

On the periodic orbits of the perturbed two and three bodies problems

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Abstract

The first and second types of periodic orbits of the rotating Kepler problem can persisted for all perturbed circular restricted three–body problem when the perturbation forces are conservative, or the perturbed motion has its own extended Jacobian integral.

Keywords: Restricted 3–body problem, First(Second) type periodic orbits, Conservative forces, First integral of motion

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1. Introduction

In *Celestial Mechanics*, the periodic orbits have a considerable importance due to the existence of direct relation among the periodic orbits and the motion of planetary systems and most of stellar systems motion too. Thus there are in the literature of celestial mechanics many researches on the periodic orbits. In 1897 Henri Poincaré studied the periodic orbits and he considered that the exploration of such orbits is an important point for understanding the dynamics of the differential systems. Through his study on the restricted three–body problem, he distinguished three classes of periodic orbits [1]:

- First the ones emerging from the circular periodic orbits of the rotating Kepler problem,
- Second the ones coming from the elliptical periodic orbits of rotating Kepler problem,
- Third the ones generated from the spatial rotating Kepler problem where the plane of motion is inclined.

The analytical studies on such three types periodic orbits are developed in [2, 3]

In space science the importance of periodic orbits is not restricted to *Celestial Mechanics* only, but also to *Astrodynamicics*. There is a necessity for finding periodic orbits for designing different types of space missions. For example in [4] the presence of periodic orbits of the first type is confirmed by using the small parameter technique, when the period of a spacecraft motion and the undisturbed circular orbit of the primaries are equal. Furthermore the obtained results can be applied for designing and ballistic analyses for electrically powered thrusting spacecraft. Also the polar periodic orbits with $\pi/2$ inclination are employed to observe the planet surface subjected to space mission [5, 6].

Periodic solutions or periodic orbits play outstanding roles not only in physical, mathematical, engineering systems but also in biological biosphere systems, where a paramount problem is to estimate or explore if the automatic oscillatory activity can be continued when it is subjected to a small external effect. Many researchers have studied the perseverance and continuation problems of the periodic orbits, although these studies are carried out from several standpoint all of them got on the same basic outcome under an assumption on the period manifold, which is so–called normal non-degeneracy, where this assumption is expressible as a condition on the unperturbed flow over the manifold [7–13].

Using Delaunay variables the periodic orbits of second type have been investigated in [14]. The presence of second type period orbits is developed under the perturbation secular potential for the perturbed three–body problem [15]. The existence of first type orbits in the framework of restricted photo–gravitational

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four-body problem has been studied in [16]. Recently in [17] the authors have studied the resonance transition structures using the continuation method. Then they have described how 1 : 1 resonance is progressively overlapped in the proximity of the first order resonance through the perturbation increases. They also proposed a technique to find these orbits as well as the transition resonance with respect to Sun–Jupiter system. Further considerable studies on periodic orbits that would serve as an excellent guide to the readers have been performed in [18–25]

The main result of this paper is included in the following theorem:

Theorem 1. *The circular and elliptic periodic orbits of the planar rotating Kepler problem can be prolonged to the periodic orbits of first and second type for the perturbed two-body problem and the perturbed planar circular restricted three-body problem when the perturbation is done with conservative forces.*

2. Equations of perturbed motion

We assume that the perturbed potential between two masses m_i and m_j can be written in the following form

$$\Gamma_{ij} = Gm_i m_j \left[\frac{1}{\rho_{ij}} + F_{ij}(\rho_{ij}, \varepsilon_{ij}) \right], \quad (1)$$

where ρ_{ij} is the distance between the two centers of the masses m_i and m_j . While $F_{ij}(\rho_{ij}, \varepsilon_{ij})$ is the perturbation force such that it tends to zero (infinity) when ρ_{ij} tends to infinity (zero), it can be also written as $F_{ij}(\rho_{ij}, \varepsilon_{ij}) = \varepsilon_{ij} \Upsilon_{ij}(\rho_{ij})$, where $\Upsilon_{ij}(\rho_{ij})$ is differentiable n -times with respect to ρ_{ij} . In addition, ε_{ij} is a small parameter and its value depends on the kind of perturbation force.

