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Research paper

On the equilateral pentagonal central configurations

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ABSTRACT

An equilateral pentagon is a polygon in the plane with five sides of equal length. In this paper we classify the central configurations of the 5-body problem having the five bodies at the vertices of an equilateral pentagon with an axis of symmetry. We prove that there are two unique classes of such equilateral pentagons providing central configurations, one concave equilateral pentagon and one convex equilateral pentagon, the regular one. A key point of our proof is the use of rational parameterizations to transform the corresponding equations, which involve square roots, into polynomial equations.

1. Introduction and statement of the result

The Newtonian planar 5-body problem describes the dynamics of five point particles of positive masses m_i at positions $\mathbf{q}_i \in \mathbb{R}^2$ moving according to the Newton's laws under their mutual gravitational forces. The equations of motion of this 5-body problem are

$$m_i\ddot{\mathbf{q}}_i = -\sum_{j=1,j\neq i}^{5} Gm_im_j\frac{\mathbf{q}_i-\mathbf{q}_j}{r_{ij}^3}, \qquad 1\le i\le 5,$$

where $r_{ij}=|\mathbf{q}_i-\mathbf{q}_j|$ is the mutual distances between the masses m_i and m_j, and G is the gravitational constant. We take conveniently the time unit so that $G=1$.

The configuration space is defined by

$$\mathcal{E}=\{\mathbf{q}=(\mathbf{q}_1,\dots,\mathbf{q}_5)\in(\mathbb{R}^2)^5 : \mathbf{q}_i\neq\mathbf{q}_j, \quad i\neq j\}.$$

The configuration $\mathbf{q}=(\mathbf{q}_1,\dots,\mathbf{q}_5)$ is called *central* if the position vector of each body with respect to the center of mass is proportional to the corresponding acceleration vector. In other words, if there exists a positive constant λ such that

$$\ddot{\mathbf{q}}_i=\lambda(\mathbf{q}_i-\mathbf{c}_m), \qquad i=1,\dots,5,$$

where $\mathbf{c}_m=(m_1\mathbf{q}_1+\cdots+m_5\mathbf{q}_5)/M$ and $M=m_1+\cdots+m_5$, being $\mathbf{c}_m$ and M the center of mass of the five bodies and the total mass, respectively. Hence a given configuration $(\mathbf{q}_1,\dots,\mathbf{q}_5)\in\mathcal{E}$ of the 5-body problem with positive masses $m_1,\dots,m_5$, is central if there exists a λ such that $(\lambda,\mathbf{q}_1,\dots,\mathbf{q}_5)$ is a solution of the system

$$\sum_{j=1,j\neq i}^{5} m_j\frac{\mathbf{q}_i-\mathbf{q}_j}{r_{ij}^3}=\lambda(\mathbf{q}_i-\mathbf{c}_m), \qquad 1\le i\le 5. \tag{1}$$

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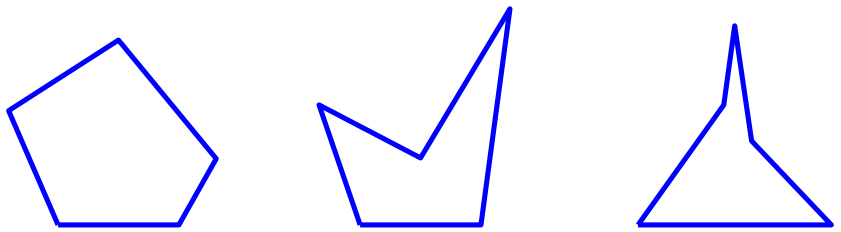

Fig. 1. Pentagonal configurations: a convex one in the left hand picture and two different type of concave ones.

A central configuration is *convex* if no body belongs to the convex hull of the other four bodies; otherwise it is called *concave*. A planar central configuration is called a *relative equilibrium* when they become equilibrium solutions in a rotating coordinate system [1].

Since Eqs. (1) are invariant under rotations, translations and dilations, when we consider the number of central configurations, this will be restricted to count the classes of central configurations modulo these mentioned transformations.

The central configurations are of special importance in Celestial Mechanics for several reasons. For instance, central configurations are the initial conditions for the homographic orbits of the n-body problem. Central configurations play an important role in the description of the topology of the integral manifolds in the n-body problem. Moreover, in the planar case the central configurations are initial positions for periodic solutions. For more information on this subject, recent advances and open questions, the reader is addressed to [2] and references therein.

In this paper we will investigate some central configurations of the 5-body problem, for which there are few known results. The first results concerning this issue take us back decades ago to the work due to Williams [3], who settled necessary and sufficient conditions for any plane central configuration of five bodies. In what follows we offer a non-exhaustive list of some interesting works concerning this topic, which have been published more recently.

Albouy and Kaloshin [4] proved that for a choice of five positive masses in the complement of a codimension-two algebraic variety in the mass space, there are only a finite number of equivalence classes of central configurations of the Newtonian 5-body problem. In [5] Chen and Hsiao provided necessary conditions for strictly convex central configurations of the planar 5-body problem.

In the last times the interest in stacked central configurations has grown a lot, that is, central configurations in which some subset of three or more masses also forms a central configuration. This concept was introduced by Hampton [6], who was arguably the first to find stacked central configurations in the 5-body problem, where two bodies can be removed and the remaining three bodies are already in a central configuration. After, several papers have been published showing the existence of other stacked central configurations in the planar 5-body problem; see, among others, [7–12].

Other studies have focused on restricting the problem to a particular shape, in [13] it was proved that the unique co-circular central configuration in the planar 5-body problem is the regular 5-gon with equal masses, while in [12] was proved the existence of three families of planar central configurations where three bodies are at the vertices of an equilateral triangle and the other two bodies are on a perpendicular bisector. Later on in [14] was studied the central configuration in a symmetric 5-body problem with three masses on an axis of symmetry and two other masses outside this axis, placed in symmetric positions. A complete classification of the isolated central configurations of the planar 5-body problem with equal masses was given in [15]. Recently in [16] were studied the central configurations of the planar 5-body problem having four bodies at the vertices of a rhombus.

For the reader's convenience, we summarize here some basic facts about the pentagon, which is a polygon with five sides and five angles. It is said that it is *convex* if all its vertices are pointing outwards, otherwise it is *concave*; see Fig. 1. A pentagon with five sides of equal length is named *equilateral*. Moreover, a pentagon is called *regular* when all the sides are equal in length, and five angles are of equal measures. If the pentagon does not have equal side length and angle measure, then it is called *irregular*. The regular pentagon is unique, up to similarity transformations, because it is equilateral and its five angles are equal.

Notice that between 5 bodies there are ten mutual distances. The pentagonal equilateral configurations satisfy that five of these distances are equal, but not all the pentagonal configurations having five equal distances are equilateral. In Fig. 2 we show one of these situations that moreover is also symmetrical with respect a vertical axis.

The goal in this paper is to characterize the equilateral pentagonal central configurations with an axis of symmetry for the planar 5-body problem whose five positive masses are at the vertices of an equilateral pentagon.

Then our main result is the following one, which will be proved in the next section.

Theorem 1. *There are only two classes of equilateral pentagonal central configuration having an axis of symmetry for the 5-body problem.*

(a) The convex regular pentagon with equal five masses, see Fig. 3(a).

