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The Solution of the Extended 16th Hilbert Problem for Some Classes of Piecewise Differential Systems

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Abstract: The limit cycles have a main role in understanding the dynamics of planar differential systems, but their study is generally challenging. In the last few years, there has been a growing interest in researching the limit cycles of certain classes of piecewise differential systems due to their wide uses in modeling many natural phenomena. In this paper, we provide the upper bounds for the maximum number of crossing limit cycles of certain classes of discontinuous piecewise differential systems (simply PDS) separated by a straight line and consequently formed by two differential systems. A linear plus cubic polynomial forms six families of Hamiltonian nilpotent centers. First, we study the crossing limit cycles of the PDS formed by a linear center and one arbitrary of the six Hamiltonian nilpotent centers. These six classes of PDS have at most one crossing limit cycle, and there are systems in each class with precisely one limit cycle. Second, we study the crossing limit cycles of the PDS formed by two of the six Hamiltonian nilpotent centers. There are systems in each of these 21 classes of PDS that have exactly four crossing limit cycles.

Keywords: discontinuous piecewise differential system; Hamiltonian nilpotent center; cubic polynomial differential system; limit cycle; vector field

MSC: 34A36; 34C07; 34C25; 37G15

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1. Introduction and Statement of the Main Results

In 1900, David Hilbert [1] shared a list of twenty-three problems at the International Congress of Mathematicians in Paris. From this list of problems, the sixteenth Hilbert problem is one of the remaining unsolved problems, together with the Riemann conjecture. The sixteenth Hilbert problem asks for the maximum number of limit cycles of a class of planar polynomial differential systems with a given degree. Recall that an isolated periodic orbit inside a planar differential system's set of all periodic orbits is known as a limit cycle. In the qualitative study, one of the main problems of planar differential systems is determining the existence and the maximum number of limit cycles, see [2,3]. This importance comes from the main role of limit cycles for understanding and explaining the behavior of a given differential system, such as the limit cycle of the Belousov Zhavotinskii model [4] or the one of the Van der Pol equations [5,6], etc.

This work focuses on a class of planar PDS with two pieces, where the separation curve is the straight line $x = 0$. Then, following the Filippov [7] conventions for defining this class of systems on the discontinuity line $x = 0$, these PDS can be written as follows

$$\begin{cases} (\dot{x},\dot{y})^T = (f^r(x,y),\ g^r(x,y))^T, & if\ \ x \in R^r; \\ (\dot{x},\dot{y})^T = (f^l(x,y),\ g^l(x,y))^T, & if\ \ x \in R^l, \end{cases} \tag{1}$$