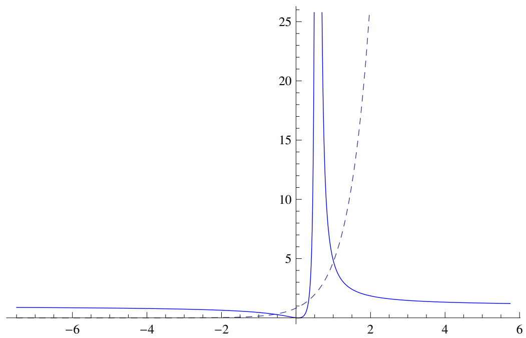


FIGURE 12. The three intersection poitns between the two functions $f_3(y_2)$ drawn in continuous line and $g_3(y_2)$ drawn in dashed line.

We consider the quadratic center

$$\begin{aligned}\dot x =&\quad -1.06383..\left(x\Big(\frac{11}{5}-\frac{1}{10}\big(1-4\big(-0.2y-0.5\big)\big)\Big)+\frac{1}{10}x^2+\frac{8}{500}y^2+\frac{28}{25}y-\frac{43}{10}\right),\\ \dot y =&\quad 1.06383..\left(-\frac{1}{4}x^2+x\Big(\frac{1}{4}\Big(1-4\Big(-\frac{1}{5}y-\frac{1}{2}\Big)\Big)+\frac{121}{25}\Big)-\frac{1}{25}y^2+\frac{19}{10}y-\frac{52}{5}\right),\end{aligned}\tag{24}$$

in the half-plane Σ^-, with its first integral

$$H_2^{(1)}(x,y)=3\left(\Big(\frac{1}{2}x-\frac{1}{5}y-\frac{1}{2}\Big)^2+\Big(-\frac{11}{5}x-y+\frac{9}{2}\Big)^2\right)-4\Big(\frac{1}{2}x-\frac{1}{5}y-\frac{1}{2}\Big)^3.$$

In the half-plane Σ^+ we consider the linear differential center

$$\dot x=\frac{1}{10}x-0.27632..y+1,\quad \dot y=x-\frac{1}{10}y-\frac{1}{2},\tag{25}$$

with the first integral

$$H(x,y)=4\Big(x-\frac{1}{10}y\Big)^2+8\Big(-\frac{1}{2}x-y\Big)+1.06528..y^2.$$

In this case system (21) has the unique solution $(y_1,y_2)=(0.978592..,6.25938..)$ which provides the unique limit cycle for the discontinuous piecewise differential system (24)–(25), see Figure 1(c). This example completes the proof of statement (I). □

Proof of statement (II) of Theorem 3. Now we must prove the statement for the discontinuous piecewise differential system formed by the linear center (3) and the quadratic center (6) of type $b+d=0$, and we distinguish the following cases:

Case 1. If $A+b=0$ and $a=0\neq C$, then $k=1$ and $j=2$ in system (21), the first integral of the quadratic center is $H_1^{(2)}(x,y)$ given in (11), the solutions of $F(y_2)=0$ are the same as the solutions of the equation $\tilde f_1(y_2)=\tilde g_1(y_2)$ where

$$\tilde f_1(y_2)=\left(\frac{m_1+m_2\ y_2}{m_3-m_2\ y_2}\right)^{r_1}\left(\frac{n_1+n_2\ y_2}{n_3-n_2\ y_2}\right)^{r_2}\quad\text{and}\quad \tilde g_1(y_2)=e^{\dfrac{k_0+k_1\ y_2}{(m_1+m_2\ y_2)(m_3-m_2\ y_2)}},$$

where

$$m_1 = -\frac{8\alpha b\gamma_2\sigma_1}{4\beta^2+\omega^2} - b\delta_2 + 1,\ m_2 = b\gamma_2 =,\ m_3 = 1 - b\delta_2,\ r_1 = b^2 + C^2,$$

$$n_1 = -b\delta_2 + C\delta_1 + 1,\ n_2 = \gamma_1 C - b\gamma_2,\ n_3 = \frac{8\alpha\sigma_1}{4\beta^2+\omega^2}(\gamma_1 C - b\gamma_2) - b\delta_2 + C\delta_1 + 1,\ r_2 = b^2,$$

$$k_0 = -\frac{8\alpha bC\sigma_1}{4\beta^2+\omega^2}(b(-b\gamma_1\delta_2 + b\gamma_2\delta_1 + \gamma_1) + \gamma_2 C),\ k_1 = 2bC(b(-b\gamma_1\delta_2 + b\gamma_2\delta_1 + \gamma_1) + \gamma_2 C).$$

The derivative of the function $\tilde{f}_1(y_2)$ and $\tilde{g}_1(y_2)$ are

$$\left(\tilde{f}_1\right)'(y_2) = \left(M_0 + M_1\ y_2 + M_2\ y_2^2\right)\frac{(m_1 + m_2\ y_2)^{r_1-1}(n_1 + n_2\ y_2)^{r_2-1}}{(m_3 - m_2\ y_2)^{r_1+1}(n_3 - n_2\ y_2)^{r_1+1}},$$

and

$$\left(\tilde{g}_1\right)'(y_2) = \frac{\left(N_0 + N_1 y_2 + N_2 y_2^2\right)}{(m_1 + m_2 y_2)^2(m_3 - m_2 y_2)^2} e^{\dfrac{k_0 + k_1\ y_2}{(m_1 + m_2\ y_2)(m_3 - m_2\ y_2)}}.$$

with

$$\begin{aligned}
M_0 =&\ m_2 n_1 n_3 r_1(m_1 + m_3) + m_1 m_3 n_2 r_2(n_1 + n_3),\\
M_1 =&\ m_2 n_2(r_1(m_1 + m_3)(n_3 - n_1) - r_2(m_1 - m_3)(n_1 + n_3)),\\
M_2 =&\ -m_2 n_2(n_2 r_1(m_1 + m_3) + m_2 r_2(n_1 + n_3)),\\
N_0 =&\ -m_3 m_2 k_0 + m_1 m_2 k_0 + m_1 m_3 k_1,\ \ N_1 = 2k_0 m_2^2,\ \ N_2 = k_1 m_2^2.
\end{aligned}$$

According to the number of the vertical asymptotes of the function $\tilde{f}_1(y_2)$ we can divide the study of this function into two parts.

If $m_1 = n_1$, $m_2 = n_2$, $m_3 = n_3$, or $r_1 = 0$ and $r_2 \neq 0$, or $r_1 \neq 0$ and $r_2 = 0$, then the function $\tilde{f}_1(y_2)$ has one vertical asymptote and the graphics $(C_{\tilde{f}_1})$ of the function $\tilde{f}_1(y_2)$ are the same as the ones of the function $f_1(y_2)$ shown in Figure 4.

If $m_1 \neq n_1$, or $m_2 \neq n_2$, or $m_3 = n_3$, or $r_1 \neq 0$ and $r_2 \neq 0$, then the function $\tilde{f}_1(y_2)$ has two vertical asymptotes. Therefore according with the derivative $\left(\tilde{f}_1\right)'(y_2)$ which depends on the parameters r_1, r_2 and with the sign of the discriminant $\Delta_1 = M_1^2 - 4M_0M_2$ also according with the positions of the roots of the numerator of $\left(\tilde{f}_1\right)'(y_2)$ with respect to the two vertical asymptotes, we obtain that all the possible topologically distinct graphics $(C_{\tilde{f}_1})$ of the function $\tilde{f}_1(y_2)$ are given as follows.

If r_1 and r_2 are even integers, or r_1 is an even integer and r_2 is rational such that $r_2 = 2p/(2q+1)$ with $p, q \in \mathbb{Z}$, all the graphics $(C_{\tilde{f}_1})$ are given in Figure 13 if $\Delta_1 > 0$. If $\Delta_1 = 0$ the graphics $(C_{\tilde{f}_1})$ are given in (a), or (b), or (c), or (d), or (e) of Figure 14. If $\Delta_1 < 0$ the graphics $(C_{\tilde{f}_1})$ are given in (f), or (g) of Figure 14.

If r_1 and r_2 are odd integers, or r_1 is an odd integer and r_2 is rational such that $r_2 = (2p+1)/(2q+1)$ with $p, q \in \mathbb{N}$, therefore if $\Delta_1 > 0$ the graphics $(C_{\tilde{f}_1})$ are given in (h), or (i), or (j), or (k), or (l) of Figure 14 and in (a), or (b), or (c), or (d), or (e), or (f), or (g) of Figure 15. If $\Delta_1 = 0$ the graphics $(C_{\tilde{f}_1})$ are given in (h), or (i), or (j), or (k), or (l) of Figure 15. If $\Delta_1 < 0$ the graphics $(C_{\tilde{f}_1})$ are given in (a), or (b) of Figure 16.

If r_1 is an odd integer and r_2 is an even integer, or r_2 is an even integer and $r_1 = (2p+1)/(2q+1)$ with $p, q \in \mathbb{Z}$, or r_1 is an odd integer and $r_2 = (2p)/(2q+1)$ with $p, q \in \mathbb{Z}$, then the sign of the derivative $\left(\tilde{f}_1\right)'(y_2)$ depends on the sign of $\left(M_0 + M_1\ y_2 + M_2\ y_2^2\right)(n_1 + n_2\ y_2)(n_3 - n_2\ y_2)$, therefore if $\Delta_1 > 0$ the graphics $(C_{\tilde{f}_1})$ are given in (c), or (d), or (e), or (f), or (g), or (h), or (i), or (j), or (k), or (l) of Figure 16 and in (a), or (b) of Figure 17. If $\Delta_1 = 0$ the graphics $(C_{\tilde{f}_1})$ are given in (c), or (d), or (e), or (f), or (g) of Figure 17. If $\Delta_1 < 0$ the graphics $(C_{\tilde{f}_1})$ are given in (h), or (i) of Figure 17.

If r_1 is an odd integer and r_2 is irrational or $r_2 = p/2q$ with $p, q \in \mathbb{Z}$, then the sign of the derivative $\big(\tilde{f}_1\big)'(y_2)$ depends on the sign of the quadratic polynomial $\big(M_0 + M_1\ y_2 + M_2\ y_2^2\big)$, therefore the graphic $(C_{\tilde{f}_1})$ are the same as the case in which r_1 and r_2 are odd integers but in their domain of definition.

If r_2 is an even integer and r_1 is irrational or $r_1 = p/2q$ with $p, q \in \mathbb{Z}$, then the sign of the derivative $\big(\tilde{f}_1\big)'(y_2)$ depends on the sign of the products $\big(M_0 + M_1\ y_2 + M_2\ y_2^2\big)\,(n_1 + n_2\ y_2)(n_3 - n_2\ y_2)$, therefore the graphics $(C_{\tilde{f}_1})$ are the same as in the case that r_1 is an odd integer and r_2 is an even integer but in their domain of definition.

If r_1 is irrational or $r_1 = p_0/2q_0$ and r_2 is irrational or rational with $r_2 = p/2q$ and p_0, p are odd integers, then the sign of the derivative $\big(\tilde{f}_1\big)'(y_2)$ depends on the quadratic polynomial $\big(M_0 + M_1\ y_2 + M_2\ y_2^2\big)$, therefore the graphics of $\tilde{f}_1(y_2)$ are the same as in the case that r_1 and r_2 are odd integers, but in their domain of definition.

If both r_1, r_2 are rational with $r_1 = (2p_0)/(2q_0+1)$ and $r_2 = (2p)/(2q+1)$ such that $p, q, p_0, q_0 \in \mathbb{Z}$, then the sign of the derivative $\big(\tilde{f}_1\big)'(y_2)$ depends on $\big(M_0 + M_1\ y_2 + M_2\ y_2^2\big)\,(m_1 + m_2\ y_2)(m_3 - m_2\ y_2)(n_1 + n_2\ y_2)(n_3 - n_2\ y_2)$, therefore the graphics of $\tilde{f}_1(y_2)$ are the same as in the case that r_1 and r_2 are even integers.

If both r_1, r_2 are rational with $r_1 = (2p_0 + 1)/(2q_0 + 1)$ and $r_2 = (2p + 1)/(2q + 1)$ such that $p, q, p_0, q_0 \in \mathbb{Z}$, then the sign of the derivative $\big(\tilde{f}_1\big)'(y_2)$ depends on $\big(M_0 + M_1\ y_2 + M_2\ y_2^2\big)$, therefore the graphics of $\tilde{f}_1(y_2)$ are the same as in the case that r_1 and r_2 are odd integers.

If both r_1, r_2 are rational with $r_1 = (2p_0+1)/(2q_0+1)$ and $r_2 = (2p)/(2q+1)$ such that $p, q, p_0, q_0 \in \mathbb{Z}$, then the sign of the derivative $\big(\tilde{f}_1\big)'(y_2)$ depends on $\big(M_0 + M_1\ y_2 + M_2\ y_2^2\big)$, therefore the graphics of $\tilde{f}_1(y_2)$ are the same as in the case of r_1 is an odd integer and r_2 is an even integer.

If r_1 is irrational or rational and r_2 rational with r_1 irrational or $r_1 = p_0/(2q_0)$ and $r_2 = (2p)/(2q+1)$ and p_0 is an odd integer and q_0, p and q are integers, therefore the graphics of $\tilde{f}_1(y_2)$ are the same in the case that r_1 is an odd integer and r_2 is an even integer but in their domain of definition.

If r_1 is irrational or rational and r_2 rational with r_1 irrational or $r_1 = p_0/(2q_0)$ and $r_2 = (2p+1)/(2q+1)$ and p_0 is an odd integer and q_0, p and q are integers, therefore the graphics of $\tilde{f}_1(y_2)$ are the same in the case that r_1 and r_2 are odd integers but in their domain of definition.

According to the sign of the derivative of the function $\tilde{g}_1(y_2)$ which depends on the sign of the quadratic polynomial $P(y_2) = N_0 + N_1 y_2 + N_2 y_2^2$, then the topologically distinct graphics of $\tilde{g}_1(y_2)$ are shown in (a) and (b) of Figure 18 if $m_3 \neq -m_1$ and $P(y_2)$ has two distinct real roots, or (c) of Figure 18 if $m_3 \neq -m_1$ and $P(y_2)$ has two complex roots, or (d) of Figure 18 if $m_3 \neq -m_1$ and $P(y_2)$ has one double real root, or (e) and (f) of Figure 18(e) if $m_3 = -m_1$.

The possible graphics of $\tilde{f}_1(y_2)$ are given in Figures 4, 13, 14, 15, 16 and 17, and the graphics of $\tilde{g}_1(y_2)$ are given in Figure 18.