2.1. Perturbed mean motion

Let $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ be the positions vectors of masses m_1 and m_2 respectively in the inertial frame, while $\boldsymbol{\rho}_{12} = \boldsymbol{\rho}_2 - \boldsymbol{\rho}_1$ is the relative position vector of m_2 with respect to m_1 . In addition, let \mathbf{F}_r ($r = 1, 2$) be the force between the masses m_2 and m_1 .

$$\mathbf{F}_1 = -\mathbf{F}_2 = \frac{\partial \Gamma_{12}}{\partial \rho_{12}} \hat{\boldsymbol{\rho}}_{12}, \quad (2)$$

where $\hat{\boldsymbol{\rho}}_{12}$ is the unit vector in the direction of $\boldsymbol{\rho}_{12}$ and Γ_{12} is the perturbed potential experienced by the second body and given by Eq. (1). Thus the equations of motion of the two bodies are

$$\begin{aligned} m_1 \ddot{\boldsymbol{\rho}}_1 &= \mathbf{F}_1, \\ m_2 \ddot{\boldsymbol{\rho}}_2 &= \mathbf{F}_2. \end{aligned} \quad (3)$$

Utilizing Eqs. (1), (2) and (3) then the vectorial equation of motion of the mass m_2 with respect to the first mass m_1 is

$$\ddot{\boldsymbol{\rho}}_{12} = - \left(\frac{m_1 + m_2}{m_1 m_2} \right) \nabla \Gamma_{12}. \quad (4)$$

In 2-dimension the operator ∇ can be written in the polar coordinates (ρ_{12}, φ) as

$$\nabla = \hat{\boldsymbol{\rho}}_{12} \frac{\partial}{\partial \rho_{12}} + \hat{\boldsymbol{\varphi}} \frac{1}{\rho_{12}} \frac{\partial}{\partial \varphi}, \quad (5)$$

where ρ_{12} is the radial distance and $\varphi \in [0, 2\pi]$ is the polar angle. Of course $\hat{\boldsymbol{\rho}}_{12}$ and $\hat{\boldsymbol{\varphi}}$ are two orthogonal units vectors such that the first refers to the direction of the second body related to the first one and $\hat{\boldsymbol{\varphi}}$ indicates to the direction of increasing the polar angle φ . Hence the vector acceleration in polar coordinates is identified by

$$\ddot{\boldsymbol{\rho}}_{12} = (\ddot{\rho}_{12} - \rho_{12} \dot{\varphi}^2) \hat{\boldsymbol{\rho}}_{12} + \frac{d}{dt} (\rho_{12}^2 \dot{\varphi}) \hat{\boldsymbol{\varphi}}. \quad (6)$$

Substituting Eqs. (5) and (6) into (4) we obtain

$$(\ddot{\rho}_{12} - \rho_{12} \dot{\varphi}^2) \hat{\boldsymbol{\rho}}_{12} + \frac{d}{dt} (\rho_{12}^2 \dot{\varphi}) \hat{\boldsymbol{\varphi}} = - \frac{(m_1 + m_2)}{m_1 m_2} \frac{\partial \Gamma_{12}}{\partial \rho_{12}} \hat{\boldsymbol{\rho}}_{12}. \quad (7)$$

From Eq. (1) we obtain

$$\frac{\partial \Gamma_{12}}{\partial \rho_{12}} = -Gm_1 m_1 \left[\frac{1}{\rho_{12}^2} - \varepsilon_{12} \frac{d\Upsilon_{12}}{d\rho_{ij}} \right]. \quad (8)$$

Now utilizing Eqs. (8) and (7) we get

$$(\ddot{\rho}_{12} - \rho_{12}\dot{\varphi}^2)\hat{\rho}_{12} + \frac{d}{dt}(\rho_{12}^2\dot{\varphi})\hat{\varphi} = -G(m_1 + m_2)\left[\frac{1}{\rho_{12}^2} - \varepsilon_{12}\frac{d\Upsilon_{12}}{d\rho_{ij}}\right]\hat{\rho}_{12}. \quad (9)$$

