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Geometric Configurations of Singularities of Planar Polynomial Differential Systems

A Global Classification in the Quadratic Case

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We dedicate this book to the memory of the mathematician

Constantin Sibirschi (1928–1990)

on the occasion of the 90th anniversary of his birth. Without the theory of algebraic invariants of polynomial differential equations, founded by Sibirschi, this book could not have been written.

Contents

Preface	xii
I Polynomial differential systems with emphasis on the quadratic ones	1
1 Introduction	3
1.1 Preliminaries	3
1.2 Problems on planar polynomial differential systems	6
1.2.1 The problem of the center	6
1.2.2 Research arising from the work of Darboux and the problem on algebraic integrability	6
1.2.3 The second part of Hilbert's 16th problem	7
1.2.4 The general finiteness problem for limit cycles or the existential Hilbert's 16th problem	9
1.2.5 The infinitesimal Hilbert's 16th problem and the Hilbert–Arnold problem	10
1.3 The contents of this book	10
2 Survey of results on quadratic differential systems	17
2.1 Brief history of quadratic differential systems	17
2.2 Some basic results obtained for quadratic differential systems	20
2.3 Study of some subclasses of the family of quadratic differential systems	23
2.4 Algebraic limit cycles in quadratic systems	26
2.5 Finiteness problems for quadratic differential systems	27
2.5.1 Basic concepts and results needed for studying the general finiteness problem	27
2.5.2 The general finiteness problem for quadratic differential systems	28
2.5.3 Application of Roussarie's ideas for the quadratic case	29

2.5.4	The infinitesimal Hilbert's 16th problem and the Hilbert–Arnold problem	32
2.5.5	The infinitesimal Hilbert's 16th problem for quadratic differential systems	33
2.6	The initial steps in the global theory of quadratic differential systems	34
3	Singularities of polynomial differential systems	37
3.1	Compactification on the Poincaré sphere, Poincaré disc and projective plane	37
3.2	Classical definitions	41
3.3	New definitions	43
3.4	The blow-up technique	45
3.4.1	The polar blow-up	45
3.4.2	The blow-up using rational functions	46
3.4.3	The blow-up technique using only one direction	50
3.5	The borsec concept	64
3.6	Equivalence relations	75
3.7	Notations for singularities of polynomial differential systems	81
3.7.1	Elemental singularities	81
3.7.2	Non-elemental singularities	82
3.7.3	Lack of singularities and complex singularities	85
3.7.4	Infinite number of singularities	86
4	Invariants in mathematical classification problems	91
4.1	Basic concepts	91
4.2	Classification problems on planar polynomial vector fields	94
4.2.1	Equivalence relations for polynomial vector fields	94
4.2.2	Classifications of some families of polynomial vector fields	96
5	Invariant theory of planar polynomial vector fields	99
5.1	Classical invariant theory	99
5.2	The work of Sibirschi's school	104
5.3	Basic concepts	109
5.3.1	Group actions on polynomial vector fields	110
5.3.2	Definition of invariant polynomials for polynomial differential systems	110
5.3.3	Assembling multiplicities of singularities in divisors of the line at infinity and in zero-cycles of the plane	113
5.3.4	Construction and geometric meaning of several basic invariant polynomials	114
5.4	Invariant polynomials associated to geometrical configurations	116
5.4.1	Building blocks for the construction of the invariant polynomials needed for the classification theorems of QS	117

5.4.2	The set of all invariant polynomials which classify geometrically the global configurations of singularities in QS	120
5.4.3	The influence of complex singularities in the study of the geometrical global configurations of singularities in QS	131
6	Main results on classifications of singularities in QS	133
6.1	Finite singularities	133
6.2	Finite weak singularities	144
6.3	Singularities of QS with an integrable saddle	145
6.4	Infinite singularities	148
7	Classifications of quadratic systems with special singularities	163
7.1	A finite star node	164
7.1.1	Conditions for the existence of at least one finite star node	165
7.1.2	Configurations of singularities with a finite star node	168
7.2	An integrable saddle	179
7.3	A center	197
7.4	A star node at infinity and another special singularity	209
7.5	Three finite special singularities	221
7.6	A weak focus of order two or three	235
7.6.1	A weak focus of order two	236
7.6.2	A weak focus of order three	245
7.7	A weak saddle of order three	250
II	Configurations of singularities of quadratic systems	261
8	QS with finite singularities of total multiplicity at most one	263
8.1	Systems without finite singularities	263
8.2	Systems with exactly one singularity	267
9	QS with finite singularities of total multiplicity two	281
9.1	Exactly one finite singularity	281
9.2	Two distinct real singularities	290
9.3	Two distinct complex singularities	323
10	QS with finite singularities of total multiplicity three	329
10.1	Exactly one singularity	329
10.2	Exactly two distinct singularities	334
10.3	Exactly three distinct singularities	345
10.3.1	Systems with zero-cycle $\mathcal{D}_{\mathbb{C}^2}(S) = p + q + r$	346
10.3.2	Systems with zero-cycle $\mathcal{D}_{\mathbb{C}^2}(S) = p + q^c + r^c$	371