Since the graphics of $\tilde{f}_1(y_2)$ can have the maximum number of local extremes in (a)–(l) of Figure 13 and due to the fact that the function $\tilde{g}_1(y_2)$ is positive and its graphics can have at most two extremes in (a) or (b) of Figure 18, we know that the maximum number of intersection points between the graphics of $\tilde{f}_1(y_2)$ and $\tilde{g}_1(y_2)$ can be precisely between (a)–(l) of Figure 13 and (a) or (b) of Figure 18. In this case the two functions $\tilde{f}_1(y_2)$ and $\tilde{g}_1(y_2)$ have $\tilde{f}_1(y_2) = \tilde{g}_1(y_2) = 1$ as an horizontal asymptote which ensures that at infinity there are no intersection points between their graphics. Therefore the graphics $(C_{\tilde{f}_1})$ and $(C_{\tilde{g}_1})$ of the two functions $\tilde{f}_1(y_2)$ and $\tilde{g}_1(y_2)$ intersect at most in six points see for example Figure 19. Then the upper bound of the maximum number of limit cycles in this case is at most three.

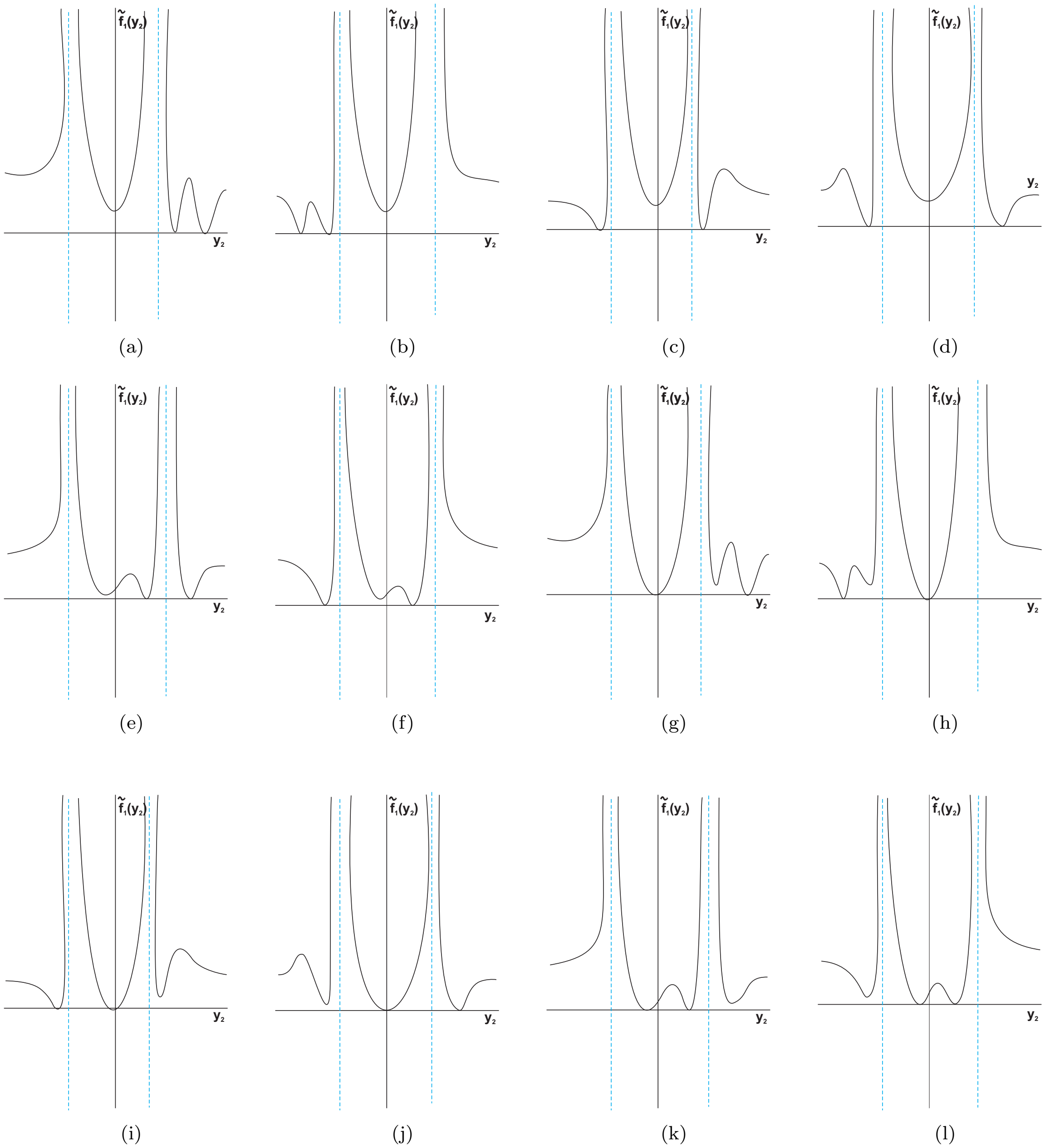


FIGURE 13. The graphics of the function $\tilde{f}_1(y_2)$.

By taking $\{k_0, k_1, m_1, m_2, m_3, r_1, r_2, n_1, n_2, n_3\} \to \{-0.21, -1.7, -1.7, -5.8, -2.5, 4, 2, -1.28, 2.75, -3.9\}$. we build an example with exaclty six intersection poitns between graphics of the functions $\tilde{f}_1(y_2)$ and $\tilde{g_1}(y_2)$. These points are shown in Figure 19.

Case 2. If $AbC\Delta \neq 0$ and $a = 0$ with $\Delta = 4b(A+b) + C^2 < 0$, then $k = 2$ and $j = 2$ in system (21), the first integral of (6) is $H_2^{(2)}(x, y)$ given in (12). The equation $F(y_2) = 0$ is equivalent to the equation

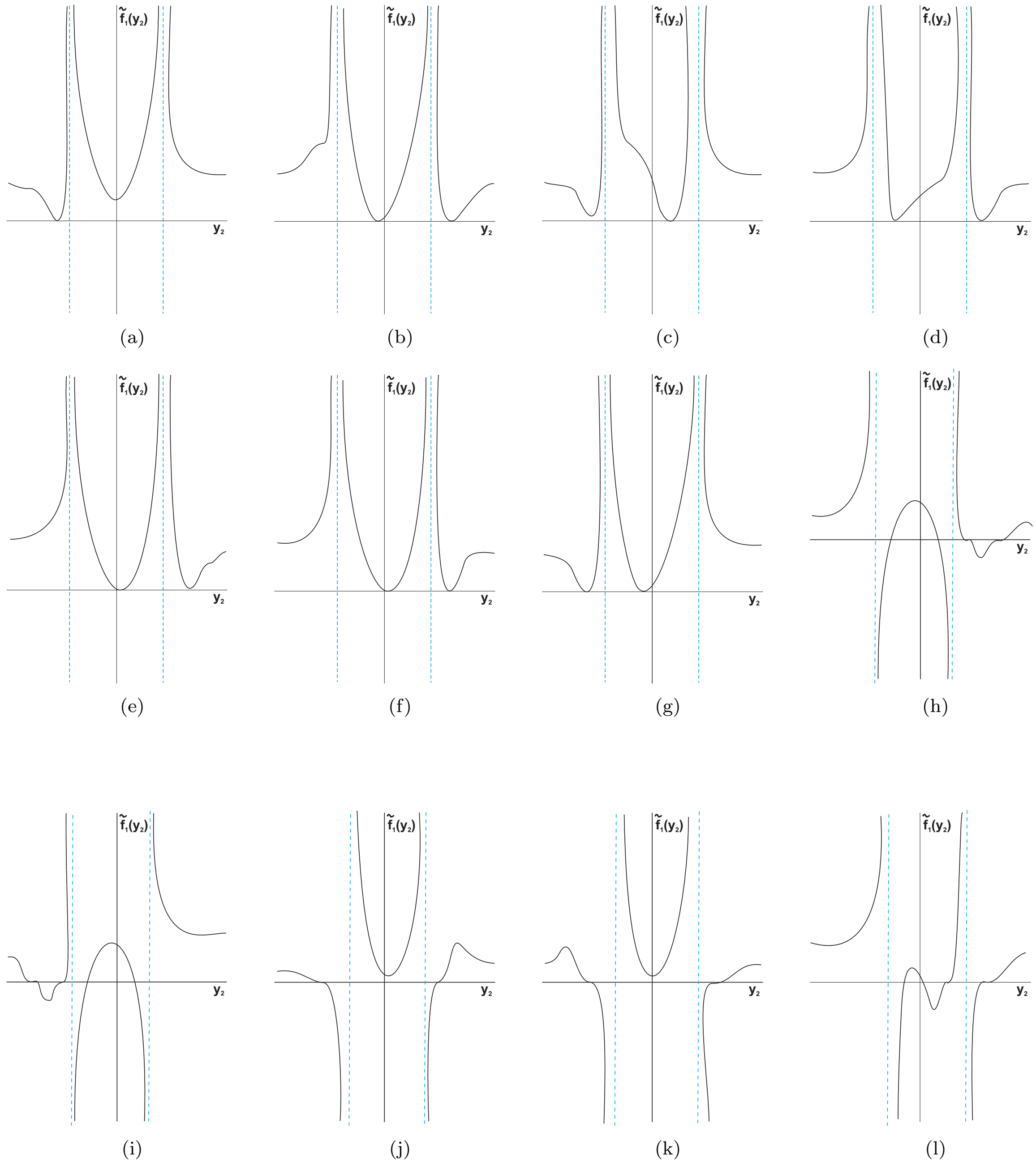


FIGURE 14. The graphics of the function $\tilde{f}_1(y_2)$.

$\tilde{f}_2(y_2) = \tilde{g}_2(y_2)$ where

$$\tilde{f}_2(y_2) = e^{m_1\left(\arctan\left(\frac{s_3 y_2 + s_4}{s_1 y_2 + s_2}\right) + \arctan\left(\frac{s_3 y_2 + s_6}{s_5 - s_1 y_2}\right)\right)},$$

and

$$\tilde{g}_2(y_2) = \left(\frac{t_1 y_2 + t_2}{-t_1 y_2 + t_3}\right)^{r_2} \left(\frac{K_1 y_2^2 + K_2 y_2 + K_3}{K_1 y_2^2 + K_4 y_2 + K_5}\right)^{r_1},$$

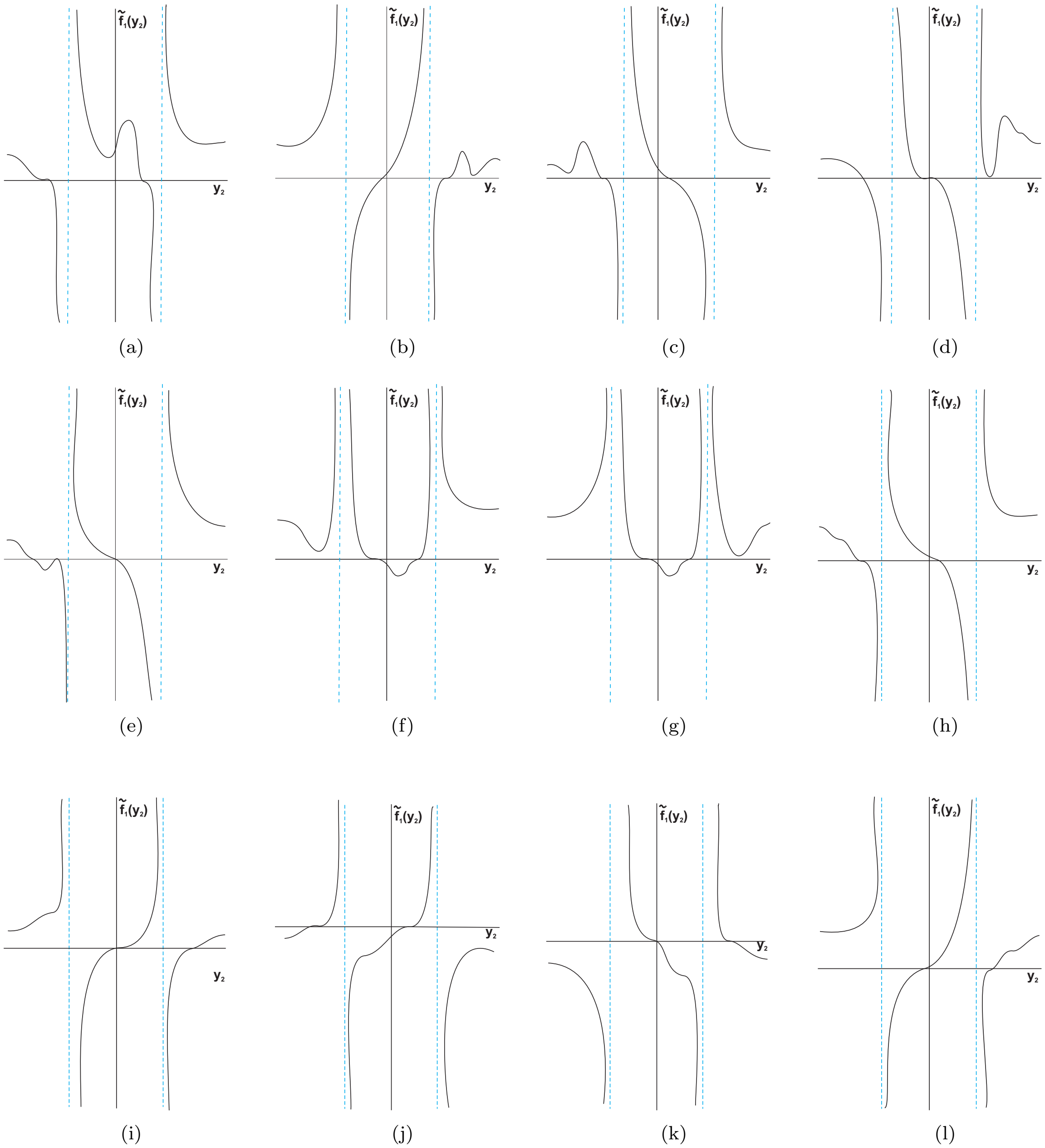


FIGURE 15. The graphics of the function $\tilde{f}_1(y_2)$.

where

$$\begin{aligned}
s_1 = &\quad \gamma_1 L,\ \ s_2 = \delta_1,\ \ s_3 = \gamma_1 C - 2b\gamma_2,\\
s_4 = &\quad -2b\delta_2 + C\delta_1 + 2,\ \ s_5 = \frac{8\alpha\gamma_1\sigma_1 L}{4\beta^2+\omega^2} + \delta_1 L,\\
s_6 = &\quad \frac{16\alpha b\gamma_2\sigma_1}{4\beta^2+\omega^2} + 2b\delta_2 - \frac{8\alpha\gamma_1 C\sigma_1}{4\beta^2+\omega^2} - C\delta_1 - 2,\ \ m_1 = \frac{2C}{bL},\\
K_1 = &\quad \frac{1}{4}\left(4b^2\gamma_2^2 - 4b\gamma_1\gamma_2 C + \gamma_1^2 C^2 + \gamma_1^2 L^2\right),
\end{aligned}$$