Eq. (9) has particular solutions as: $\rho_{12} = \text{constant}$ and $\dot{\varphi} = \varpi$, where ϖ is the mean motion which can be written as

$$\varpi^2 = \frac{G(m_1 + m_2)}{\rho_{12}}\left[\frac{1}{\rho_{12}^2} - \varepsilon_{12}\frac{d\Upsilon_{12}}{d\rho_{ij}}\right]. \quad (10)$$

2.2. Perturbed three-body motion

The following vectorial equation of motion of the infinitesimal body in the framework of classical restricted three-body problem in normalized synodic coordinates is given in [26]:

$$\ddot{\boldsymbol{\rho}} + 2\boldsymbol{\varpi} \wedge \dot{\boldsymbol{\rho}} = \nabla V, \quad (11)$$

where

$$V = \frac{1}{2}|\boldsymbol{\varpi} \wedge \boldsymbol{\rho}|^2 + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}. \quad (12)$$

It is assumed that the universal constant of gravitation, the distance between the two primaries, the sum of their masses and the angular velocity on the circular orbit is equal to one, see for details [26]. Thus the mass of the massive big primary is $m_1 = 1 - \mu$ and the small primary is $m_2 = \mu$, with $\mu = m_2/(m_1 + m_2) \in (0, 1/2)$, and that in synodical coordinates the position of the massive primary is $(\xi_1, \eta_1, \zeta_1) = (-\mu, 0, 0)$ and of the small one is $(\xi_2, \eta_2, \zeta_2) = (1 - \mu, 0)$, while the position of the infinitesimal body is (ξ, η, ζ) . Further, we denote the magnitudes of vector $\boldsymbol{\rho}$, $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ by ρ , ρ_1 and ρ_2 respectively, under these assumptions we get

$$\begin{aligned} \boldsymbol{\varpi} &= [0 \quad 0 \quad \varpi]^T, \\ \boldsymbol{\rho}_1 &= [(\xi + \mu) \quad \eta \quad \zeta]^T, \\ \boldsymbol{\rho}_2 &= [(\xi + \mu - 1) \quad \eta \quad \zeta]^T, \\ \boldsymbol{\rho} &= [\xi \quad \eta \quad \zeta]^T, \\ \dot{\boldsymbol{\rho}} &= [\dot{\xi} \quad \dot{\eta} \quad \dot{\zeta}]^T, \end{aligned} \quad (13)$$

here ϖ equals one and in Eqs. (13) the distances among the infinitesimal body and the primaries are

$$\begin{aligned} \rho_1^2 &= (\xi + \mu)^2 + \eta^2 + \zeta^2, \\ \rho_2^2 &= [(\xi + \mu - 1)^2 + \eta^2 + \zeta^2], \\ \rho^2 &= \xi^2 + \eta^2 + \zeta^2. \end{aligned} \quad (14)$$

In the case that the motion of the infinitesimal body is perturbed by very small forces with respect to the main gravitational forces of the primaries bodies the infinitesimal body will be affected by an additional acceleration in synodic frame, which is called the perturbing acceleration \mathcal{A} . Using Eqs. (8), (11) and (12) the perturbed motion of the infinitesimal body in synodic frame is given by

$$\ddot{\boldsymbol{\rho}} + 2\boldsymbol{\varpi} \wedge \dot{\boldsymbol{\rho}} = \nabla \Psi, \quad (15)$$

where $\nabla \Psi = \nabla(V + \mathcal{A})$, and \mathcal{A} may be written as $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3$, here the magnitudes of the perturbing accelerations are $\mathcal{A}_1 = \varepsilon_1 \Upsilon(\rho_1)$ and $\mathcal{A}_2 = \varepsilon_2 \Upsilon(\rho_2)$ due to the massive and smaller primaries respectively, while $\mathcal{A}_3 = \varepsilon_3 \Upsilon(\rho)$. Hence Ψ will be given by

$$\Psi = \frac{1}{2}\varpi^2(\xi^2 + \eta^2) + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} + \mathcal{A}, \quad (16)$$

where ρ_1 and ρ_2 are given in (14), and by using Eq. (10) then the perturbed mean motion ϖ can be calculated by

$$\varpi^2 = 1 - \varepsilon_{12}\Upsilon'_{12}, \quad (17)$$

where $\Upsilon'_{12} = d\Upsilon_{12}/d\rho_{12}$ when $\rho_{12} = 1$.