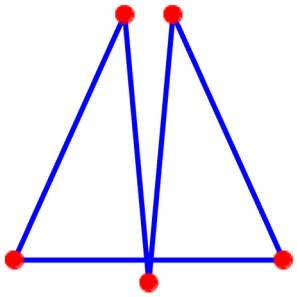

Fig. 2. A possible pentagonal central configuration, with five equal mutual distances but not equilateral.

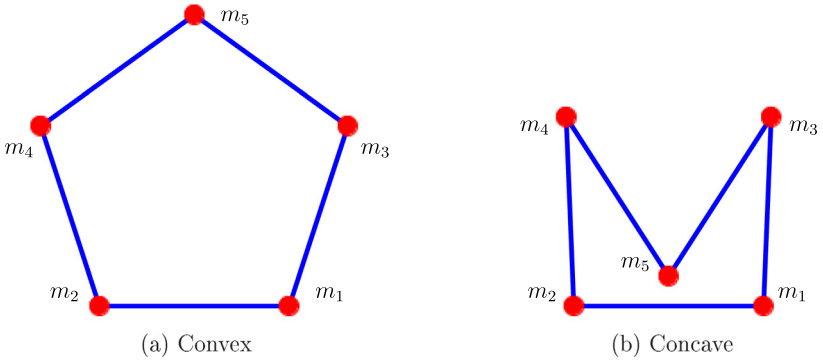

Fig. 3. The two equilateral pentagonal central configurations with an axis of symmetry of the 5-body problem.

(b) The equilateral concave pentagon with the masses normalized, i.e. $\sum_{i=1}^{5} m_i = 1$, *equal to* $m_1 = m_2 \approx 0.0922539749$, $m_3 = m_4 \approx 0.3860948766$ *and* $m_5 \approx 0.04330242730$. *In Fig. 3(b) we show a representative of its class where the bodies of masses* m_1 *and* m_2 *are fixed at* $(1/2, 0)$ *and* $(-1/2, 0)$, *respectively. The other bodies of masses* m_3, m_4 *and* m_5 *are located at* (x_3, y_3), $(-x_3, y_3)$ *and* $(0, y_5)$, *respectively, where* $x_3 \approx 0.5402091568$, $y_3 \approx 0.9991912848$ *and* $y_5 \approx 0.1576604970$. *See Remark 2 for some comments of how these values are obtained.*

Remark 2. *From the proof of Theorem 1 and the results of Appendix B we obtain a value* $t = t^* \approx 0.7332148086$ *which is the smallest root of the quadratic polynomial* $t^2 - u^* t + 1 = 0$, *where* $u = u^* \approx 2.0970716051$ *is the unique root in the interval* $[205/100, 210/100]$ *of the polynomial of degree* 60 *with integer coefficients,* $R_{60}(u)$, *given in Appendix B. This polynomial is constructed from a reciprocal polynomial of degree* 120 *and also with integer coefficients that appears in (10). Then* $y_5 = (1 - (t^*)^2)/(4t^*)$ *and all the other values in the theorem,* $x_3, y_3, m_j, j = 1, 2, \ldots, 5$, *can be obtained from this* t^* *by elementary computations: sums, subtractions, multiplications, divisions and square roots. Recall that, although we do not know the exact value of* u^*, *the classical use of Sturm sequences allows to obtain explicit intervals, with rational endpoints, containing* u^* *and with arbitrarily small length.*

Although we have not approached next question, with minor changes, the tools introduced in this work are also applicable to characterize symmetric 5-body pentagonal configurations having five equal mutual distances.

At this stage the reader should be warned that there is a previous work by Perko and Walter [17], who showed that n equal masses at the vertices of a regular polygon, for $n \geq 4$, forms a central configuration if and only if the masses are equal. Therefore it was known that the regular pentagon with equal masses is a central configuration for the 5-body problem, but it was unknown that it is the unique equilateral convex pentagonal central configuration with an axis of symmetry.

One of the key points of our approach is the use of rational parameterizations to eliminate some of the square roots that appear in the equations governing the central configurations, converting in this way these equations into polynomial ones. Then these equations can be treated analytically by using some classical tools, like for instance the Sturm sequences or the computation of resultants.

2. Preliminaries

Central configurations are invariant under composition of translations, rotations, and scaling through its center of mass, hence without loss of generality we can assume that the position of the masses $m_i > 0$ for $i = 1 \ldots, 5$ at the vertices of

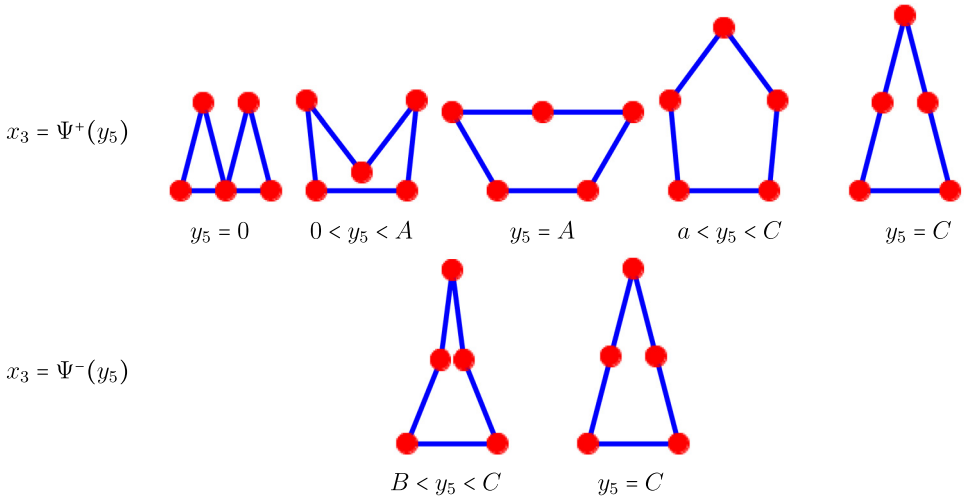

Fig. 4. Gallery of possible pentagon equilateral configurations, according whether $x_3 = \Psi^+(y_5)$ or $x_3 = \Psi^-(y_5)$ and the value of y_5, together with the boundary cases, that are no more pentagons.

an equilateral pentagon with an axis of symmetry are $p_k = (x_k, y_k)$ for $k = 1, 2, 3, 4, 5$, where $y_1 = 0$, $x_2 = -x_1$, $y_2 = 0$, $x_4 = -x_3$, $y_4 = y_3$ and $x_5 = 0$. Note that we can assume that $x_1 > 0$, $x_3 > 0$, $y_3 > 0$, $y_5 > 0$ and $y_3 \neq y_5$, because we want that the points (x_i, y_i) be the vertices of a pentagon. In particular, the cases $x_3 < 0$ or $y_5 < 0$ are excluded because they would give rise to symmetrical pentagonal configurations, but not equilateral.

Next we will obtain the coordinates for the equilateral pentagon vertices. By a suitable scaling we may assume that $r_{12} = 1$, so $x_1 = 1/2$. Now we substitute this value into the equation $r_{13} = 1$, obtaining that $y_3^2 = (3 + 4x_3 - 4x_3^2)/4$. These values replaced in the equation $r_{35} = 1$ provides x_3 in terms of y_5,

$$x_3 = \Psi^{\pm}(y_5) = \frac{1}{4} \pm \frac{y_5}{2}\Phi(y_5), \quad \text{where} \quad \Phi(y) = \sqrt{\frac{15 - 4y^2}{1 + 4y^2}}. \tag{2}$$

Since we are interested in central configurations, with an axis of symmetry, modulus rotations and homothetic transformations, without loss of generality, we have the next result.

