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On the Length, Area and Volume of Lattice Figures

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Abstract. Some explicit formulas for length, area and volume of a large class of lattice figures are given and short proofs are presented.

Let L denote the fundamental lattice in the real space R^3 consisting of all points with integer coordinates. A unit segment will be any open segment of unit length whose endpoints belong to L. A unit rectangle (resp. cube) will be any open rectangle (resp. cube) of unit area (resp. volume) whose vertices belong to L.

An L–path p will be the closure of a finite union of unit segments. An L–surface S will be the closure of a finite union of unit rectangles. An unbranched L–surface U is an L–surface which has the additional property that none of the unit segments contained in U is incident with more than two of its closed unit rectangles. Lastly, an L–polyhedron P will be the closure of a finite union of unit cubes. Then the following hold:

$$\text{length}\,(p) = L(p) - \chi(p), \tag{1}$$

$$\text{area}\,(S) = \frac{1}{2}[L_2(S) - 2L(S) + \chi(S)], \tag{2}$$

$$\text{area}\,(U) = L(U) - \chi(U) - \frac{1}{2}\text{length}\,(\partial U), \tag{3}$$

$$\text{volume}\,(P) = \frac{1}{6}[L_2(P) - 2L(P) + \chi(P) - \text{area}\,(\partial P)], \tag{4}$$

where $L(K)$ denotes the number of points of L which belong to the set $K, \chi(K)$ denotes the Euler–Poincaré characteristic of K (see [CF]), ∂_K denotes the boundary of $K, L_2(K)$ denotes the number of points of L_2 which belong to K, and L_2 denotes the lattice defined as follows: the point (a,b,c) belongs to L_2 if and only if $(2a,2b,2c)$ belongs to L.

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