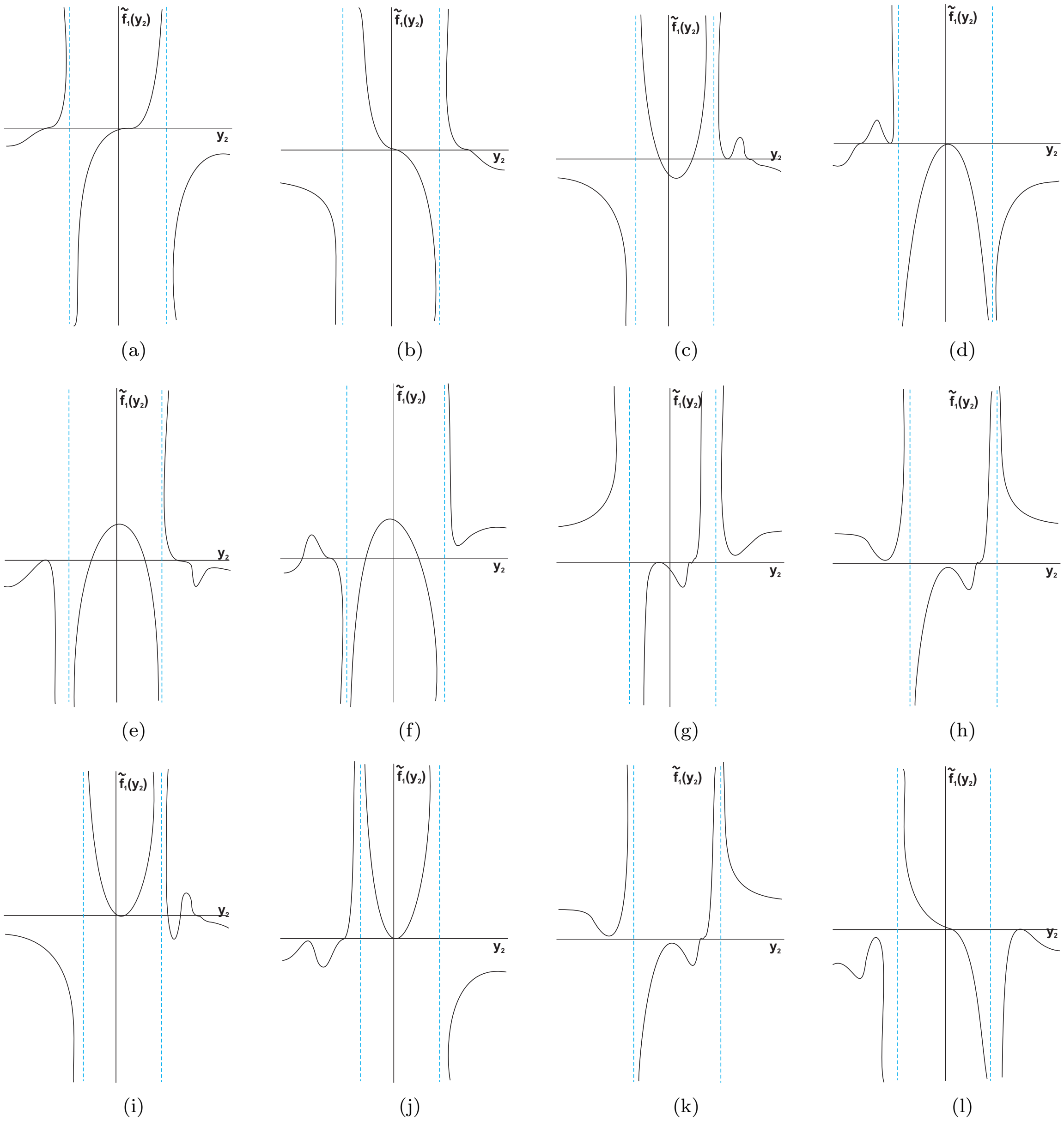


FIGURE 16. The graphics of the function $\tilde{f}_1(y_2)$.

$$
\begin{aligned}
K_2 = \; & -\frac{1}{2(4\beta^2+\omega^2)}\Big(8\alpha\sigma_1\left(4b^2\gamma_2^2+\gamma_1^2L^2\right)+\left(4\beta^2+\omega^2\right)\left(4b^2\gamma_2\delta_2-4b\gamma_2+\gamma_1\delta_1L^2\right)-2C(4\alpha b\gamma_1\gamma_2\sigma_1 \\
& +(4\beta^2+\omega^2)(b\gamma_1\delta_2+b\gamma_2\delta_1-\gamma_1))+\gamma_1C^2(8\alpha\gamma_1\sigma_1+\delta_1(4\beta^2+\omega^2))\Big), \\
K_3 = \; & \frac{1}{4\left(4\beta^2+\omega^2\right)^2}\Big(-4C\left(\delta_1\left(4\beta^2+\omega^2\right)+8\alpha\gamma_1\sigma_1\right)\left(4\beta^2(b\delta_2-1)+8\alpha b\gamma_2\sigma_1+b\delta_2\omega^2-\omega^2\right)+4\big(4\beta^2(b\delta_2-1) \\
& +8\alpha b\gamma_2\sigma_1+b\delta_2\omega^2-\omega^2\big)^2+C^2\left(\delta_1\left(4\beta^2+\omega^2\right)+8\alpha\gamma_1\sigma_1\right)^2+L^2\left(\delta_1\left(4\beta^2+\omega^2\right)+8\alpha\gamma_1\sigma_1\right)^2\Big),
\end{aligned}
$$

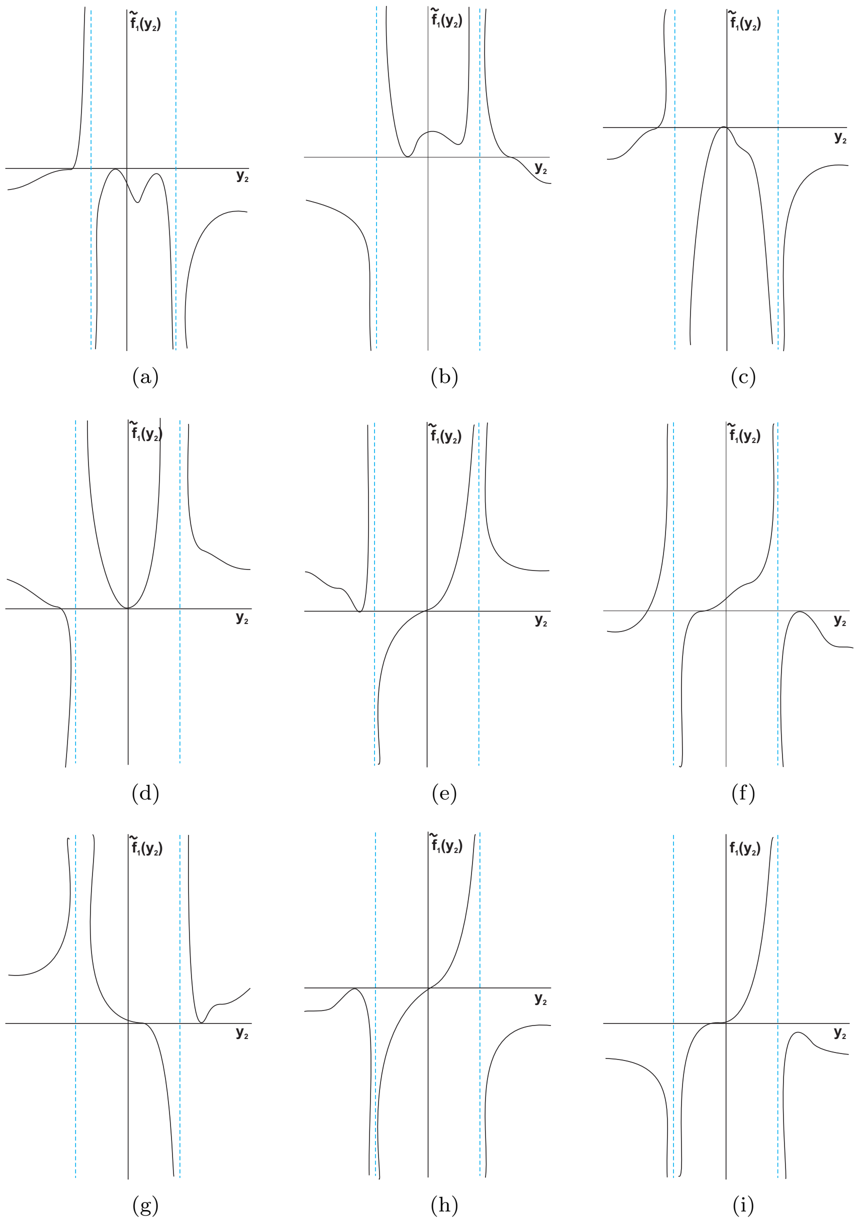


FIGURE 17. The graphics of the function $\tilde{f}_1(y_2)$.

$$
\begin{aligned}
K_4 = &\quad \frac{1}{2}\left(4b^2\gamma_2\delta_2 - 4b\gamma_2 - 2C(\gamma_1(b\delta_2-1)+b\gamma_2\delta_1)+\gamma_1C^2\delta_1+\gamma_1\delta_1L^2\right),\\
K_5 = &\quad \frac{1}{4}\left(-4C\delta_1(b\delta_2-1)+4(b\delta_2-1)^2+C^2\delta_1^2+\delta_1^2L^2\right),\ r_1=\frac{1}{b},\ r_2=-\frac{8b}{4b^2+C^2+L^2},\\
t_1 = &\quad \frac{\gamma_2\left(4b^2+C^2+L^2\right)}{4b},\ t_2=-\frac{1}{4b}\Big(\frac{8\alpha\gamma_2\sigma_1\left(4b^2+C^2+L^2\right)}{4\beta^2+\omega^2}+4b^2\delta_2-4b+C^2\delta_2+\delta_2L^2\Big),\\
t_3 = &\quad -\frac{C^2\delta_2}{4b}-b\delta_2-\frac{\delta_2L^2}{4b}+1.
\end{aligned}
$$

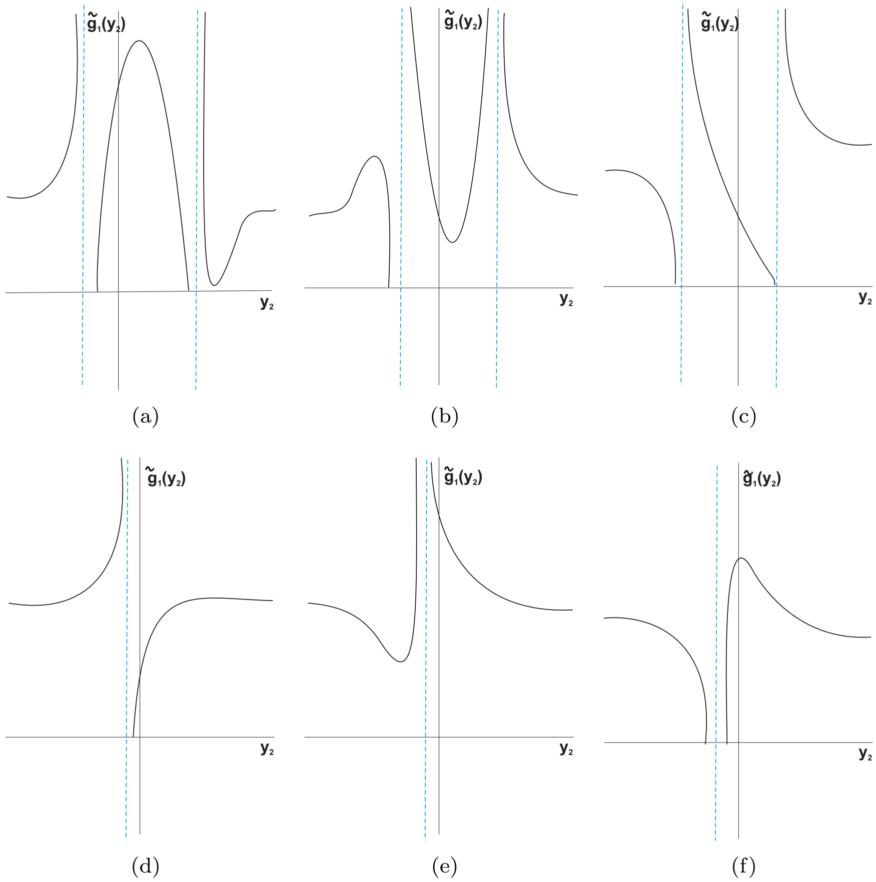


FIGURE 18. The graphic of the function $\tilde{g}_1(y_2)$.

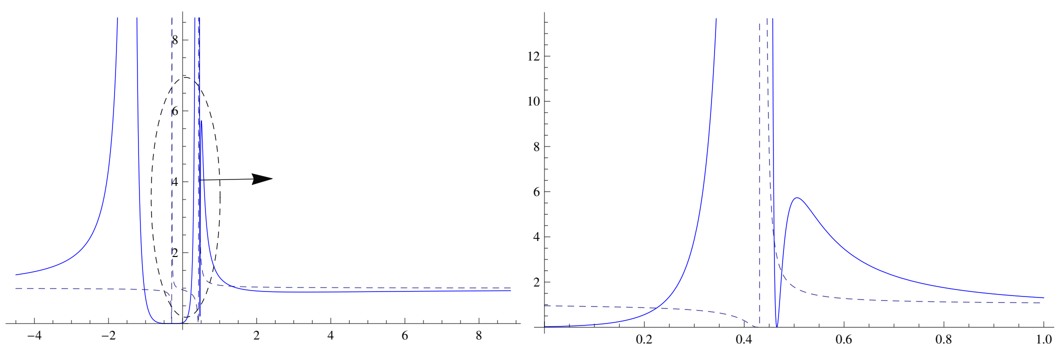


FIGURE 19. The six intersection poitns between the two functions $\tilde{f}_1(y_2)$ drawn in continuous line and $\tilde{g}_1(y_2)$ drawn in dashed line.