Proposition 3. *Suppose that* $\mathbf{q}_1, \ldots, \mathbf{q}_5$ *form an equilateral pentagon, with* $\mathbf{q}_1$ *and* $\mathbf{q}_2$ *fixed on the x-axis and with the y-axis as an axis of symmetry. Then*

$$(x_1, y_1) = (1/2, 0), \quad (x_2, y_2) = (-1/2, 0), \quad (x_3, y_3), \quad (-x_3, y_3) \quad \textit{and} \quad (0, y_5),$$

where

(a) either $x_3 = \Psi^+(y_5) = \dfrac{1}{4} + \dfrac{y_5}{2}\Phi(y_5), y_3 = \dfrac{y_5}{2} + \dfrac{1}{4}\Phi(y_5), \quad$ *and* $\quad y_5 \in \left(0, \sqrt{15}/2\right)$,

(b) or $x_3 = \Psi^-(y_5) = \dfrac{1}{4} - \dfrac{y_5}{2}\Phi(y_5), y_3 = \dfrac{y_5}{2} - \dfrac{1}{4}\Phi(y_5), \quad$ *and* $\quad y_5 \in \left(1 + \sqrt{3}/2, \sqrt{15}/2\right)$.

The geometrically distinct equilateral pentagon are shown in Fig. 4. It follows from the case (a) of Proposition 3 that the equilateral pentagon is concave when $y_5 \in (0, \sqrt{3}/2)$, and convex if $y_5 \in (\sqrt{3}/2, \sqrt{15}/2)$. While in case (b) of Proposition 3 the equilateral pentagon is always concave.

In the next section we will see that the only values of y_5 that give rise to central configurations will be $x_3 = \Psi^+(y_5)$, where $y_5 = (\sqrt{5 + 2\sqrt{5}})/2 \approx 1.539$ is associated to the regular pentagon, while $y_5 \approx 0.1576605$ gives a concave central configuration.

3. Proof of Theorem 1

By a suitable scaling we may assume that $m_1 + m_2 + m_3 + m_4 + m_5 = 1$. Then the center of mass of the five bodies is

$$\mathbf{c}_m = (x_m, y_m) = \left(\frac{m_1 - m_2 + 2(m_3 - m_4)x_3}{2}, \frac{(m_3 + m_4)\sqrt{3 + 4x_3 - 4x_3^2} + 2m_5 y_5}{2} \right).$$

Since we are studying equilateral pentagons we can assume that $r_{12} = r_{13} = r_{35} = r_{45} = r_{24} = 1$. Then the others mutual distances are

$$r_{14} = r_{2,3} = \sqrt{1+2x_3}, \qquad r_{1,5} = r_{2,5} = \sqrt{y_5^2 + 1/4}, \qquad r_{3,4} = 2x_3. \tag{3}$$

From (1) we obtain the ten equations for the central configurations of the 5-body problem in the plane:

$$\begin{aligned} e_j &= \sum_{j=1, j\neq i}^{5} \frac{m_j(x_i - x_j)}{r_{i,j}^3} - \lambda(x_j - x_m) = 0, \quad 1 \le j \le 5, \\ e_{j+5} &= \sum_{j=1, j\neq i}^{5} \frac{m_j(y_i - y_j)}{r_{i,j}^3} - \lambda(y_j - y_m) = 0, \quad 1 \le j \le 5. \end{aligned} \tag{4}$$

Substituting into (4) the values and expressions of the mutual distances and taking $m_5 = 1 - m_1 - m_2 - m_3 - m_4$, it is seen that

$$e_8 - e_9 = -\frac{(m_1 - m_2)\sqrt{3 + 4x_3 - 4x_3^2}(-1 + \sqrt{1+2x_3} + 2x_3\sqrt{1+2x_3}\,)}{2(1+2x_3)^{3/2}} = 0. \tag{5}$$

A straightforward computation shows that

$$\sqrt{3 + 4x_3 - 4x_3^2}(-1 + \sqrt{1+2x_3} + 2x_3\sqrt{1+2x_3}) = 0$$

for $x_3 = -1/2, 0, 3/2$. However, any of these values is good, because $x_3 > 0$ and $x_3 = 3/2$ implies that $r_{1,3} = \sqrt{1+y_3^2}$, but this is impossible because we have assumed that $y_3 \neq 0$ and $r_{1,3} = 1$. So $m_2 = m_1$.

Since $e_3 + e_4 = (m_3 - m_4)(1 + 8\lambda x_3^3)/(4x_3^2) = 0$. It follows that either $m_4 = m_3$, or $\lambda = -1/(8x_3^3)$. In this last case we obtain

$$e_6 - e_7 = \frac{(m_3 - m_4)\sqrt{3 + 4x_3 - 4x_3^2}(-1 + \sqrt{1+2x_3} + 2x_3\sqrt{1+2x_3}\,)}{2(1+2x_3)^{3/2}} = 0.$$

Hence as in (5) we have that $m_4 = m_3$. Therefore we do not need to consider $\lambda = -1/(8x_3^3)$, and in what follows we consider that $m_4 = m_3$, such that, $e_6 - e_7 = 0$.

In summary, we have that $e_1 + e_2 = 0$, $e_5 = 0$ and $e_6 - e_7 = 0$. Hence we conclude that from the ten Eqs. (4) only $e_1, e_3, e_6, e_8, e_{10}$ remain independent. These equations are

$$\begin{aligned} f_1 = &\ -\frac{\lambda}{2} - \frac{4}{E_3} + m_1\left(-1 + \frac{8}{E_3}\right) + m_3\left(-\frac{1}{2} + x_3 - \frac{1}{2E_1} + \frac{8}{E_3}\right), \\ f_2 = &\ -(1+\lambda)x_3 + \frac{1}{4}m_3\left(8 - \frac{1}{x_3^3}\right)x_3 + m_1\left(\frac{1}{2} + x_3 - \frac{1}{2E_1}\right), \\ f_3 = &\ y_5\left(\lambda + \frac{8}{E_3}\right) + m_1\left(-2\lambda y_5 - \frac{16y_5}{E_3}\right) + m_3\left(\frac{1}{2}E_2\left(1 + \frac{1}{(1+2x_3)E_1}\right) + \lambda\Big(E_2 - 2y_5\Big) - \frac{16y_5}{E_3}\right), \\ f_4 = &\ -\frac{1}{2}(1+\lambda)\Big(E_2 - 2y_5\Big) + (1+\lambda)m_3\Big(E_2 - 2y_5\Big) + m_1\left(\frac{1}{2}E_2\left(1 - \frac{1}{(1+2x_3)E_1}\right) - 2(1+\lambda)y_5\right), \\ f_5 = &\ (1+\lambda)m_3\,(E_2 - 2y_5) + m_1\left(-2\lambda y_5 - \frac{16y_5}{E_3}\right), \end{aligned}$$

where $E_1 = \sqrt{1+2x_3}$, $E_2 = \sqrt{3 + 4x_3 - 4x_3^2}$ and $E_3 = (1 + 4y_5^2)^{3/2}$.

