KNEADING THEORY FOR A CLASS OF CIRCLE MAPS*

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Abstract. We give a kneading theory for the class of continuous maps of the circle of degree one with a single maximum and a single minimum. For a map of this class we characterize the set of itineraries depending on the rotation interval. From this result we obtain lower and upper bounds of the topological entropy and of the number of periodic orbits of each period. These lower bounds appear to be valid for a general continuous map of the circle of degree one.

We denote by $e : \mathbf{R} \longrightarrow S^1 = \{z \in \mathbf{C} : |z| = 1\}$ the natural projection $e(x) = exp(2\pi i x)$. A continuous map $F : \mathbf{R} \longrightarrow \mathbf{R}$ is called a *lifting* of a continuous map $f : S^1 \longrightarrow S^1$ if $e \circ F = f \circ e$. Note that if F is such a map, then there is $k \in \mathbf{Z}$ such that $F(x+1) = F(x) + k$ for all $x \in \mathbf{R}$. This k is called the *degree* of F. Also we note that every lifting of f is of the form $F + m$ with $m \in \mathbf{Z}$.

A continuous map $F : \mathbf{R} \longrightarrow \mathbf{R}$ will be called *old* if $F(x+1) = F(x) + 1$ for all $x \in \mathbf{R}$ (old stands for "degree one lifting"; see [M]). It is easy to see that if F is old, then $F(x+k) = F(x) + k$ for all $x \in \mathbf{R}$ and $k \in \mathbf{Z}$ and that iterates of old maps are old maps. In this paper, unlike in [M], every old map will be continuous. We also note that every old map is a lifting of a continuous map of the circle into itself of degree one.

We shall say that a point $x \in \mathbf{R}$ is *periodic (mod. 1) of period q with rotation number* $\frac{p}{q}$ for an old map F if $F^q(x) - x = p$ and $F^i(x) - x \notin \mathbf{Z}$ for $i = 1, \ldots, q-1$. A periodic (mod. 1) point of period 1 will be called *fixed (mod. 1)*. Clearly if F is a lifting of f then x is periodic (mod. 1) for F if and only if $e(x)$ is periodic for f and their periods are equal.

Let F be an old map. For $x \in \mathbf{R}$ we define its *rotation number* as $\limsup_{n\to\infty} \frac{F^n(x)-x}{n}$ and we denote it by $\rho(x)$ or $\rho_F(x)$ (see [NPT]). We note that if $e(x) = e(y)$, then $\rho_F(x) = \rho_F(y)$ and if x is a periodic (mod. 1) point of F with rotation number $\frac{p}{q}$, then $\rho_F(x) = \frac{p}{q}$. We denote by L_F the set of all rotation numbers of F.

From [I] it follows that L_F is a closed interval on $\mathbf{R}$ (perhaps degenerated to one point). Thus, from now on L_F will be called the *rotation interval* of F.

Let F be an old map. Assume that F is a lifting of f. We define the *topological entropy* of F, $h(F)$, as the topological entropy of f (see [AKM] or [DGS]).

In this paper we study the following class $\mathcal{A}$ of maps (see Figure 1). We say that $F \in \mathcal{A}$ if:

(1) F is old.
(2) There exists $c_F \in (0,1)$ such that F is strictly increasing in $[0, c_F]$ and strictly decreasing in $[c_F, 1]$.

*This is a summary of [AM]