We note that $\left(\tilde{f}_2\right)'(y_2)$ and $\left(\tilde{g}_2\right)'(y_2)$ are the derivatives of the functions $\tilde{f}_2(y_2)$ and $\tilde{g}_2(y_2)$, respectively, where

$$\left(\tilde{f}_2\right)'(y_2) = m_1 \tilde{f}_2(y_2) \frac{P_1(y_2)}{\left((s_1y_2+s_2)^2+(s_3y_2+s_4)^2\right)\left((s_5-s_1y_2)^2+(s_3y_2+s_6)^2\right)},$$

and

$$(\tilde{g}_2)'(y_2) = -\frac{P_2(y_2)}{(t_1y_2+t_2)(-t_1y_2+t_3)\left(K_1y_2^2+K_2y_2+K_3\right)\left(K_1y_2^2+K_4y_2+K_5\right)}\left(\frac{t_1y_2+t_2}{-t_1y_2+t_3}\right)^{r_2}$$
$$\left(\frac{K_1y_2^2+K_2y_2+K_3}{K_1y_2^2+K_4y_2+K_5}\right)^{r_1},$$

with

$$\begin{aligned}
P_1(y_1) = \;& y_2^2\left(s_1^2+s_3^2\right)\left(s_1(s_4-s_6)-s_2s_3-s_3s_5\right)-2y_2\left(s_1^2+s_3^2\right)\left(s_2s_6+s_4s_5\right)-s_1s_2^2s_6-s_1s_4^2s_6+s_1s_4s_5^2\\
&+s_1s_4s_6^2-s_2^2s_3s_5-s_2s_3s_5^2-s_2s_3s_6^2-s_3s_4^2s_5, \quad \textit{and}\\
P_2(y_2) = \;& y_2^4\left(K_1^2r_2t_1t_2+K_1^2r_2t_1t_3+K_1K_2r_1t_1^2-K_1K_4r_1t_1^2\right)+y_2^3\big(K_1K_2r_1t_1t_2-K_1K_2r_1t_1t_3+K_1K_2r_2t_1t_2
\end{aligned}$$

$$\begin{aligned}
&+K_1K_2r_2t_1t_3+2K_1K_3r_1t_1^2-K_1K_4r_1t_1t_2+K_1K_4r_1t_1t_3+K_1K_4r_2t_1t_2+K_1K_4r_2t_1t_3-2K_1K_5r_1t_1^2\big)\\
&+y_2^2\big(-K_1K_2r_1t_2t_3+2K_1K_3r_1t_1t_2-2K_1K_3r_1t_1t_3+K_1K_3r_2t_1t_2+K_1K_3r_2t_1t_3+K_1K_4r_1t_2t_3\\
&-2K_1K_5r_1t_1t_2+2K_1K_5r_1t_1t_3+K_1K_5r_2t_1t_2+K_1K_5r_2t_1t_3+K_2K_4r_2t_1t_2+K_2K_4r_2t_1t_3\\
&-K_2K_5r_1t_1^2+K_3K_4r_1t_1^2\big)+y_2(-2K_1K_3r_1t_2t_3+2K_1K_5r_1t_2t_3-K_2K_5r_1t_1t_2+K_2K_5r_1t_1t_3\\
&+K_2K_5r_2t_1t_2+K_2K_5r_2t_1t_3+K_3K_4r_1t_1t_2-K_3K_4r_1t_1t_3+K_3K_4r_2t_1t_2+K_3K_4r_2t_1t_3)\\
&+K_2K_5r_1t_2t_3-K_3K_4r_1t_2t_3+K_3K_5r_2t_1t_2+K_3K_5r_2t_1t_3.
\end{aligned}$$

We denoted by $(C_{\tilde{f}_2})$ and $(C_{\tilde{g}_2})$ the graphics of $\tilde{f}_2$ and $\tilde{g}_2$, respectively.

According with the sign of $\left(\tilde{f}_2\right)'(y_2)$ which depends on m_1 and with the sign of $\delta_1 = (s_1s_4-s_2s_3)(s_1s_6+s_3s_5)$, we obtain that and all the possible topologically distinct graphics $(C_{\tilde{f}_2})$ of the function $\tilde{f}_2(y_2)$ are given in what follows.

For $s_5 \neq -s_2$ the function $\tilde{f}_2(y_2)$ can have two vertical asymptotes and all the distinct topologically equivalent graphics of the function $\tilde{f}_2(y_2)$ are given in Figure 20 as follows.

If $\delta_1 > 0$ the graphics $(C_{\tilde{f}_2})$ are given in (a), or (b), or (c), or (d), or (e), or (f) of Figure 20.

If $\delta_1 < 0$ the graphic $(C_{\tilde{f}_2})$ is shown in Figure 20(g).

If $\delta_1 = 0$ the graphic $(C_{\tilde{f}_2})$ is shown in Figure 20(h).

For $s_5 = -s_2$ the graphic $(C_{\tilde{f}_2})$ has only one vertical asymptote, then the graphics $(C_{\tilde{f}_2})$ depends only on the sign of δ_1.

If $\delta_1 \leq 0$, the graphic is given in Figure 20(h).

If $\delta_1 > 0$, the graphic is given in (i)$, or($j$), or(k)$ of Figure 20.

According with the derivative $\left(\tilde{g}_2\right)'(y_2)$ and the parameters r_1, r_2, and due to the fact that the sign of the discriminants Δ_1 and Δ_2 of the equations $K_1y_2^2+K_2y_2+K_3=0$, and $K_1y_2^2+K_4y_2+K_5=0$, respectively, are negative, where $\Delta_1 = \Delta_2 = -L^2(b\gamma_1\delta_2 - b\gamma_2\delta_1 - \gamma_1)^2$, and knowing the different kind of the roots x_i with $i \in \{1,\ldots,4\}$ of the polynomial $P_2(y_2)$, we get all the possible topologically distinct graphics $(C_{\tilde{g}_2})$ of the function $\tilde{g}_2(y_2)$ which are given in Figures 4, 13, 14, 15, 16 and 17 if $\Delta_1 = 0$ as we proved in the first case of statements (I) and (II). For $\Delta_1 < 0$ the topologically distinct possible graphics of $\tilde{g}_2(y_2)$ are given in what follows.

If either r_2 is an even integer or $r_2 = (2p)/(2q+1)$ with $p, q \in \mathbb{Z}$, and $P_2(y_2)$ has four simple real roots, then the graphic of $\tilde{g}_2(y_2)$ is given in (a), or (b), or (c), or (d), or (e), or (f) of Figure 21.

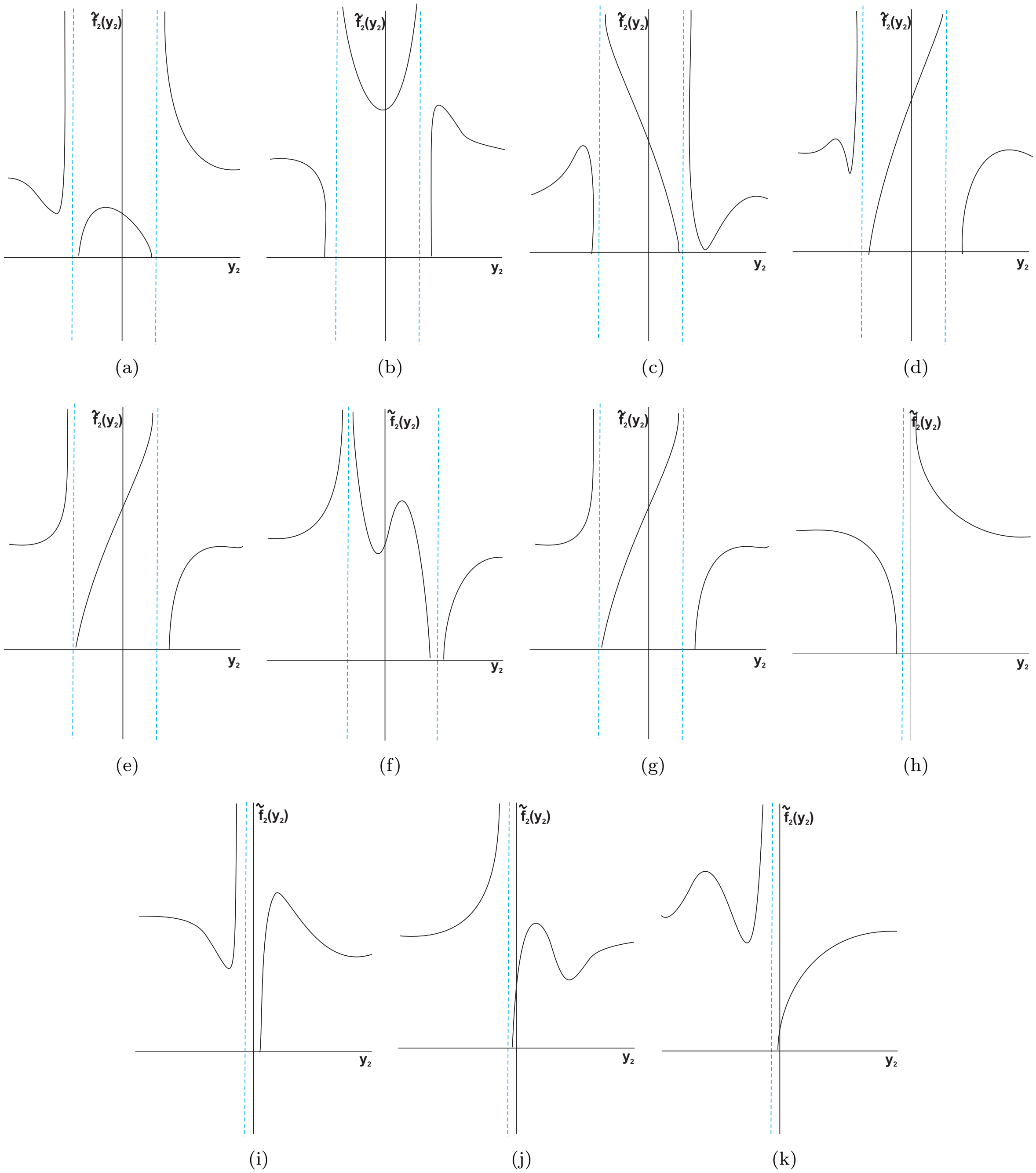


FIGURE 20. The graphic of the function $\tilde{f}_2(y_2)$.

If $P_2(y_2)$ has two complex and two simple real roots, then the graphic of $\tilde{g}_2(y_2)$ is given in (g), or (h), or (i), or (j) of Figure 21.

If $P_2(y_2)$ has four complex roots, then the graphic of $\tilde{g}_2(y_2)$ is shown in (k), or (l) of Figure 21.

If $P_2(y_2)$ has one double and two complex roots, then the graphic of $\tilde{g}_2(y_2)$ is given in (a), or (b), or (c) of Figure 22.

If $P_2(y_2)$ has two double real roots, then the graphic of $\tilde{g}_2(y_2)$ is shown in (d), or (e), or (f), or (g) of Figure 22.

If $P_2(y_2)$ has one triple and one simple real root, or one double and two simple real roots, then the graphic of $\tilde{g}_2(y_2)$ is given in (h) of Figure 22.

If r_2 is an odd integer or $r_2 = (2p+1)/(2q+1)$ with $p, q \in \mathbb{Z}$ we have the same graphics as the case when r_2 is an even integer where $x_0 = -(t_2/t_1)$ represent an inflexion point of the function $\tilde{g}_2(y_2)$.

if r_2 is irrational or $r_2 = p/(2q)$ with $p, q \in \mathbb{Z}$ the sign of the derivative $\left(\tilde{g}_2\right)'(y_2)$ depends only on the sign of $P_2(y_2)$, then the possible graphics of the function $\tilde{g}_2(y_2)$ are the same as the ones of the case where r_2 is an odd integer on its definition domain.

For $r_2 < 0$ and by a similar way we find the same graphics as in the case $r_2 > 0$.

The graphics of $\tilde{f}_2(y_2)$ are given in Figure 10 and the graphics of $\tilde{g}_2(y_2)$ are given in Figures 4, 13, 14, 15, 16, 17, 21 and 22.

For the function $\tilde{g}_2(y_2)$ we remark that its graphics can have at most five local extremes in (a), or (b), or (c), or (d), or (e) or (f) of Figure 21, we know also that the function $\tilde{f}_2(y_2)$ is positive and its graphics can have at most two extremes in (a), or (b), or (c), or (d), or (f), or (i), or (j) or (k) of Figure 10. Therefore we guarantee that the maximum number of intersection points between the graphics of $\tilde{f}_2(y_2)$ and $\tilde{g}_2(y_2)$ can be precisely between (a), or (b), or (c), or (d), or (e) or (f) of Figure 21 and (a), or (b), or (c), or (d), or (f), or (i), or (j) or (k) of Figure 10. Due to the fact that there are no intersection points at infinity because of the common horizontal asymptote $\tilde{f}_2(y_2) = \tilde{g}_2(y_2) = 1$. Then the maximum number of solutions of system (21) is at most seven see for example Figure 23, this provides at most three limit cycles of the discontinuous piecewise differential system (3)–(6).

Now we construct an example with exactly seven intersections points between the two functions $\tilde{f}_2(y_2)$ and $\tilde{g}_2(y_2)$ by taking $\{K_1, K_2, K_3, K_4, K_5, t_1, t_2, t_3, r_1, r_2, s_1, s_2, s_3, s_4, s_5, s_6, m_1\} \rightarrow \{0.576282, -3.2, 4.1, 0.1, 0.02, -5, 5.4, 93, 1, 2, 1.2, 4, 1, 100, 5, 2, -3.8.\}$, see Figure 23.

Case 3. If $AbC\Delta \neq 0$ and $a = 0$ with $\Delta = 4b(A+b) + C^2 > 0$, then $k = 3$ and $j = 2$ in system (21), the first integral of the quadratic center (6) is $H_3^{(2)}(x, y)$ given in (13). Then the solutions y_2 satisfying $F(y_2) = 0$ are equivalent to the solutions y_2 of the equation $\tilde{f}_3(y_2) = \tilde{g}_3(y_2)$ with

$$\tilde{f}_3(y_2) = f_1(y_2) \quad \text{and } \tilde{g}_3(y_2) = \tilde{f}_1(y_2),$$

where

$$\begin{aligned}
m_1 = &\ \frac{1}{2}\delta_1\left(C + \sqrt{4b(A+b)+C^2}\right) - b\delta_2 + 1,\ m_2 = b\gamma_2 - \frac{1}{2}\gamma_1\left(C + \sqrt{4b(A+b)+C^2}\right),\\
m_3 = &\ \frac{1}{2}\left(C + \sqrt{4b(A+b)+C^2}\right)\left(\frac{8\alpha\gamma_1\sigma_1}{4\beta^2+\omega^2} + \delta_1\right) - \frac{8\alpha b\gamma_2\sigma_1}{4\beta^2+\omega^2} - b\delta_2 + 1,\\
n_1 = &\ \frac{1}{2}\delta_1\left(C - \sqrt{4b(A+b)+C^2}\right) - b\delta_2 + 1,\ n_2 = b\gamma_2 - \frac{1}{2}\gamma_1\left(C - \sqrt{4b(A+b)+C^2}\right),\\
n_3 = &\ \frac{1}{2}\left(C - \sqrt{4b(A+b)+C^2}\right)\left(\frac{8\alpha\gamma_1\sigma_1}{4\beta^2+\omega^2} + \delta_1\right) - \frac{8\alpha b\gamma_2\sigma_1}{4\beta^2+\omega^2} - b\delta_2 + 1,\\
r_1 = &\ \frac{1}{2b}\Big(1 + \frac{C}{\sqrt{4b(A+b)+C^2}}\Big),\ r_2 = \frac{1}{2b}\Big(1 - \frac{C}{\sqrt{4b(A+b)+C^2}}\Big),\\
L_1 = &\ A\left(\frac{8\alpha\gamma_2\sigma_1}{4\beta^2+\omega^2} + \delta_2\right) + 1,\ L_2 = A\gamma_2,\ L_3 = A\delta_2 + 1,\ r = \frac{1}{A}.
\end{aligned}$$

The graphics of the function $\tilde{g}_3(y_2)$ are shown in Figures 4, 13, 14, 15, 16 and 17. For the function $\tilde{f}_3(y_2)$ all its graphics are given in Figure 4.