Utilizing Eqs. (15) and (16) the three dimensional equations of motions in the cartesian synodic frame are controlled by

$$\begin{aligned}\ddot{\xi} - 2\varpi\dot{\eta} &= \Psi_\xi, \\ \ddot{\eta} + 2\varpi\dot{\xi} &= \Psi_\eta, \\ \ddot{\zeta} &= \Psi_\zeta,\end{aligned}\tag{18}$$

where $\Psi_\xi = \partial\Psi/\partial\xi$, $\Psi_\eta = \partial\Psi/\partial\eta$, $\Psi_\zeta = \partial\Psi/\partial\zeta$.

Eqs. (18) represents the dynamical system of the perturbed motion of the infinitesimal body in the framework of the spatial restricted three-body problem, when the perturbing forces are conservative. Thus, Eqs. (16) and (17) admit a Jacobian integral of the following form

$$\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2 = 2\Psi - \mathcal{C},\tag{19}$$

where \mathcal{C} is the Jacobian constant, which can be used to identify the real motion and the possible regions of motion as well.

3. The Hamiltonian of perturbed motion

From [27] and using Eqs. (11) and (12), the Hamiltonian (\mathcal{H}_0) of the *spatial unperturbed circular restricted 3-body problem* in synodical coordinates is

$$\mathcal{H}_0(\xi, \eta, \zeta, \mathcal{P}_\xi, \mathcal{P}_\eta, \mathcal{P}_\zeta) = \frac{1}{2}(\mathcal{P}_\xi^2 + \mathcal{P}_\eta^2 + \mathcal{P}_\zeta^2) + \eta\mathcal{P}_\xi - \xi\mathcal{P}_\eta - \frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2},\tag{20}$$

where $(\mathcal{P}_\xi, \mathcal{P}_\eta, \mathcal{P}_\zeta)$ is the conjugate momentum of the infinitesimal body at the position (ξ, η, ζ) .

While in the case that the infinitesimal body is moving under some conservative perturbing force or its motion is governed by system (18) and admits the Jacobian integral (19), then the Hamiltonian of the perturbed motion is given by

$$\mathcal{H}(\xi, \eta, \zeta, \mathcal{P}_\xi, \mathcal{P}_\eta, \mathcal{P}_\zeta) = \mathcal{H}_0(\xi, \eta, \zeta, \mathcal{P}_\xi, \mathcal{P}_\eta, \mathcal{P}_\zeta) + \varepsilon_{12}\Upsilon'_{12}(\eta\mathcal{P}_\xi - \xi\mathcal{P}_\eta) + \varepsilon_1\Upsilon(\rho_1) + \varepsilon_2\Upsilon(\rho_2).\tag{21}$$

It is clear that the Hamiltonians (20) and (21) are equivalent in the absence of the perturbed forces.

Since the quantities ε_{12} , ε_1 , ε_2 and ε_3 are small such that they can be written with a factor $\lambda \ll 1$, i.e. $\varepsilon_{12} = \lambda\bar{\varepsilon}_{12}$, $\varepsilon_1 = \lambda\bar{\varepsilon}_1$, $\varepsilon_2 = \lambda\bar{\varepsilon}_2$ and $\varepsilon_3 = \lambda\bar{\varepsilon}_3$. In this context the Hamiltonian (21) becomes

$$\mathcal{H}(\xi, \eta, \zeta, \mathcal{P}_\xi, \mathcal{P}_\eta, \mathcal{P}_\zeta) = \frac{1}{2}(\mathcal{P}_\xi^2 + \mathcal{P}_\eta^2 + \mathcal{P}_\zeta^2) + \eta\mathcal{P}_\xi - \xi\mathcal{P}_\eta - \frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2} + \mathcal{O}(\lambda).\tag{22}$$