Solving $f_2 = 0$ and $f_5 = 0$, we obtain the following expressions for m_1 and m_3

$$m_1 = \big(2E_1E_3(1+\lambda)^2x_3^3(E_2 - 2y_5)\big)/m, \qquad m_3 = \big(4E_1(1+\lambda)(8 + E_3\lambda)x_3^3y_5\big)/m, \tag{6}$$

where

$$m = 2E_1\,(E_2E_3\lambda + 2\lambda y_5E_3 + E_2E_3 - 2E_3y_5 + 32y_5)\,x_3{}^3 + E_3\,(1+\lambda)\,(E_2 - 2y_5)\,(E_1 - 1)\,x_3{}^2 - E_1y_5\,(\lambda E_3 + 8)\,.$$

We substitute the values of m_1 and m_3 into the equations $f_1 = 0$, $f_3 = 0$ and $f_4 = 0$, and taking only the numerators of these three equations because the denominators do not vanish, the former system, reduce to

$$\begin{aligned} g_1 = &8E_1E_3\,(1+\lambda)\,(\lambda E_3 + 8)\,y_5x_3{}^4 + \Big(\big(\,16E_1E_3{}^2\lambda - 4\lambda^2E_3{}^2 + 8E_1E_3{}^2 - 192\lambda E_1E_3 \\ &- 4\lambda E_3{}^2 - 64E_1E_3 + 512E_1\lambda - 32\lambda E_3 - 32E_3\,\big)\,y_5 - 2E_1E_2E_3\,(1+\lambda)\,(3\lambda E_3 + 2E_3 - 16\lambda - 8)\Big)\,x_3{}^3 \end{aligned}$$

$$+ \left(2E_3\left(1+\lambda\right)\left(\lambda E_3+8\right)\left(E_1-1\right)y_5 - E_2E_3\left(1+\lambda\right)\left(\lambda E_3+8\right)\left(E_1-1\right)\right)x_3{}^2 + E_1\left(\lambda E_3+8\right)^2 y_5,$$

$$g_3 = (8+E_3\lambda)y_5(L_2\lambda - L_1),$$

$$g_4 = (1+\lambda)(2y_5 - E_2)(L_2\lambda - L_1),$$

where

$$L_1 = -2\left(1+2x_3\right)\left(2E_1E_3x_3{}^3 - E_1E_3x_3{}^2 + x_3{}^2E_3 - 4E_1\right)y_5 - E_2E_3x_3{}^2\left(2x_3E_1 + E_1 - 1\right),$$

$$L_2 = \left(1+2x_3\right)\left(12E_1E_3x_3{}^3 - 2E_1E_3x_3{}^2 - 64x_3{}^3E_1 + 2x_3{}^2E_3 - E_1E_3\right)y_5 + E_2E_3x_3{}^2\left(2x_3E_1 + E_1 - 1\right).$$

Computing λ from equation $g_3 = 0$ we obtain the two solutions

$$\lambda_1 = -\frac{8}{E_3}, \qquad \lambda_2 = \frac{L_1}{L_2}. \tag{7}$$

The solution $\lambda = \lambda_1$ is not suitable because then $m_3 = 0$. Substituting the solution $\lambda = \lambda_2$ in the equations $g_1 = 0$ and $g_4 = 0$, we get that $g_4 \equiv 0$, and the equation $g_1 = 0$ reduces to

$$\bar{h}_1 = (2x_3^3y_5E_1E_3)(E_3-8)^2(2x_3-1)(1+2x_3+4x_3^2)h_1 = 0,$$

where

$$h_1 = -4\left(1+2x_3\right)^2\left(4E_1{}^2\left(E_3-16\right)x_3{}^4 - 8E_1\left(E_1E_3 - 4E_1 - 4\right)x_3{}^3 - E_3\left(E_1-1\right)^2x_3{}^2 + E_1{}^2\left(E_3-8\right)\right)y_5{}^2 - 2E_2\left(1+2x_3\right)\left(16E_1{}^2\left(E_3-4\right)x_3{}^4 + 4E_1\left(3E_1E_3 - 8E_1 - E_3 - 8\right)x_3{}^3 + 2E_3\left(E_1-1\right)^2x_3{}^2 - 2E_1{}^2\left(E_3-8\right)x_3 - E_1{}^2\left(E_3-8\right)\right)y_5 + E_2{}^2E_3x_3{}^2\left(2x_3E_1 + E_1 - 1\right)^2.$$

Notice that $2x_3^3y_5E_1E_3(1+2x_3+4x_3^2)$ does not vanish because $x_3 \in (0,1)$, $y_5 > 0$ and $E_1 = r_{1,4} > 0$.

At this step we shall prove that condition $(E_3-8)(2x_3-1) = 0$ implies that $m_5 = 0$. This is so, because

$$m_5 = 1 - (m_1+m_2+m_3+m_4) = 1 - 2(m_1+m_3)$$

$$= \frac{2E_1y_5x_3{}^3E_2E_3\left(2x_3-1\right)\left(4x_3{}^2+2x_3+1\right)\left(E_3-8\right)}{L_2^2}$$

$$\times\Big(-\left(1+2x_3\right)\Big[8E_1{}^2\left(E_3-16\right)x_3{}^4 + 8E_1\left(E_3-8\right)\left(E_1+1\right)x_3{}^3 + 2E_3\left(E_1-1\right)^2x_3{}^2 - 2E_1{}^2\left(E_3-16\right)x_3 - E_1\left(E_1E_3 - 16E_1 + E_3\right)\Big]y_5 + E_2E_3x_3{}^2\left(2E_1x_3 + E_1 - 1\right)^2\Big),$$

where we have used the expressions (6) for m_1, m_3 and substituted $\lambda = \lambda_2$ where λ_2 is given in Eq. (7). In short we have proved that the equations $\bar{h}_1 = 0$ and $h_1 = 0$ are equivalent.

From Proposition 3 we have that

$$x_3 = \frac{1}{4} \pm \frac{y_5}{2}\sqrt{\frac{15-4y_5^2}{1+4y_5^2}},$$

or equivalently,

$$h_2 = (1-4x_3)^2(1+4y_5^2) + 4y_5^2(4y_5^2 - 15) = 0. \tag{8}$$

Hence the central configurations are the solutions of the simultaneous solution of both equations $h_1 = 0$ and $h_2 = 0$, with the two unknowns x_3 and y_5. Indeed it provides positive masses m_j. In order to avoid the square roots which appear in E_1, E_2 and E_3 in h_1, we do a change of variables such that the expressions appearing inside each square root are equal to some new squared expressions. These changes of variables are given by the so called rational parameterizations and correspond to parameterizations of planar algebraic curves given by rational functions. Due to the famous Cayley–Riemann's Theorem [18,19] they exist if and only if the corresponding surfaces have genus zero. There are effective methods to find one of these parameterizations see for instance [20, Chap. 4 & 5]. In fact, many programs of symbolic calculus have implemented some methods and algorithms for obtaining them. For more information on this subject the reader is addressed to [21] and references therein, where there are several examples of applications of this approach.