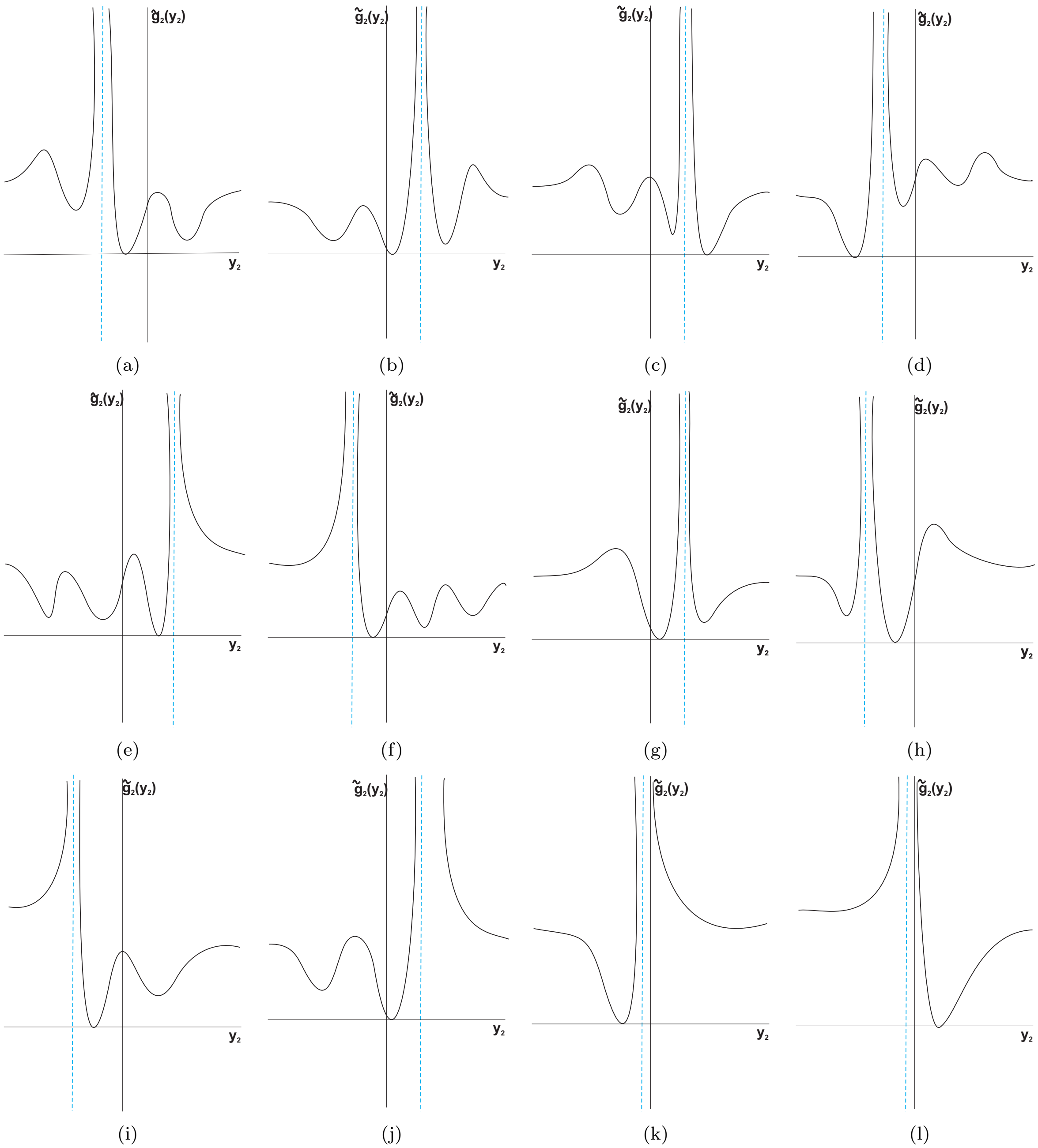


FIGURE 21. The graphic of the function $\tilde{g}_2(y_2)$.

It is obvious that the graphics of $\tilde{f}_3(y_2)$ can have at most four local extrem in Figure 13 and due to the fact that the function $\tilde{g}_3(y_2)$ is positive and its graphics can have at most one extremes in (a) or (b) of Figure 4, we guarantee that the maximum number of intersection points between the graphics of $\tilde{f}_3(y_2)$ and $\tilde{g}_3(y_2)$ take place between Figure 13 and (a) or (b) of Figure 4. In this case we know that the two functions have the common horizontal asymptote $\tilde{f}_3(y_2) = \tilde{g}_3(y_2) = 1$. Hence the maximum number of intersection points between these two functions is at most seven see for example Figure 24. Due to the

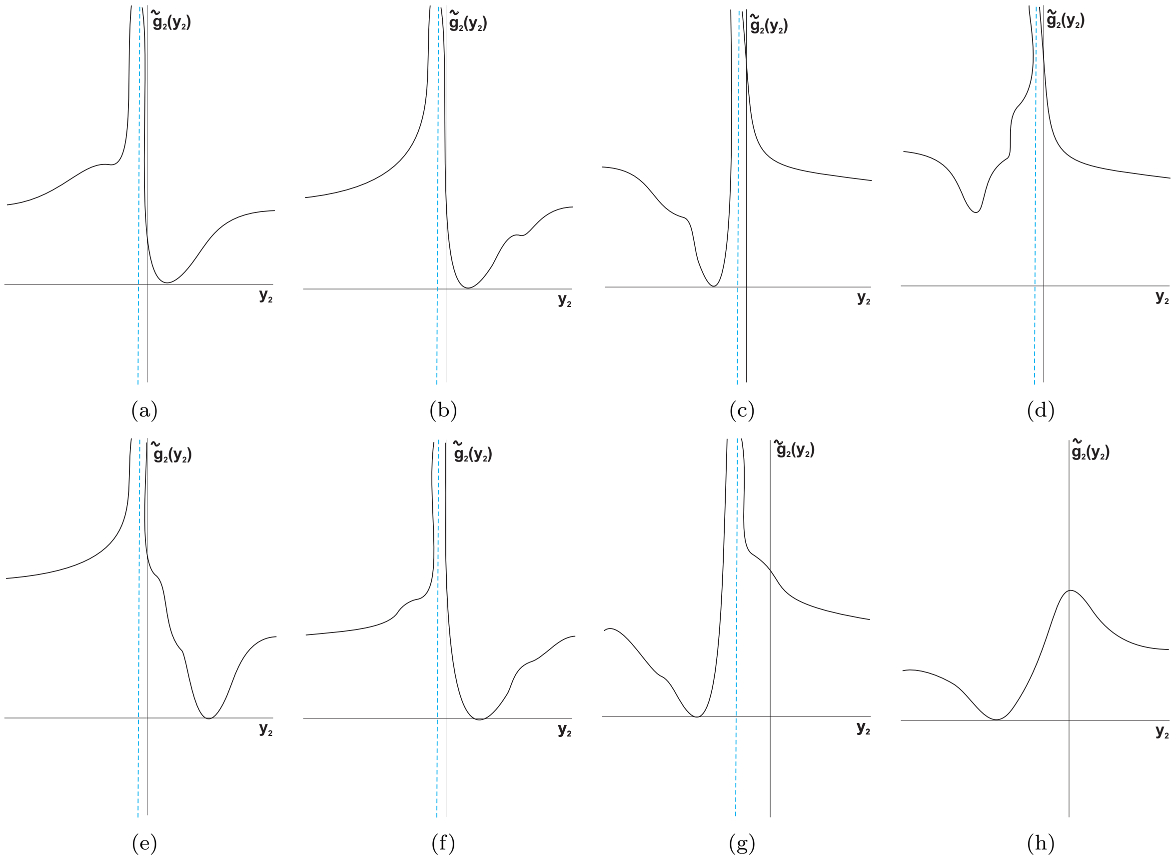


FIGURE 22. The graphics of the function $\tilde{g}_2(y_2)$.

symmetry of solutions of system (21) we conclude that the maximum number of limit cycles is at most three.

In Figure 24 we build an example that shows exactly seven points of intersection between the two functions $\tilde{f}_3(y_2)$ and $\tilde{g}_3(y_2)$ by choosing $\{m_1, m_2, m_3, r_1, r_2, n_1, n_2, n_3, L_1, L_2, L_3, r\} \to \{7, -1, 2, 2, 6, -0.53, -0.17, -1.71, 4, -3.25, -0.15, 2\}$.

Case 4. If $b = C = 0$ then $k = 4$ and $j = 2$ in system (21), (14) is the first integral of the quadratic center (6). Then the solutions of $F(y_2) = 0$ are the solutions of the equation $\tilde{f}_4(y_2) = \tilde{g}_4(y_2)$, where

$$\tilde{f}_4(y_2) = \left(\frac{m_0\ y_2^2 + m_1\ y_2 + m_2}{m_0\ y_2^2 + n_1\ y_2 + n_2} \right)^r$$

and

$$\tilde{g}_4(y_2) = e^{k_1 + k_2 y_2 + 2A\left(\tanh^{-1}\left(\frac{S_1 y_2 + S_2}{S_1 y_2 + S_4}\right) - \tanh^{-1}\left(\frac{S_5 - S_1 y_2}{S_6 - S_1 y_2}\right)\right)}.$$

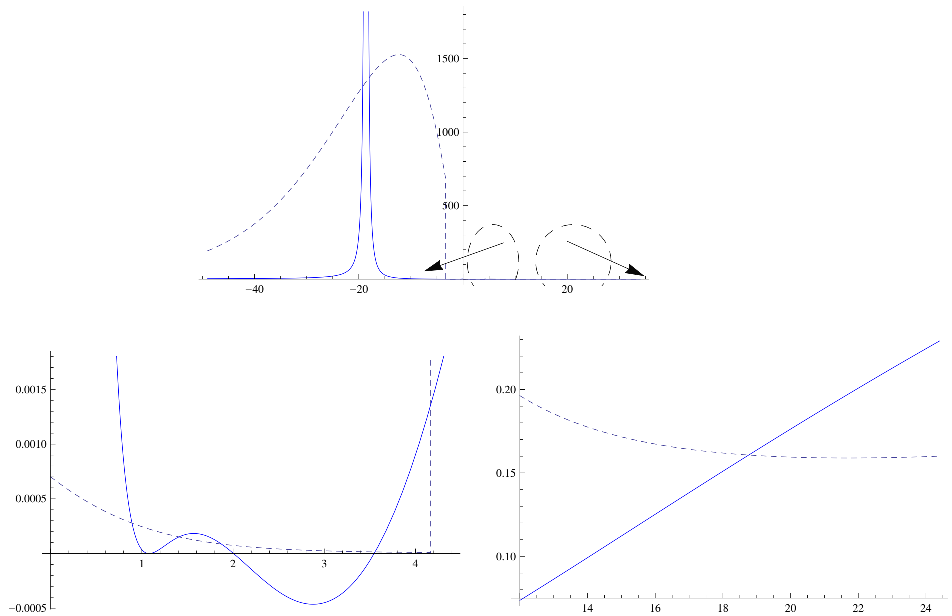


FIGURE 23. The seven intersection poitns between the functions $\tilde{f}_2(y_2)$ drawn in continuous line and $\tilde{g}_2(y_2)$ drown in dashed line.

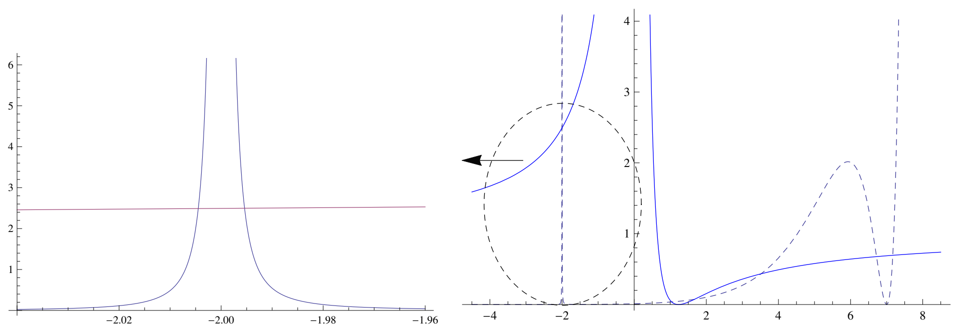


FIGURE 24. The seven intersection poitns between the functions $\tilde{f}_3(y_2)$ drawn in continuous line and $\tilde{g}_3(y_2)$ drown in dashed line.