11 QS with finite singularities of total multiplicity four	389
11.1 Exactly one singularity	390
11.2 Exactly two distinct singularities	398
11.2.1 One triple and one simple real singularities	398
11.2.2 Two double real singularities	428
11.2.3 Two double complex singularities	433
11.3 Exactly three distinct singularities	437
11.3.1 One double and two elemental real singularities	437
11.3.2 One double real and two elemental complex singularities	484
11.4 Exactly four distinct finite singularities	489
11.4.1 Four real elemental singularities	489
11.4.2 Two real and two complex elemental singularities	546
11.4.3 Four complex elemental finite singularities	608
12 Degenerate quadratic systems ($m_f = \infty$)	613
13 Conclusions	629
13.1 New concepts	629
13.2 The classical versus the new way	630
13.3 Algorithm to study the singularities of quadratic differential systems	631
13.4 The topological configurations of singularities	632
13.5 The study of the quadratic differential systems modulo limit cycles	632
Appendix A Table of notation	635
Appendix B Manual of Mathematica tools	645
B.1 How to initiate the program	646
B.2 Notation	646
B.3 Examples	648
Bibliography	669
Index	697

Preface

In this book we consider planar polynomial differential systems, i.e. systems of the form

$$\frac{dx}{dt} = p(x, y), \quad \frac{dy}{dt} = q(x, y)$$

where $p(x, y), q(x, y)$ are polynomials in x, y with real coefficients. To each such system there corresponds a point in \mathbb{R}^N determined by its $N = (n + 1)(n + 2)$ coefficients, where n is the degree of the system, i.e. $n = \max(\deg(p), \deg(q))$. A system of degree 2 is called *quadratic*.

The study of these differential systems always begins with the study of their singularities, finite or infinite, followed by the study of separatrix connections and of limit cycles. Also in some particular cases, the study of first integrals, algebraic invariant curves and period function is of great interest.

Our main goal in this book is to classify in a geometrical way the global schemes of singularities, finite and infinite, of quadratic differential systems and to obtain their bifurcation diagram in the 12-dimensional space \mathbb{R}^{12} . This global classification and its bifurcation diagram is completely algebraic, and we provide the algorithm that computes, for every family of quadratic systems, the global bifurcation diagram of its corresponding schemes of singularities. The study of singularities is the first step in the topological classification of the phase portraits of these differential systems and their bifurcation diagram. The geometrical equivalence relation between singularities considered here, is deeper than the topological one, including features of an algebraic-geometric meaning that play a significant role in studying bifurcations of the systems.

This was a long-term project. Our work began seven or even eight years ago. Every year we met in the spring in Barcelona, then in the fall in Montreal, in order to work on the project. During the past three years, two of us met in late summer in Chişinău, Moldova. We were happy to have the opportunity to work together and in the acknowledgements we mention the institutions and grants that supported us.

Over the years, we published partial results such as the study of infinite singularities, then of quadratic systems with total multiplicity of finite singularities less than or equal to one, or with total multiplicity of finite singularities equal to

two or three. From the class of quadratic differential systems with total multiplicity of finite singularities equal to four, those with total number of distinct finite singularities less than or equal to three, were also published. On one of these last published articles, we worked together with Alex C. Rezende, and we thank him for his contribution to our project.

The original results appearing in the book in Chapters 7, 12 and in Section 11.4 of Chapter 11 have never been published before and so they appear here for the first time. Section 11.4 contains the most generic and most difficult cases. This classification yielded 1765 distinct geometrical configurations of singularities, finite or infinite, plus at most 8 other such configurations (sharing the same finite part) that we conjecture are not realizable.

We give in the final chapter of the book some concluding comments with a view towards the future.

We are thankful to the editors and referees for the improvements they suggested and their advice was followed by us.

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