In our case, for instance, we have that $E_1 = \sqrt{1+2x_3}$, $E_2 = \sqrt{3+4x_3-4x_3^2}$. Hence if we write $x_3 = (u^2-1)/2$ we get that $E_1 = \sqrt{u^2}$. Then

$$3 + 4x_3 - 4x_3^2\Big|_{x_3=(u^2-1)/2} = u^2(4-u^2).$$

Consider now the algebraic curve $F(u,v) = u^2(4-u^2) - v^2 = 0$. It has genus 0, and by the Cayley–Riemann's theorem it admits a rational parameterization. For example, for all s,

$$F\Big(\frac{4\left(2s-1\right)s}{5s^2-4s+1}, \frac{8s\left(s-1\right)\left(3s-1\right)\left(2s-1\right)}{\left(5s^2-4s+1\right)^2}\Big) = 0.$$

As a consequence, by taking

$$x_3 = u^2(4-u^2)\Big|_{u=\frac{4(2s-1)s}{5s^2-4s+1}} = \frac{\left(3s^2-1\right)\left(13s^2-8s+1\right)}{2\left(5s^2-4s+1\right)^2}$$

we obtain that

$$E_2 = \sqrt{v^2} = \sqrt{\frac{\left(8s\,(s-1)\,(3s-1)\,(2s-1)\right)^2}{\left(5s^2-4s+1\right)^4}}.$$

The rational parameterization of $E_3 = (1+4y_5^2)^{3/2}$ is much simpler and can be obtained similarly. It suffices to consider the algebraic curve of genus 0, $G(y_5, w) = 1 + 4y_5^2 - w^2 = 0$. A good parameterization for y_5 is $y_5 = (1-t^2)/(4t)$.

We are interested on values $y_5 \in (0, \sqrt{15}/2)$ and $x_3 \in (0, 1)$. In short, we do the change of variables $(x_3, y_5) \to (s, t)$ where

$$x_3 = \frac{\left(3s^2-1\right)\left(13s^2-8s+1\right)}{2\left(5s^2-4s+1\right)^2}, \quad y_5 = \frac{1-t^2}{4t}.$$

Then we have that t varies in the interval $T := (4-\sqrt{15}, 1)$ and similarly, the values of $x_3 \in (0, 1)$ are covered for instance for $s \in S := (\sqrt{3}/3, (6+\sqrt{3})/11)$. Hence,

$$E_1 = \frac{4s(2s-1)}{5s^2-4s+1}, \quad E_2 = \frac{-8s(s-1)(3s-1)(2s-1)}{\left(5s^2-4s+1\right)^2}, \quad E_3 = \frac{\left(t^2+1\right)^3}{8t^3}.$$

In the variables (s, t) the two equations $h_1 = 0$ and $h_2 = 0$ become

$$h_1 = \frac{8s^2(2s-1)^2}{\left(5s^2-4s+1\right)^2 t^3} H_1 = 0, \qquad h_2 = \frac{1}{16\left(5s^2-4s+1\right)^4 t^4} H_2 = 0, \tag{9}$$

respectively, where

$$\begin{aligned} H_2 = {} & \left(5s^2-4s+1\right)^4\left(t^8+1\right) - 4\left(47s^4-152s^3+150s^2-56s+7\right) \\ & \times \left(153s^4-168s^3+58s^2-8s+1\right)\left(t^6+t^2\right) + \left(101222s^8-258784s^7+326904s^6\right. \\ & \left. - 286240s^5+183428s^4-79776s^3+21432s^2-3168s+198\right)t^4 \end{aligned}$$

and H_1 ia a huge polynomial of total degree 34 and with $\deg_s(H_1) = 24$ and $\deg_t(H_1) = 10$, which is given in Appendix A.

Taking into account that $t \in T$ and $s \in S$ to solve system (9), we see that this can be reduced to solve the system $H_1 = 0$, $H_2 = 0$, because $s(2s-1)(s^2+2s-1) \neq 0$. Hence, to find the real solutions of the systems $h_1 = 0$, $h_2 = 0$, is equivalent to find the real solutions, $(s, t) \in S \times T$, of the polynomial system $H_1 = 0$, $H_2 = 0$.

To study the above planar systems of equations we will use a mixture of the classical approach applying resultants together with simple inequalities in the original variables x_3 and y_5. For our problem this approach is very suitable because of the simplicity of the equation $h_2 = 0$, given in (8).

We start with the polynomials system $H_1(s, t) = 0$, $H_2(s, t) = 0$. Recall that if $(\hat{s}, \hat{t})$ is one of its solutions (real or complex), then $t = \hat{t}$ must be a zero of the one variable polynomial

$$P(t) = \mathrm{Res}_s(H_1, H_2),$$

where $\mathrm{Res}_s(\cdot,\cdot)$ denotes the resultant of two polynomials with respect the variable s, see for instance [20]. After some computations (implemented for instance in Maple or Mathematica) we get that

$$P(t) = (1+t^2)^6 p_4(t) q_4(t) p_{120}(t) p_{132}(t), \tag{10}$$

where $p_4(t) = 1 + 4t - 14t^2 + 4t^3 + t^4$, $q_4(t) = 1 - 4t - 14t^2 - 4t^3 + t^4$ and p_k denotes a polynomial with integer coefficients of degree k that we do not detail. We only remark that precisely p_{120} is the polynomial that gives rise to the polynomial R_{60}, detailed in Appendix B and that gives rise to the values t^* and u^* that appear in Remark 2. Hence, by computing the Sturm sequences of each of the four polynomials, p_4, q_4, p_{120} and p_{132}, we get that they have respectively, 4,4, 28 and 32 real roots and, moreover, that all them are simple. Furthermore, since we are only interested on the roots $t \in T \sim (0.127, 1)$, we consider a slightly bigger interval $T \subset T' = (3/25, 100)$, with rational endpoints. Again, the corresponding Sturm sequences allow to prove that their number of roots in T' are 1, 1, 7 and 9, respectively. We will denote them by t_1; t_2; $t_3, \ldots, t_9$ and $t_{10}, \ldots, t_{18}$, where for each p_k the roots are ordered. Their approximated ordered value are

$$\begin{array}{lllll} t_{10} \approx 0.1278827, & t_3 \approx 0.1296657, & t_{11} \approx 0.1318307, & t_{12} \approx 0.1535285, & t_2 \approx 0.1583844, \\ t_{13} \approx 0.1690804, & t_4 \approx 0.1818971, & t_5 \approx 0.1871837, & t_{14} \approx 0.4693713, & t_1 \approx 0.5095254, \\ t_{15} \approx 0.5490528, & t_{16} \approx 0.5930556, & t_6 \approx 0.7095411, & t_7 \approx 0.7332148, & t_8 \approx 0.9432977, \\ t_{17} \approx 0.9681690, & t_9 \approx 0.9958185, & t_{18} \approx 0.9962499. & & \end{array}$$

In fact, the roots t_1 and t_2 are $t_1 = -1+\sqrt{5}-\sqrt{5-2\sqrt{5}} \approx 0.5095254$ and $t_2 = 1+\sqrt{5}-\sqrt{5+2\sqrt{5}} \approx 0.1583844$, while each of the other sixteen roots can be obtained, again using the Sturm sequences, with any desired error.