Where

$$
\begin{aligned}
m_0 =&\quad a^2\gamma_1^2 - a^2\gamma_2^2 + aA\gamma_1\gamma_2,\ m_1 = 2a^2\gamma_1\delta_1 - 2a^2\gamma_2\delta_2 + aA\gamma_1\delta_2 + aA\gamma_2\delta_1 + 2a\gamma_1 + A\gamma_2,\\
m_2 =&\quad a^2\delta_1^2 - a^2\delta_2^2 + aA\delta_1\delta_2 + 2a\delta_1 + A\delta_2 + 1,\\
n_1 =&\quad -a(\gamma_1(2a\delta_1 + A\delta_2 + 2) + \gamma_2(A\delta_1 - 2a\delta_2)) - \frac{16a\alpha\sigma_1(a(\gamma_1-\gamma_2)(\gamma_1+\gamma_2) + A\gamma_1\gamma_2)}{4\beta^2+\omega^2} - A\gamma_2,\\
n_2 =&\quad a^2\delta_1^2 - a^2\delta_2^2 + \frac{1}{(4\beta^2+\omega^2)^2}(64a\alpha^2\sigma_1^2(a(\gamma_1-\gamma_2)(\gamma_1+\gamma_2) + A\gamma_1\gamma_2)) + \frac{1}{4\beta^2+\omega^2}(8\alpha\sigma_1((a\delta_1\\
&\quad +1)(2a\gamma_1 + A\gamma_2) + a\delta_2(A\gamma_1 - 2a\gamma_2))) + aA\delta_1\delta_2 + 2a\delta_1 + A\delta_2 + 1,
\end{aligned}
$$

$$
\begin{aligned}
r = &\quad -\sqrt{4a^2+A^2},\ k_1 = \frac{16a\beta^2\gamma_1\sqrt{4a^2+A^2}}{4\beta^2+\omega^2} + \frac{4a\gamma_1\omega^2\sqrt{4a^2+A^2}}{4\beta^2+\omega^2},\ k_2 = -\frac{16a\alpha\gamma_1\sigma_1\sqrt{4a^2+A^2}}{4\beta^2+\omega^2},\\
S_1 = &\quad aA\gamma_1 - 2a^2\gamma_2,\ S_2 = -2a^2\delta_2 + aA\delta_1 + A,\ S_3 = a\gamma_1\sqrt{4a^2+A^2},\\
S_4 = &\quad a\delta_1\sqrt{4a^2+A^2} + \sqrt{4a^2+A^2},\ S_5 = -\frac{16\alpha a^2\gamma_2\sigma_1}{4\beta^2+\omega^2} - 2a^2\delta_2 + \frac{8\alpha aA\gamma_1\sigma_1}{4\beta^2+\omega^2} + aA\delta_1 + A,\\
S_6 = &\quad \frac{8a\alpha\gamma_1\sqrt{4a^2+A^2}}{4\beta^2+\omega^2} + a\delta_1\sqrt{4a^2+A^2} + \sqrt{4a^2+A^2}.
\end{aligned}
$$

We see that the discriminants of the numerator and the denominator of $\tilde{f}_4(y_2)$ are equal, and it is $\Delta = \left(4a^2+A^2\right)(a\gamma_1\delta_2 - a\gamma_2\delta_2 - \gamma_2)^2$. Since $\Delta \geq 0$ we obtain that all the possible topologically different graphics $(C_{\tilde{f}_4})$ of the function $\tilde{f}_4(y_2)$ are given in Figures 4, 13, 14, 15, 16 and 17 as we proved in the first case of statements (I) and (II).

Now we study the function $\tilde{g}_4(y_2)$ where its derivative is

$$
(\tilde{g}_4)'(y_2) = \tilde{g}_4(y_2)\frac{P_1(y_2)}{P_2(y_2)},
$$

with

$$
\begin{aligned}
P_1(y_2) = &\quad -2AS_1^3S_4y_2^2 - 2AS_1^3S_6y_2^2 + 2AS_1^2S_2S_3y_2^2 - 4AS_1^2S_2S_6y_2 + 2AS_1^2S_3S_5y_2^2 + 4AS_1^2S_4S_5y_2\\
&-2AS_1S_2^2S_6 + 2AS_1S_3^2S_4y_2^2 + 2AS_1S_3^2S_6y_2^2 + 2AS_1S_4^2S_6 - 2AS_1S_4S_5^2 + 2AS_1S_4S_6^2\\
&+2AS_2^2S_3S_5 - 2AS_2S_3^3y_2^2AS_2S_3^2S_6y_2 + 2AS_2S_3S_5^2 - 2AS_2S_3S_6^2 - 2AS_3^3S_5y_2^2 - 4AS_3^2S_4S_5y_2\\
&-2AS_3S_4^2S_5 + k_2S_1^4y_2^4 + 2k_2S_1^3S_2y_2^3 - 2k_2S_1^3S_5y_2^3 + k_2S_1^2S_2^2y_2^2 - 4k_2S_1^2S_2S_5y_2^2 - 2k_2S_1^2S_3^2y_2^4\\
&-2k_2S_1^2S_3S_4y_2^3 + 2k_2S_1^2S_3S_6y_2^3 - k_2S_1^2S_4^2y_2^2 + k_2S_1^2S_5^2y_2^2 - k_2S_1^2S_6^2y_2^2 - 2k_2S_1S_2^2S_5y_2\\
&-2k_2S_1S_2S_3^2y_2^3 + 4k_2S_1S_2S_3S_6y_2^2 + 2k_2S_1S_2S_5^2y_2 - 2k_2S_1S_2S_6^2y_2 + 2k_2S_1S_3^2S_5y_2^3\\
&+4k_2S_1S_3S_4S_5y_2^2 + 2k_2S_1S_4^2S_5y_2 - k_2S_2^2S_3^2y_2^2 + 2k_2S_2^2S_3S_6y_2 + k_2S_2^2S_5^2 - k_2S_2^2S_6^2 + k_2S_3^4y_2^4\\
&+2k_2S_3^3S_4y_2^3 - 2k_2S_3^3S_6y_2^3 + k_2S_3^2S_4^2y_2^2 - 4k_2S_3^2S_4S_6y_2^2 - k_2S_3^2S_5^2y_2^2 + k_2S_3^2S_6^2y_2^2 + k_2S_4^2S_6^2\\
&-2k_2S_3S_4^2S_6y_2 - 2k_2S_3S_4S_5^2y_2 + 2k_2S_3S_4S_6^2y_2 - k_2S_4^2S_5^2,
\end{aligned}
$$

and

$$
P_2(y_2) = (y_2(S_1-S_3) + S_2 - S_4)(y_2(S_1+S_3) + S_2 + S_4)(y_2(S_3-S_1) + S_5 - S_6)(-y_2(S_1+S_3) + S_5 + S_6).
$$

According with the sign of $(\tilde{g}_4)'(y_2)$ and the kind of roots of the quartic polynomial $P_1(y_2)$ and with their position with respect to the two vertical asymptotes $y_{21} = \dfrac{S_5+S_6}{S_1+S_3}$ and $y_{22} = \dfrac{S_4-S_2}{S_1-S_3}$, we give all the possible topologically different graphics of the function $\tilde{g}_4(y_2)$ in what follows.

If $P_1(y_2)$ has four simple real roots the graphics of $\tilde{g}_4(y_2)$ are given in (a), or (b), or (c), or (d), or (e), or (f), or (g), or (h), or (i) of Figure 25.

If $P_1(y_2)$ has two simple real roots and two complex roots the graphics of $\tilde{g}_4(y_2)$ are given in (j), or (k), or (l) of Figure 25, or in (a) of Figure 26.

If $P_1(y_2)$ has four complex roots the unique graphic of $\tilde{g}_4(y_2)$ is shown in (b) of Figure 26.

If $P_1(y_2)$ has one triple and one simple real root the unique graphic of $\tilde{g}_4(y_2)$ is shown in (c) of Figure 26.

If $P_1(y_2)$ has one double real root and two complex roots the unique graphic of $\tilde{g}_4(y_2)$ is shown in (d) of Figure 26.

If $P_1(y_2)$ has one double and two simple real roots the graphics of $\tilde{g}_4(y_2)$ are shown in (e), or (f), or (g) of Figure 26.

If $P_1(y_2)$ has one real root of order four or two double real roots the unique graphic of $\tilde{g}_4(y_2)$ is shown in (h) of Figure 26.

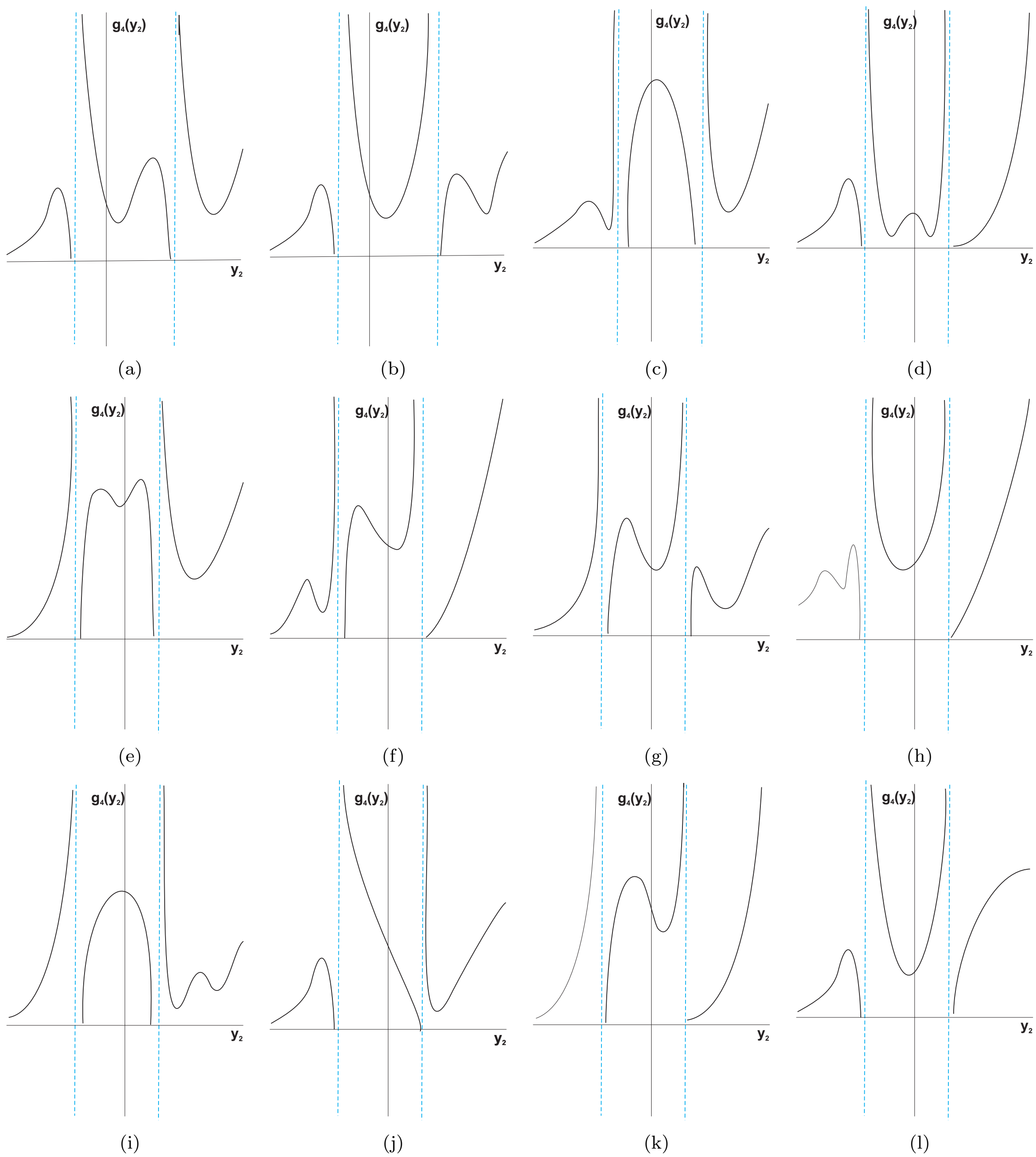


FIGURE 25. The graphics of the function $\tilde{g}_4(y_2)$.

Since the graphics of $\tilde{f}_4(y_2)$ can have at most four local extremes in Figure 13 and by knowing that the function $\tilde{g}_4(y_2)$ is positive and its graphics can have at most four local extremes in (a), or (b), or (c),

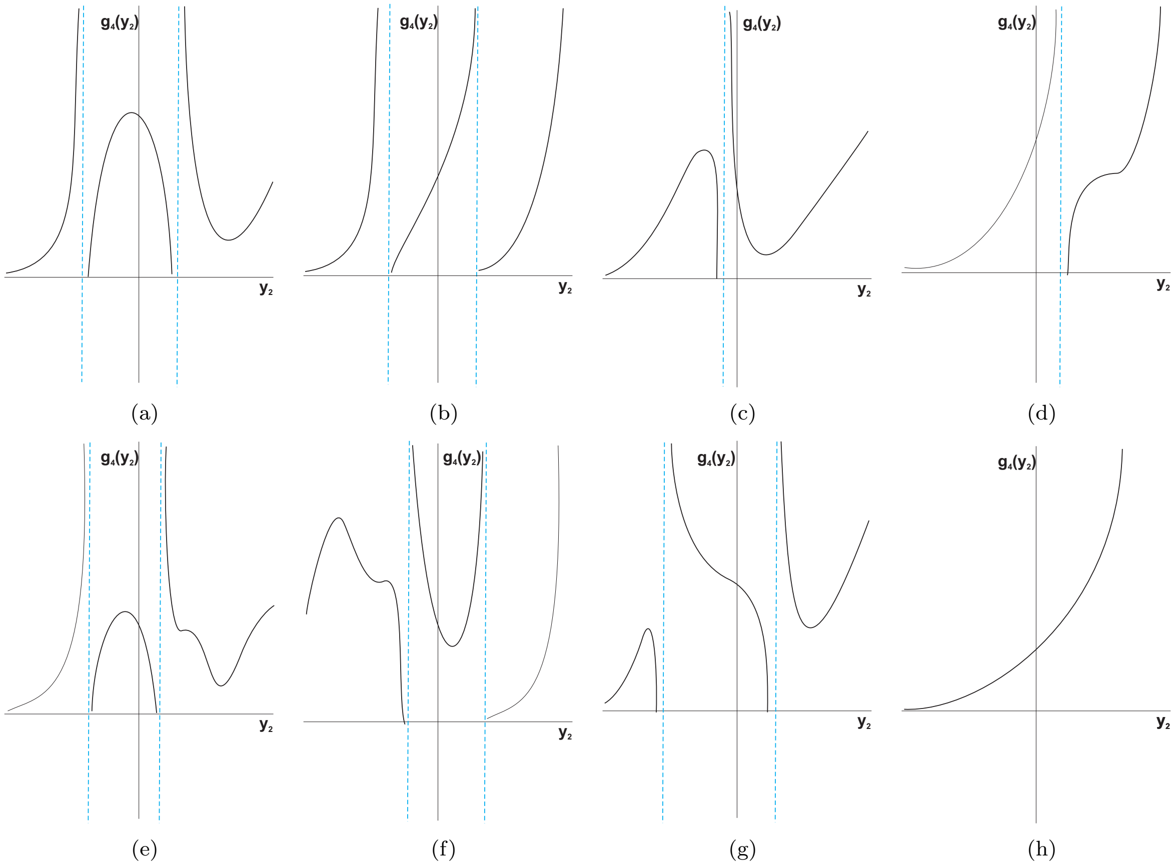


FIGURE 26. The graphics of the function $\tilde{g}_4(y_2)$.

or (d), or (e), or (f), or (g), or (f) of Figure 25, therefore we conclude that the maximum number of intersection points between the graphics of $\tilde{f}_4(y_2)$ and $\tilde{g}_4(y_2)$ can be precisely between Figure 13 and (a), or (b), or (c), or (d), or (e), or (f), or (g), or (h) of Figure 25. It results that the maximum number of intersection points between the functions $\tilde{f}_4(y_2)$ and $\tilde{g}_4(y_2)$ is at most seven see for example Figure 27. Due to symmetry of solutions (y_1, y_2) of (21), we know that the maximum number of limit cycles of the discontinous piecewise differential system (3)–(6) is at most three.