Therefore, each $t_j, j = 3, \dots, 18$, can be bounded by $\underline{t}_j < t_j < \overline{t}_j$, with $\underline{t}_j, \overline{t}_j \in \mathbb{Q}$ and $0 < \overline{t}_j - \underline{t}_j$ as small, as desired. For each of these values of t_j we can use that the function $t \to (1-t^2)/(4t)$ is decreasing in $(0, 1)$ and that $y_5 = (1-t^2)/(4t)$ to get that if $(1-t_j^2)/(4t_j) = y_5(j)$, then

$$\underline{y}_5(j) = \frac{1-\overline{t}_j^2}{4\overline{t}_j} < y_5(j) < \frac{1-\underline{t}_j^2}{4\underline{t}_j} = \overline{y}_5(j), \quad \text{with} \quad \underline{y}_5(j), \overline{y}_5(j) \in \mathbb{Q}.$$

Recall that in (2) we have introduced the functions

$$\Psi^{\pm}(y) = \frac{1}{4} \pm \frac{y}{2}\sqrt{\frac{15-4y^2}{1+4y^2}}.$$

By simple derivation we get that Ψ^+ (resp. Ψ^-) is increasing (resp. decreasing) for $y \in (0, \sqrt{3}/2)$ and decreasing (resp. increasing) for $y \in (\sqrt{3}/2, 1)$. Hence, if we define $x_3^{\pm} = \Psi^{\pm}(y_5)$ it holds that

- If j is such that $y_5(j) < \sqrt{3}/2$ then

$$\Psi^+(\underline{y}_5(j)) < x_3^+(j) < \Psi^+(\overline{y}_5(j)), \quad \Psi^-(\overline{y}_5(j)) < x_3^-(j) < \Psi^-(\underline{y}_5(j)).$$

- If j is such that $y_5(j) > \sqrt{3}/2$ then

$$\Psi^+(\overline{y}_5(j)) < x_3^+(j) < \Psi^+(\underline{y}_5(j)), \quad \Psi^-(\underline{y}_5(j)) < x_3^-(j) < \Psi^-(\overline{y}_5(j)).$$

From these inequalities it is easy to find rational values $\underline{x}_3^{\pm}(j)$ and $\overline{x}_3^{\pm}(j)$ such that

$$\underline{x}_3^{\pm}(j) < x_3^{\pm}(j) < \overline{x}_3^{\pm}(j),$$

and with $0 < \overline{x}_3^{\pm}(j) - \underline{x}_3^{\pm}(j)$ as small as desired. Finally, the functions that define E_1, E_2 and E_3, given by $\psi_1(x) = \sqrt{1+2x}$, $\psi_2(x) = \sqrt{3+4x-4x^2}$ and $\Psi_3(y) = (1+4y^2)^{3/2}$ are increasing, increasing for $x \in (0, \sqrt{3}/3)$ and decreasing for $x \in (\sqrt{3}/3, 1)$, and increasing, respectively. Similarly that for $x^{\pm}(j)$ we can find rational bounds, $\underline{E}_j$ and $\overline{E}_j$, $j = 1, 2, 3$, as sharp as desired and satisfying

$$\underline{E}_1 < E_1 < \overline{E}_1, \quad \underline{E}_2 < E_2 < \overline{E}_2, \quad \underline{E}_3 < E_3 < \overline{E}_3.$$

In short, we have found sharp rational upper and lower bound of any possible solution $y_5 = y_5(j)$, $x_3^{\pm}(j)$, corresponding to each $t = t_j, j = 1, 2, \dots, 18$. These rational bounds give also rational bounds for E_1, E_2 and E_3. Gluing these bounds we prove that some candidates to be solutions of our system can be discarded. That other candidates are actual solution of our problem can be easily proved from Bolzano's theorem. We detail two examples, one of each type and skip the computations for all the rest.

A suitable way is to write h_1 in function of the variables x_3, y_5, E_1, E_2 and E_3, namely

$$\begin{aligned} h_1 = &- 128x_3^3y_5^2\left(4x_3^2+4x_3+1\right)E_1 + 4x_3^2y_5^2\left(4x_3^2+4x_3+1\right)E_3 \\ &+ 32y_5^2\left(32x_3^6+16x_3^5-8x_3^4-4x_3^3+4x_3^2+4x_3+1\right)E_1^2 \\ &+ 64x_3^3y_5\left(1+2x_3\right)E_1E_2 - 8x_3^2y_5^2\left(4x_3^2+4x_3+1\right)E_1E_3 - 4x_3^2y_5\left(1+2x_3\right)E_3E_2 \\ &+ 16y_5\left(16x_3^5+16x_3^4+4x_3^3-4x_3^2-4x_3-1\right)E_1^2E_2 \\ &- 4y_5^2\left(16x_3^6-16x_3^5-32x_3^4-12x_3^3+3x_3^2+4x_3+1\right)E_1^2E_3 \\ &+ 8x_3^2y_5\left(2x_3^2+3x_3+1\right)E_1E_2E_3 + E_2^2E_3x_3^2 - 2y_5\left(32x_3^5+40x_3^4+16x_3^3-2x_3^2-4x_3-1\right)E_1^2E_2E_3 \\ &- 2x_3^2\left(1+2x_3\right)E_1E_2^2E_3 + x_3^2\left(4x_3^2+4x_3+1\right)E_1^2E_2^2E_3. \end{aligned}$$

We remark that the addends of h_1 can be bounded using the inequalities given above. For instance

$$\underline{E}_2^2\underline{E}_3\underline{x}_3^2 < E_2^2E_3x_3^2 < \overline{E}_2^2\overline{E}_3\overline{x}_3^2,$$

and

$$-4\overline{x}_3^2\overline{y}_5\left(1+2\overline{x}_3\right)\overline{E}_3\overline{E}_2 < -4x_3^2y_5\left(1+2x_3\right)E_3E_2 < -4\underline{x}_3^2\underline{y}_5\left(1+2\underline{x}_3\right)\underline{E}_3\underline{E}_2.$$

Let us prove for instance that the value $y_5 = y_5(5)$ corresponding to $t = t_5 \approx 0.1871837$, together with $x_3 = x_3^+(5) = \Psi^+(y_5(5))$ does not provide a solution of our system. From the Sturm sequence we get that

$$\frac{1871}{10000} < t_5 < \frac{1872}{10000}.$$

From these inequalities and all the above considerations we obtain that

$$1.2886701 \approx \frac{94234}{73125} < y_5 < \frac{96499359}{74840000} \approx 1.2894088, \quad \frac{9235}{10000} < x_3^+ < \frac{9238}{10000},$$
$$\frac{16873}{10000} < E_1 < \frac{16873}{10000}, \quad \frac{18118}{10000} < E_2 < \frac{18115}{10000}, \quad \frac{21128}{1000} < E_3 < \frac{21161}{1000}.$$

By using all these inequalities we obtain that the corresponding value of $h_1 > 242$ and the system has no solution for the value of y_5, and its corresponding $x_3^+(5)$ associated to $t = t_5$.

On the other hand, let us prove that the value $y_5 = y_5(7)$ corresponding to $t = t_7 = t^* \approx 0.7332148$ together with $x_3 = x_3^+(7) = \Psi^+(y_5(7))$ does provide an actual solution. This is a simple consequence of Bolzano's theorem, because if we denote as $h_1(\tau)$ the value of the expression of h_1 when all the values y_5, x_5^+, E_1, E_2 and E_3 are obtained when $t = \tau$ we get for instance that

$$h_1\Big(\frac{7332}{10000}\Big)h_1\Big(\frac{7333}{10000}\Big) < 0.$$

We carry out similar computations for $y_5 = y_5(j)$, corresponding to $t = t_j$, and $x_3 = x_3^{\pm}(j) = \Psi^{\pm}(y_5(j))$. We conclude that among the 36 candidates that could be a solution of the system $h_1(x_3, y_5) = 0$, $h_2(x_3, y_5) = 0$ the only couples (x_3, y_5) that do solve it are:

(I) $(x_3^+(2), y_5(2)) = \left(\frac{1+\sqrt{5}}{4}, \frac{\sqrt{5+2\sqrt{5}}}{2}\right) \approx (0.8090170, 1.5388418)$ corresponding to $t = t_2$,
(II) $(x_3^+(7), y_5(7)) \approx (0.5402091, 0.1576605)$ corresponding to $t = t_7 = t^*$,
(III) $(x_3^+(9), y_5(9)) \approx (0.2540572, 0.0020951)$ corresponding to $t = t_9$,
(IV) $(x_3^-(4), y_5(4)) \approx (-0.4091526, 1.3289291)$ corresponding to $t = t_4$,
(V) $(x_3^-(14), y_5(14)) \approx (-0.3542470, 0.4152845)$ corresponding to $t = t_{14}$,
(VI) $(x_3^-(18), y_5(18)) \approx (0.2463622, 0.0018786)$ corresponding to $t = t_{18}$.

Clearly, solutions in items (IV) and (V) can be discarded because the corresponding values of x_3 are negative. The solutions given in items (III) and (VI) are not good either, since both options result in negative m_5 values. In short the central configurations are:

(I) $(x_3, y_3) = \left(\frac{1+\sqrt{5}}{4}, \frac{\sqrt{10+2\sqrt{5}}}{4}\right)$ and $y_5 = \frac{\sqrt{5+2\sqrt{5}}}{2}$, with masses $m_j = \frac{1}{5}$, for all j.
(II) $(x_3, y_3) = (x_3^+(7), y_3^+(7)) \approx (0.5402091, 0.9991913)$ and $y_5 = y_5(7) \approx 0.1576605$ with masses $m_1 = m_2 \approx 0.0922539$, $m_3 = m_4 \approx 0.3860949$ and $m_5 \approx 0.04330243$.

The values of y_3 and m_j are obtained from Proposition 3 and the expressions (6) and (7).

The first solution provides the regular pentagon of Fig. 3(a) as a convex central configuration of the 5-body problem with masses equal to 1/5. While the second one provides the equilateral concave pentagon of Fig. 3(b) as a concave central configuration of the 5-body problem with the masses given in the statement (b) of the theorem. This completes the proof of Theorem 1.

3.1. Alternative approaches to solve system $h_1 = 0, h_2 = 0$

In this section we comment about to alternative approaches two solve this system and its equivalent one $H_1 = 0$, $H_2 = 0$.

A first one consists on computing the Gröbner basis of the two polynomials H_1 and H_2 with respect to the two variables s and t. Doing this we obtain three polynomial equations whose common solutions are also the solutions of system $H_1 = 0$, $H_2 = 0$. We do not provide explicitly these three polynomials, but only comment that they are huge and their expressions need many pages. The first one essentially coincides with $P(t)$ given in (10). The second one $P_2(s, t) = (1 + t^2)^2 p_{259}(s, t)$. depends on both variables s and t, and P_2 is linear in the variable s. Consequently each root $t = t_j$ of the polynomial P provides a single value of s from $P_2(s, t_j) = 0$, say $s = s_j$. Then we only need to keep the j's such that $s_j \in S$. Finally, the third polynomial $P_3(s, t)$ of the Gröbner basis has degree 262 but it is only cubic in the variable s. By keeping only the values $s_j \in S$ that also satisfy $P_3(s_j, t_j) = 0$ we arrive to the actual solutions. We have used our approach instead of this one because it is not easy to check analytically all the above facts because only two of the eighteen roots of P are known analytically. Moreover, we prefer our point of view because the computation of a resultant is simple and self contained while the computation of a Gröbner basis is implemented in the computer algebra systems but the user has no control on what the algorithm is doing.

A second alternative approach would consist on computing also $Q(s) = \mathrm{Res}_t(H_1, H_2)$. In this case we arrive to

$$Q(s) = q_2^6(s) q_4^2(s) q_{120}(s) q_{132}(s),$$

for some polynomials q_k of degree k, where here q_4 is different to the one given in (10). Their respective number of real roots are 0, 4, 28 and 32. Moreover, only 1, 0, 4 and 6 of them are in S. Call them s_m, $m = 1, 2, \ldots, 11$. Hence all the possible solutions of system $H_1 = 0, H_2 = 0$ in $S \times T$ are given by 11×18 values (s_m, t_j). Then, a discarding process, similar to the one done in our proof of Theorem 1 can be done. On the other hand, a proof that the non discarded candidates to be solutions are actual solutions can be done for instance by using the nice Poincaré–Miranda theorem. See for instance [22] to have more details of how to utilize this approach. We have not used it in our work because the expression of h_2 is much simpler that the one of Q.

CRediT authorship contribution statement

M. Alvarez-Ramírez: Conceptualization, Methodology, Software, Writing – original draft, Writing – review & editing, Project administration, Funding acquisition. **A. Gasull:** Conceptualization, Methodology, Software, Writing – original draft, Writing – review & editing, Project administration, Funding acquisition. **J. Llibre:** Conceptualization, Methodology, Software, Writing – original draft, Writing – review & editing, Project administration, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. The expression of H_1

The polynomial H_1 in (9) writes as

$$H_1(s,t) = R_0(s)\left(t^{10}+1\right) - R_1(s)\left(t^9+4t^7-t\right) + R_2(s)\left(t^8+t^2\right) + R_3(s)\left(t^7+t^3\right) + \sum_{j=4}^{6} R_j(s)t^j,$$

where

$$\begin{aligned}
R_0 =& s^2(2s-1)^2\Big(140137001s^{20} - 473336800s^{19} - 77771662s^{18} + 3288160224s^{17} \\
&- 8637659443s^{16} + 12537556864s^{15} - 12225124968s^{14} + 8691543680s^{13} \\
&- 4785798270s^{12} + 2180211392s^{11} - 880204628s^{10} + 324883264s^9 - 105646862s^8 \\
&+ 28078976s^7 - 5656040s^6 + 796288s^5 - 67435s^4 + 1568s^3 + 306s^2 - 32s + 1\Big), \\
R_1 =& 2s(s-1)(3s-1)(2s-1)\Big(402088273s^{20} - 1176961940s^{19} + 169440330s^{18} \\
&+ 4244422908s^{17} - 9214008723s^{16} + 10491664368s^{15} - 7879610248s^{14} \\
&+ 4350977648s^{13} - 1966298574s^{12} + 810673640s^{11} - 313304452s^{10} \\
&+ 104139528s^9 - 26338798s^8 + 4329392s^7 - 227208s^6 - 97872s^5 + 33277s^4 - 5620s^3 + 586s^2 - 36s + 1\Big), \\
R_2 =& 2610735845s^{24} - 15402964740s^{23} + 36242909513s^{22} - 34656361080s^{21} \\
&- 19279444736s^{20} + 100341955724s^{19} - 143714126267s^{18} + 121841849904s^{17} \\
&- 68503362257s^{16} + 27259846072s^{15} - 9818466526s^{14} + 5237546480s^{13} \\
&- 3390994016s^{12} + 1681689656s^{11} - 535976190s^{10} + 78574400s^9 + 17032379s^8 \\
&- 14081204s^7 + 4575253s^6 - 963768s^5 + 142992s^4 - 15044s^3 + 1081s^2 - 48s + 1, \\
R_3 =& 4s(2s-1)\Big(4011437555s^{22} - 17997365760s^{21} + 47815887103s^{20} - 119864245704s^{19} \\
&+ 262594976801s^{18} - 427430598888s^{17} + 498867338773s^{16} - 424920266400s^{15} \\
&+ 272105947982s^{14} - 136189975712s^{13} + 56438427910s^{12} - 20882782960s^{11}
\end{aligned}$$