In figure 27 we build an example that shows exactly seven intersection points between the functions $\tilde{f}_4(y_2)$ and $\tilde{g}_4(y_2)$ by considering $\{m_0, m_1, m_2, n_1, n_2, r, k_1, k_2, A, S_1, S_2, S_3, S_4, S_5, S_6\} \to \{1, -1.23283, 0.265293, -1.65, 6.63, 4, 5, -4.5, 4, 4, -0.5, 0.5, 1, 0.5, 9.4\}$

Case 5. If $A = a = 0$, $C \neq 0$ and $b \neq 0$, then $k = 5$ and $j = 2$ in system (21), (6) has the first integral $H_5^{(2)}(x, y)$ given in (15), studying the solutions of $F(y_2) = 0$ is equivalent to study the solutions of the equation $\tilde{f}_5(y_2) = \tilde{g}_5(y_2)$ where

$$\tilde{f}_5(y_2) = f_2(y_2) \quad \text{and } \tilde{g}_5(y_2) = \tilde{f}_1(y_2),$$

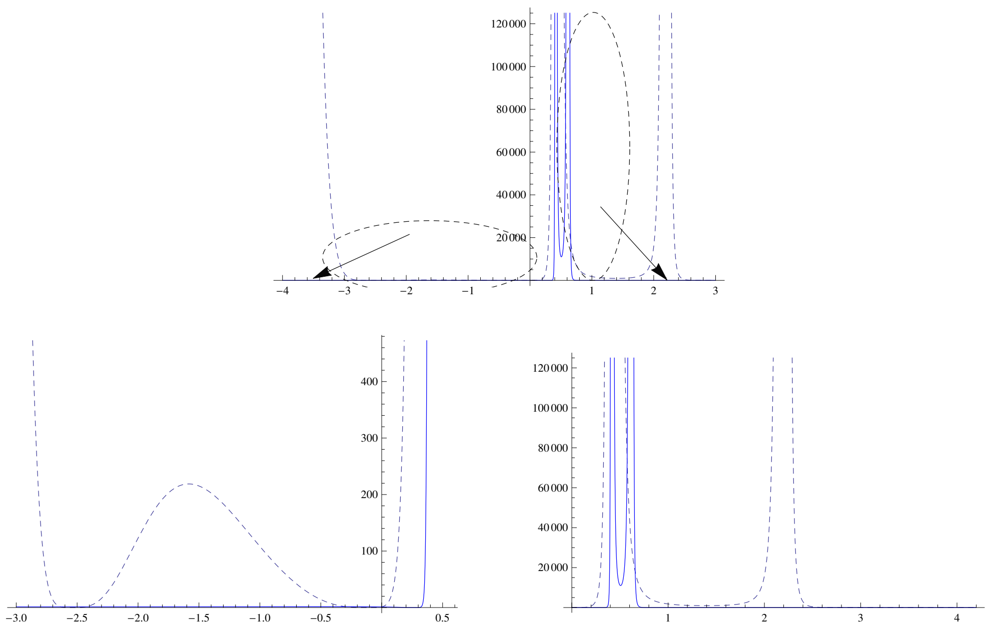


FIGURE 27. The seven intersection poitns between the functions $\tilde{f}_4(y_2)$ drawn in continuous line and $\tilde{g}_4(y_2)$ drawn in dashed line.

where

$$
\begin{aligned}
m_1 = &\quad \frac{1}{2}\delta_1\left(C+\sqrt{4b^2+C^2}\right)-b\delta_2+1,\ m_2=\frac{1}{2}\gamma_1\left(C+\sqrt{4b^2+C^2}\right)-b\gamma_2,\\
m_3 = &\quad \frac{4\alpha\gamma_1\sigma_1}{4\beta^2+\omega^2}\left(C+\sqrt{4b^2+C^2}\right)+\frac{1}{2}\delta_1\left(C+\sqrt{4b^2+C^2}\right)-\frac{8\alpha b\gamma_2\sigma_1}{4\beta^2+\omega^2}-b\delta_2+1,\\
r_1 = &\quad \frac{1}{2b}\Big(1+\frac{C}{\sqrt{4b^2+C^2}}\Big),\ r_2=\frac{1}{2b}\Big(1-\frac{C}{\sqrt{4b^2+C^2}}\Big),\\
n_1 = &\quad \frac{1}{2}\delta_1\left(C-\sqrt{4b^2+C^2}\right)-b\delta_2+1,\ n_2=\frac{1}{2}\gamma_1\left(C-\sqrt{4b^2+C^2}\right)-b\gamma_2,\\
n_3 = &\quad \frac{4\alpha\gamma_1\sigma_1}{4\beta^2+\omega^2}\left(C-\sqrt{4b^2+C^2}\right)+\frac{1}{2}\delta_1\left(C-\sqrt{4b^2+C^2}\right)-\frac{8\alpha b\gamma_2\sigma_1}{4\beta^2+\omega^2}-b\delta_2+1,
\end{aligned}
$$

$$
l_0 = \quad -2\gamma_2,\ l_1=\frac{8\alpha\gamma_2\sigma_1}{4\beta^2+\omega^2}.
$$

We know that the graphics of $\tilde{f}_5(y_2)$ are shown in Figure 10, and the graphics of $\tilde{g}_5(y_2)$ are shown in Figures 4, 13, 14, 15, 16 and 17.

As in the previous case we ensure that the graphics $(C_{\tilde{g}_5})$ shown in Figure 13 are the ones that guarentee the maximum number of intersections points between the graphics of the functions $g_5(y_2)$ and $f_5(y_2)$ which has the horizontal asymptote $\tilde{f}_5(y_2)=0$. Then we guarantee that the maximum number of intersection points between the graphics (C_{f_5}) and (C_{g_5}) takes place between Figure 10 and Figure 13. Thus the maximum number of the intersection points of these graphics is at most seven, which provide at most three limit cycles of the discontinuous piecewise differential system (3)–(6).

In what follows we build an example provides the seven intersection points between the tow functions $\tilde{f}_5(y_2)$ and $\tilde{g}_5(y_2)$ when we consider $\{m_1,m_2,m_3,n_1,n_2,n_3,r_1,r_2,l_0,l_1\}\to\{5,-1.3,3,-0.5,-0.1703,$ $-1.1,-2,-4,2.19,-0.18\}$, these points are shown in Figure 28.

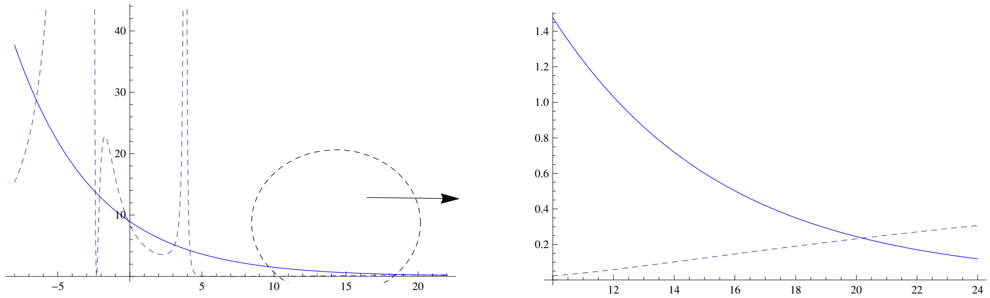


FIGURE 28. The seven intersection poitns between graphics of the functions $\tilde{f}_5(y_2)$ drawn in continuous line and $\tilde{g}_5(y_2)$ drawn in dashed line.

Case 6. If $b = a = 0$, $C \neq 0$ and $A \neq 0$, then $k = 6$ and $j = 2$ in system (21), the first integral of (6) is $H_6^{(2)}(x, y)$ given in (16). The equation $F(y_2) = 0$ is equivalent to the equation $\tilde{f}_6(y_2) = \tilde{g}_6(y_2)$ where

$$\tilde{f}_6(y_2) = f_2(y_2) \quad \text{and } \tilde{g}_6(y_2) = \tilde{f}_1(y_2),$$

where

$$\begin{aligned} l_0 = & -\frac{16\alpha AC\sigma_1}{4\beta^2+\omega^2}(A\gamma_1+\gamma_2 C),\ l_1 = 4AC(A\gamma_1+\gamma_2 C), \\ m_1 = & \ A\delta_2+1,\ m_2 = A\gamma_2,\ m_3 = \frac{8\alpha A\gamma_2\sigma_1}{4\beta^2+\omega^2}+A\delta_2+1,\ r_1 = 2C^2, \\ n_1 = & \ C\delta_1+1,\ n_2 = \gamma_1 C,\ n_3 = \frac{8\alpha\gamma_1 C\sigma_1}{4\beta^2+\omega^2}+C\delta_1+1,\ r_2 = 2A^2. \end{aligned}$$

The graphics of $\tilde{f}_6(y_2)$ are given in Figure 10 and the graphics of $\tilde{g}_6(y_2)$ are given in Figures 4, 13, 14, 15, 16 and 17. Then the maximum number of solutions of system (21) is at most seven which provides at most three limit cycles of the discontinuous piecewise differential system (3)–(6).

Since the maximum number of limit cycles of all these six cases is at most three, we will build only an example with three limit cycles of the discontinuous piecewise differential system (3)–(6) of type $b+d = 0$ with $A = a = 0$, $C \neq 0$ and $b \neq 0$.

In the half-plane Σ^+ we consider the quadratic center

$$\begin{aligned} \dot{x} = & -0.0273949..x^2 + x(0.0133241..y + 3.85736..) + (-0.00150606.y - 0.845815..)y + 3.3482.., \\ \dot{y} = & -0.273949..x^2 + x(0.133241..y + 18.5736..) + (-0.0150606..y - 3.7726..)y + 25.7332.., \end{aligned} \tag{26}$$

with its corresponding first integral

$$H_6^{(2)}(x, y) = \ -0.0000384239..e^{\frac{1}{200}(y-10x)}(x - 0.307716..y - 34.9503..)(x - 0.178658..y + 27.1508..)^2.$$

In the half-plane Σ^- we consider the linear differential center

$$\dot{x} = \frac{1}{5}x - \frac{29}{120}y + \frac{19}{20}, \quad \dot{y} = \frac{6}{5}x - \frac{1}{5}y + \frac{11}{5}, \tag{27}$$

with the first integral

$$H(x, y) = 4\left(\frac{6}{5}x - \frac{1}{5}y\right)^2 + \frac{48}{5}\left(\frac{11}{5}x - \frac{19}{20}y\right) + y^2.$$

In this case system (21) has the three solutions $(y_1, y_2) = (0.592968..., 7.2691..)$, $(y_3, y_4) = (1.31716..., 6.54491...)$ and $(y_5, y_6) = (2.34295..., 5.51911...)$ which provide the three limit cycles for the discontinuous piecewise differential system (26)–(27), see Figure 2(b).

Case 7. If $a = 0 \neq C$ and $\Delta = 4b(A+b)+C^2 = 0$, then $k=7$ and $j=2$ in system (21), (17) is the first integral of the quadratic center (6). Then the solutions of $F(y_2)=0$ are the same as the solutions of the equation $\tilde{f}_7(y_2) = \tilde{g}_7(y_2)$ where

$$\tilde{f}_7(y_2) = \tilde{f}_1(y_2) \quad \text{and } \tilde{g}_7(y_2) = \tilde{g}_1(y_2),$$

where

$$\begin{aligned} m_1 =& \ \frac{8\alpha\sigma_1}{4\beta^2+\omega^2}(\gamma_1 C - 2b\gamma_2) - 2b\delta_2 + C\delta_2 + 2,\ m_2 = 2b\gamma_2 - \gamma_1 C,\ m_3 = -2b\delta_2 + C\delta_2 + 2, \\ n_1 =& \ 1 - \frac{\left(4b^2+C^2\right)}{4b\left(4\beta^2+\omega^2\right)}\left(\delta_2\left(4\beta^2+\omega^2\right) + 8\alpha\gamma_2\sigma_1\right),\ n_2 = \frac{\gamma_2}{4b}\left(4b^2+C^2\right),\ n_3 = 1 - \frac{\delta_2}{4b}\left(4b^2+C^2\right), \\ r_1 =& \ 1,\ r_2 = -\frac{4b^2}{4b^2+C^2}, \\ k_0 =& \ -\frac{16\alpha C\sigma_1}{4\beta^2+\omega^2}(b\gamma_1\delta_2 - b\gamma_2\delta_2 - \gamma_1),\ k_1 = 4C(b\gamma_1\delta_2 - b\gamma_2\delta_2 - \gamma_1). \end{aligned}$$

Since $r_1 = 1$, the graphics of the function $\tilde{f}_7(y_2)$ are given in (h), (i), (j), (k) and (l) of Figure 14 and in Figures 4, 15, 16 and 17. All the graphics of the function $\tilde{g}_7(y_2)$ are shown in Figure 18.

Therefore the graphics of the two functions $\tilde{f}_7(y_2)$ and $\tilde{g}_7(y_2)$ intersect at most in five points see for example Figure 29. Consequently, the maximum number of limit cycles of the discontinuous piecewise differential system (3)–(6) is at most two.

Now we construct an example with exactly five intersection points between $\tilde{f}_7(y_2)$ and $\tilde{g}_7(y_2)$ by taking $\{m_1, m_2, m_3, r_1, r_2, n_1, n_2, n_3, k_0, k_1\} \to \{-0.5, 7.2, -2, 1, -2, 0.12, -9.5, -0.3, 1.2, 0\}$. These points are shown in Figure 29.

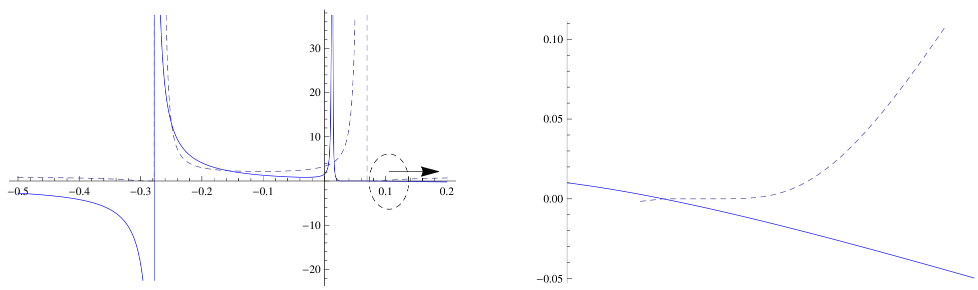


FIGURE 29. The five intersection poitns between the functions $\tilde{f}_7(y_2)$ drawn in continuous line and $\tilde{g}_7(y_2)$ drawn in dashed line.