$$
\begin{aligned}
&+ 7255673746s^{10} - 2308064816s^{9} + 623396538s^{8} - 133508384s^{7} + 21674927s^{6}\\
&- 2595360s^{5} + 227627s^{4} - 15240s^{3} + 861s^{2} - 40s + 1\Big),\\
R_4 =& - 128824963717s^{24} + 476911508392s^{23} + 165019719846s^{22} - 4572358588576s^{21}\\
&+ 13580634656322s^{20} - 22953475876792s^{19} + 26679405170926s^{18}\\
&- 23043609894160s^{17} + 15476921278317s^{16} - 8354767360048s^{15} + 3726249539948s^{14}\\
&- 1406580864896s^{13} + 458981618588s^{12} - 132125593264s^{11} + 34342034540s^{10}\\
&- 8246853600s^{9} + 1837269525s^{8} - 367787384s^{7} + 62442990s^{6} - 8479328s^{5}\\
&+ 879394s^{4} - 67480s^{3} + 3750s^{2} - 144s + 3,\\
R_5 =& - 1024s^{3}(2s-1)^{3}\Big(5478853s^{18} - 19634676s^{17} + 61412857s^{16} - 189035488s^{15}\\
&+ 395309060s^{14} - 534006384s^{13} + 487648452s^{12} - 316317344s^{11} + 154130630s^{10}\\
&- 61395832s^{9} + 22312398s^{8} - 7766560s^{7} + 2406420s^{6} - 594416s^{5}\\
&+ 108148s^{4} - 13664s^{3} + 1117s^{2} - 52s + 1\Big),\\
R_6 =& 138883898747s^{24} - 544790344792s^{23} + 35213761702s^{22} + 4234516514656s^{21}\\
&- 132185008214 38s^{20} + 22675637356616s^{19} - 26464806060818s^{18}\\
&+ 228126186790 24s^{17} - 15226596524179s^{16} + 8152770613200s^{15}\\
&- 3614885715604s^{14} + 1368000514176s^{13} - 453304331364s^{12} + 133617865040s^{11}\\
&- 35152022420s^{10} + 8179417120s^{9} - 1643965931s^{8} + 272577160s^{7} - 34266642s^{6}\\
&+ 2694560s^{5} - 25822s^{4} - 22424s^{3} + 2726s^{2} - 144s + 3
\end{aligned}
$$

Appendix B. The polynomial R_{60}

The polynomial $p_{120}(t)$ is reciprocal, that is $p_{120}(t) - t^{120}p_{120}(1/t) \equiv 0$. Notice that if $\hat{t}$ is one of its roots, $1/\hat{t}$ is another one. Hence there is a standard trick to "reduce" its degree to the half. Consider the numerator of $t + 1/t = u$, that is $t^2 - ut + 1$ and compute the resultant between it and $p_{120}(t)$. We obtain that

$$\mathrm{Res}_t\big(p_{120}(t), t^2 - ut + 1\big) = (R_{60}(u))^2,$$

where R_{60} is the polynomial of degree $120/2 = 60$,

$$
\begin{aligned}
&u^{60} + 4u^{59} - 480u^{58} - 2368u^{57} + 102656u^{56} + 661504u^{55} - 12378112u^{54} - 114393088u^{53}\\
&+ 813367296u^{52} + 13487570944u^{51} - 6779043840u^{50} - 1119258411008u^{49}\\
&- 4587041849344u^{48} + 63761809408000u^{47} + 556458915659776u^{46} - 2085902406909952u^{45}\\
&- 38793866899357696u^{44} - 14062083704356864u^{43} + 1871701598900584448u^{42}\\
&+ 6613561341561012224u^{41} - 6343943608 1954553856u^{40} - 4688791470348435456 00u^{39}\\
&+ 1328797291115756650496u^{38} + 2099331883823001082 2656u^{37} - 2305632465931114381312u^{36}\\
&- 6807377826227033126993 92u^{35} - 1102380299141445066948608u^{34} + 1643380206977791982095 5648u^{33}\\
&+ 5170531182181291723954 5856u^{32} - 2874548392902863516378 07104u^{31}\\
&- 1467660179422186654016733184u^{30} + 3159833116868066015124127744u^{29}\\
&+ 3023582460165037659602393 4976u^{28} - 3647821652057127970278473728u^{27}\\
&- 4787868819797446834982276 30080u^{26} - 6921463691588692637217938472 96u^{25}\\
&+ 6031919651486804311277903544320u^{24} + 1700905067117134541295575891 9680u^{23}\\
&- 6265738827283917318163240740 4544u^{22} - 2493635100602370958781779919 50336u^{21}\\
&+ 5583944304187734352801309623 58272u^{20} + 2603575445697403174765988761 567232u^{19}\\
&- 4366163704989486187475739888058368u^{18} - 2018389990605551664588201602 6329088u^{17}\\
&+ 2941713603864244058850535383339008u^{16} + 1171214192230470385462066249 94820096u^{15}\\
&- 1626808944477931782056031540163379 20u^{14} - 5056824227310874466357607560 82081792u^{13}\\
&+ 7005327984217623022789750219205509 12u^{12} + 1597276835067625721595839817 342517248u^{11}
\end{aligned}
$$

$$
\begin{aligned}
&- 22402453820485365839598367660750929 92u^{10} - 35745667979451694825988575798346383 36u^9\\
&+ 50532203758825881244334900271518842 88u^8 + 53363830963591690952222176023559536 64u^7\\
&- 74236864334901698018914771267024322 56u^6 - 46730671726813448656774466962980864 00u^5\\
&+ 59347953093053079794103573042985697 28u^4 + 15992274324287269095873928693997895 68u^3\\
&- 99692099683868690467785529521025843 2u^2 + 33230699894622896822595176507008614 4u\\
&- 13292279957849158729038070602803445 76.
\end{aligned}
$$

Then all the roots of p_{120} can be obtained simply by computing the roots of R_{60} and then for each one of them, say $u = \hat{u}$, two roots of p_{120} are given by the solutions of the quadratic equation $t^2 - \hat{u}t + 1 = 0$.

By computing its Sturm sequence we get that R_{60} has exactly 14 real roots, all them simple (of course half the number the real roots of p_{120}). One of them is $u = u^* \approx 2.0970716051$ and the value $t = t^*$ appearing in Remark 2 is the smallest root of $t^2 - u^*t + 1 = 0$.

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