Case 8. If $A = b = 0$ and $a = 0 \neq C$, then $k=8$ and $j=2$ in system (21), the first integral of (6) is $H_8^{(2)}(x,y)$ given in (18), and the solutions of $F(y_2)=0$ are the same as the solutions of the equation $\tilde{f}_8(y_2) = \tilde{g}_8(y_2)$ where

$$\tilde{f}_8(y_2) = M^2 f_2(y_2) \quad \text{and } \tilde{g}_8(y_2) = f_1(y_2), \quad \text{with } \ M = \frac{1}{4\beta^2+\omega^2},$$

and

$$\begin{aligned} l_0 =& \ -M^2(16\alpha C\sigma_1)\left(\left(4\beta^2+\omega^2\right)(\gamma_1 + \gamma_2 C\delta_2) + 4\alpha\gamma_2^2 C\sigma_1\right), \\ l_1 =& \ M(4C\left(\left(4\beta^2+\omega^2\right)(\gamma_1 + \gamma_2 C\delta_2) + 4\alpha\gamma_2^2 C\sigma_1\right)), \\ L_1 =& \ C\delta_1 + 1,\ L_2 = \gamma_1 C,\ L_3 = C\left(\frac{8\alpha\gamma_1\sigma_1}{4\beta^2+\omega^2} + \delta_1\right) + 1,\ r = 2. \end{aligned}$$

The graphics of $\tilde{f}_8(y_2)$ are given in Figure 10. Since $\tilde{g}_8(y_2)$ is a sub-case of $f_1(y_2)$ with the particular parameters given previously, then its graphics are shown in Figures 4(a) and 4(b). Clearly that the maximum number of intersection points of their corresponding graphics is at most three see for example Figure 30. Then the upper bound of the number of limit cycles in this case is at most one.

In what follows we consider $\{l_0, l_1, L_1, L_2, L_3, r, M\} \to \{-0.2, -1.75, 1, -5.6, -0.4, 2, 1.53\}$ for building an example with exactly three intersection points between the two functions $\tilde{f}_8(y_2)$ and $\tilde{g}_8(y_2)$ see Figure 30.

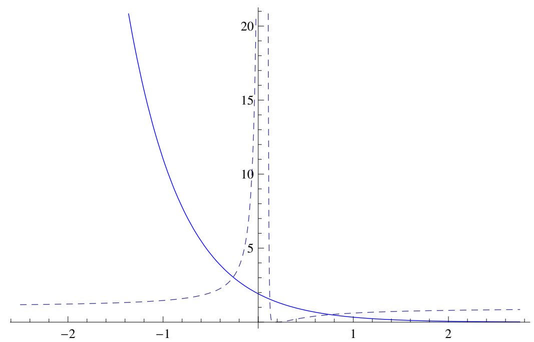


FIGURE 30. The three intersection poitns between the functions $\tilde{f}_8(y_2)$ drawn in continuous line and $\tilde{g}_8(y_2)$ drawn in dashed line.

Now we will prove that the result of case 8 is reached by giving an example of system (3)–(6) of type $b + d = 0$ with $A = b = 0$ and $a = 0 \neq C$.

In the half-plane Σ^+ we consider the quadratic center

$$(28) \qquad \begin{aligned} \dot{x} = & \ \frac{1}{770}\left(-24x^2 + x(1537 - 2490y) - 20(5y(90y - 151) + 241)\right), \\ \dot{y} = & \ \frac{3320}{3850}\Big(xy + x(32x - 1921) + 12000y^2 - 7300y + 10\Big), \end{aligned}$$

this system has the first integral

$$H_8^{(2)}(x, y) = \ e^{-(1/100)(8x+30y+1)^2 + ((1/5)x+20y-10}(x + 100y - 60)^2.$$

In the half-plane Σ^- we consider the linear differential center

$$(29) \qquad \dot{x} = (1/2)x - 1.57968..y + 2, \quad \dot{y} = 2x - (1/2)y + 1/2,$$

with the first integral

$$H(x, y) = x((1/2) \ - (1/2)y) + x^2 + (0.789842..y - 2)y.$$

In this case system (21) has the unique solution $(y_1, y_2) = (0.894524.., 1.63763..)$ which provides the unique limit cycle for the discontinuous piecewise differential system (28)–(29), see Figure 2(b). This example completes the proof of statement (II). □

Proof of statement (III) *of Theorem 3.* In this statement $F(y_2) = 0$ is the following cubic equation in the variable y_2

$$F(y_2) = \frac{1}{(4\beta^2+\omega^2)^3}\Big(2\left(y_2\left(4\beta^2+\omega^2\right)-4\alpha\sigma_1\right)\Big(64\alpha^2\sigma_1^2\left(a\left(\gamma_1^3-3\gamma_1\gamma_2^2\right)+3b\gamma_1^2\gamma_2+\gamma_2^3 d\right)-4\alpha\sigma_1$$
$$\Big(4\beta^2+\omega^2\Big)\Big(\gamma_1^2(-6a\sigma_1-6b\delta_2+6b\gamma_2y_2-3)-6\gamma_1\gamma_2(-2a\delta_2+a\gamma_2y_2+2b\delta_1)$$
$$+\gamma_2^2(6a\delta_1-6d\delta_2+2\gamma_2dy_2-3)+2a\gamma_1^3y_2\Big)+\left(4\beta^2+\omega^2\right)^2\Big(a(3\gamma_1(\delta_1-\delta_2)(\delta_1$$
$$+\delta_2)-6\gamma_2\delta_1\delta_2+y_2^2\Big(\gamma_1^3-3\gamma_1\gamma_2^2\Big)\Big)+3b(\delta_1(2\gamma_1\delta_2+\gamma_2\delta_1)+\gamma_1^2\gamma_2y_2^2)+3\gamma_1\delta_1$$
$$+3\gamma_2\delta_2+3\gamma_2 d\delta_2^2+\gamma_2^3dy_2^2\Big)\Big)\Big).$$

Therefore this equation has at most three real solutions. Eventually the planar discontinuous piecewise differential system (3)–(6) has at most one limit cycle.

To confirm we present in what follows a discontinuous piecewise differential systems with exactly one limit cycle. In the half-plane Σ^- we consider the quadratic center

$$\begin{aligned}\dot{x} &= \frac{1}{3400}\Big(-262(174x+19)y+20(x(841x+1963)-3198)+24185y^2\Big),\\ \dot{y} &= \frac{1}{1700}\Big(6500x^2+20x(665-841y)+y(11397y-19630)+9000\Big),\end{aligned} \tag{30}$$

its first integral is

$$H_1^{(3)}(x,y) = -393(174x+19)y^2+60(x(841x+1963)-3198)y+(x(x(130x+399)+540)-2869)$$
$$+24185y^3.$$

In the half-plane Σ^+ we consider the linear differential center

$$\dot{x} = x+0.618871..y-1, \quad \dot{y} = -2.01981..x-y+1.3, \tag{31}$$

with the first integral

$$H(x,y) = 16.3185..x^2+x(16.1584..y-21.006..)+5(y-3.23169..)y.$$

In this case system (21) has the unique solution $(y_1,y_2) = (0.567633..,2.66406..)$ which provides the unique limit cycle for the discontinuous piecewise differential system (30)–(31), see Figure 3(a). This example completes the proof of statement (III). □

Proof of statement (IV) of Theorem 3. In this statement the solutions of $F(y_2)=0$ are equivalent to the solutions of an equation of degree nine and due to the big expression of this equation we omit it. This equation has at most nine real solutions which provide at most four limit cycles for the discontinuous piecewise differential system (3)–(6).

In what follows we give a discontinuous piecewise differential system of the class (3)–(6) of type (IV) with four limit cycles. In the half-plane Σ^- we consider the quadratic center

$$\begin{aligned}\dot{x} &= x(525.153..-2.477..y)-0.0005176..x^2+y(5365.y-850504.)+66467.,\\ \dot{y} &= x(0.0686512..-0.0004798..y)+y(104.576..-0.024..y)-1.924..\cdot 10^{-8}x^2-13667.3..,\end{aligned} \tag{32}$$

with its first integral

$$H_1^{(4)}(x,y) = (0.990099(1.x^3+x^2(3.62455\cdot 10^6-33461.5y)+x(y(3.73225\cdot 10^8y-8.09237$$
$$\cdot 10^{10})+4.38777\cdot 10^{12})+y((4.51686\cdot 10^{14}-1.38763\cdot 10^{12}y)y-4.90232\cdot 10^{16})$$
$$+1.77406\cdot 10^{18})^2)/((1.x^2+x(2.41637\cdot 10^6-22307.7y)+y(1.24408\cdot 10^8y-10^{10}$$
$$\cdot 2.69973)+1.46548\cdot 10^{12})^3).$$

In the half-plane Σ^+ we consider the linear differential center

$$\dot{x} = -8x-(25601/40)y+50, \quad \dot{y} = (1/10)x+8y+20, \tag{33}$$

with the first integral

$$H(x,y) = 4(x+80y)^2+800(2x-5y)+y^2.$$

In this case system (21) has the four solutions $(y_1, y_2) = (0.00817805.., 0.148066..)$, $(y_3, y_4) = (0.0177713.., 0.138473..)$, $(y_5, y_6) = (0.0292114.., 0.127033..)$ and $(y_7, y_8) = (0.0443241.., 0.11192..)$ which provide the four limit cycles for the discontinuous piecewise differential system (32)–(33), see Figure 3(b). This example completes the proof of statement (IV). □

Acknowledgements

The third author is partially supported by the Agencia Estatal de Investigación grant PID2019-104658GB-I00, and the H2020 European Research Council grant MSCA-RISE-2017-777911.

References

[1] A. Andronov, A. Vitt, S. Khaikin, Theory of oscillations, Pergamon Press, Oxford, 1996.
[2] J.C. Artés, J. Llibre, J.C. Medrado and M.A.Teixeira, Piecewise linear differential systems with two real saddles. Math.Comput. Simul. **95**, 13–22 (2013).
[3] N. N. Bautin, On the number of limit cycles that appear with the variation of the coefficients from an equilibrium position of focus or center type, American Math. Soc. **1954** 19 pp, (1954).
[4] R. Benterki, L. Damene and J. Llibre, The limit cycles of discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubic curves II. Differ. Equ. Dyn. Syst. (2021).
[5] R. Benterki, J. Jimenez and J. Llibre. Limit cycles of planar discontinuous piecewise linear Hamiltonian systems without equilibria separated by reducible cubics. Electron. J. Qual. Theory Differ. Equ. 2021, Paper No. 69, 38 p. (2021).
[6] R. Benterki and J. Llibre, The limit cycles of discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubic curves I. Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis. **28** , 153-192 (2021).
[7] M. Bernardo, C.J. Budd, A.R. Champneys and P. Kowalczyk, Piecewise-Smooth Dynamical Systems: Theory and Applications. Applied Mathematical Sciences. Springer-Verlag, London. **163**, (2008).
[8] D.C. Braga and L.F. Mello, Limit cycles in a family of discontinuous piecewise linear differential systems with two zones in the plane. Nonlinear Dyn. **73**, 1283–1288 (2013).
[9] J. Castillo, J. Llibre, F. Verduzco, The pseudo-Hopf bifurcation for planar discontinuous piecewise linear differential systems. Nonlinear Dyn. **90**, 1829–1840 (2017).
[10] F. Dumortier, J. Llibre and J. C. Artés, Qualitative Theory of Planar Differential Systems, UniversiText, Springer-Verlag, New York, (2006).
[11] R.D. Euzébio and J. Llibre, On the number of limit cycles in discontinuous piecewise linear differential systems with two pieces separated by a straight line J. Math. Anal. Appl. **424**, 475–486 (2015).
[12] A. F. Filippov, *Differential Equations with Discontinuous Righthand Side*, Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, (1988).
[13] E. Freire, E. Ponce, F. Rodrigo and F. Torres, Bifurcation sets of continuous piecewise linear systems with two zones. Int.J. Bifurc. Chaos **8**, 2073–2097 (1998).
[14] D. Hilbert, Problems in Mathematics. Bull. (New Series) Amer. Math. Soc. **37** (2000), 407–436 Sci.Eng. **13**, 47–106 (2003).
[15] Yu. Ilyashenko , Centennial history of Hilbert's 16 th problem. Bull. (New Series) Am. Math. Soc. **39**, 301–354 (2002).
[16] W. Kapteyn, On the midpoints of integral curves of differential equations of the first degree, Nederl. Akad. Wetensch. Verslag. Afd. Natuurk. Konikl. Nederland, (Dutch), 1446–1457, (1911).
[17] W. Kapteyn, New investigations on the midpoints of integrals of differential equations of the first degree, Nederl. Akad. Wetensch. Verslag Afd. Natuurk, **20** 1354–1365, (1912).
[18] J. Li, Hilbert's 16 th problem and bifurcations of planar polynomial vector fields. Int. J. Bifurc. Chaos Appl Sci.Eng. **13**, 47–106 (2003).
[19] J. Llibre, Limit cycles of planar continuous piecewise differential systems separated by a parabola and formed by an arbitrary linear and quadratic centers. Continuous discret dynamical systems-Series S, doi:10.3934/dcdss.2022034.
[20] J. Llibre, M. A. Teixeira, Piecewise linear differential systems with only centers can create limit cycles?. Nonlinear Dynam. **91**, 249–255 (2018).
[21] J. Llibre, x. Zhang, Limit cycles for discontinuous planar piecewise linear differential systems separated by one straight line and having a center. J. Math. Anal. Appl. **467**, 537-–549 (2018).
[22] O. Makarenkov, and J.S.W. Lamb, Dynamics and bifurcations of nonsmooth systems: a survey. Phys. D. **241**, 1826-–1844 (2012).
[23] D.J.W. Simpson, Bifurcations in piecewise-smooth continuous systems. World Scientific Series on Nonlinear Science A. World Scientific, Singapore vol. **69** (2010).

[1] Mathematical Analysis and Applications Laboratory, Department of Mathematics, University Mohamed El Bachir El Ibrahimi of Bordj Bou Arréridj 34000, El Anasser, Algeria

Email address: imane.benabdallah@univ-bba.dz and r.benterki@univ-bba.dz

[2] Departament de Matematiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Email address: jllibre@mat.uab.cat