Moreover if the mass ratio μ is very small, such that $\mu = \lambda\bar{\mu}$, we have that the Hamiltonian (22) becomes

$$\mathcal{H}(\xi, \eta, \zeta, \mathcal{P}_\xi, \mathcal{P}_\eta, \mathcal{P}_\zeta) = \frac{1}{2}(\mathcal{P}_\xi^2 + \mathcal{P}_\eta^2 + \mathcal{P}_\zeta^2) + \eta\mathcal{P}_\xi - \xi\mathcal{P}_\eta - \frac{1}{\sqrt{\xi^2 + \eta^2 + \zeta^2}} + \mathcal{O}(\lambda).\tag{23}$$

We must take care with $\mathcal{O}(\lambda)$ because it has terms that go to infinity near the primaries; so we must exclude a neighborhood of the primaries. When the infinitesimal body is moving in the primaries' plane we have that $\zeta = 0$ and $\mathcal{P}_\zeta = 0$, hence the Hamiltonian (23) becomes

$$\mathcal{H}(\xi, \eta, \mathcal{P}_\xi, \mathcal{P}_\eta) = \frac{1}{2}(\mathcal{P}_\xi^2 + \mathcal{P}_\eta^2) + \eta\mathcal{P}_\xi - \xi\mathcal{P}_\eta - \frac{1}{\sqrt{\xi^2 + \eta^2}} + \mathcal{O}(\lambda).\tag{24}$$

Then equations of motion related to the Hamilton (24) will take the following form

$$\begin{aligned}\ddot{\xi} &= 2n\dot{\eta} + n^2\xi - \frac{\xi}{(\xi^2 + \eta^2)^{3/2}} + \mathcal{O}(\lambda), \\ \ddot{\eta} &= -2n\dot{\xi} + n^2\eta - \frac{\eta}{(\xi^2 + \eta^2)^{3/2}} + \mathcal{O}(\lambda).\end{aligned}\tag{25}$$

Note that the Hamiltonian (24) and system (25) for $\lambda = 0$ coincides with the planar Kepler problem in a rotating reference frame.

4. Proof of Theorem 1

In this section we study the existence of first and second type periodic orbits for the perturbed problem. It is clear that system (25) represents Kepler motion in the rotating reference frame or the so-called rotation Kepler problem when the parameter of mass ratio (μ equals to zero. This means that the third body or the infinitesimal body will be moving under the effect of the gravitational field of the massive body only. In this case, the massive body will take a place at the origin of reference frame.

We also remark that if $\lambda = 0$ the periodic solutions of system (25) are circular or elliptical orbits, namely $(\xi(t), (\eta(t)))$ when the motion of the two-body is bounded. In general, these orbits are called the first (second) type when the unperturbed motion is represented by circular (elliptical) orbits.

We emphasize that system (25) is the same or similar to the system in [2, page 4, Eqs. (6)]. Therefore in the case of $\lambda = 0$ (or $\lambda \neq 0$, but small, this means that also the parameter of mass ratio μ is very small and $\mu \neq 0$) the obtained results in [2] can be applied to our system of perturbed motion (25). Hence the results of Theorems 3.1, 3.2 and 5.1 in [2] can be applied to the perturbed circular restricted three-body motion. These theorems will be stated and indexed in the present work as Theorems 2, 3 and 4 in Appendix A.

Furthermore the proof of Theorem 1 comes direct from the proofs of Theorems 3.1, 5.1 in [2], here Theorems 2 and 4. Thereby the circular (elliptical) periodic orbits of the planar rotating Kepler problem can be prolonged to the periodic orbits of first (second) types for the perturbed two-body problem and also for the perturbed restricted three-body problem by conservative perturbing forces. This means that both theorems are satisfied when the perturbed model has a first integral or the perturbed motion has the so-called Jacobian constant.

Now we show that Theorem 1 can be applied to many models, for example to the perturbed two-body problem by different perturbed conservative forces, some of these models are listed in what follows:

1. Perturbed two-body problem where both bodies are oblate [28].
2. An anisotropic perturbed Kepler problem [29].
3. Relativistic effect is a source of perturbation [30].
4. Perturbed two-body problem when the main body has a continuation fraction potential [31].

While we can list also some related works for the perturbed restricted three-body problem as the following ones:

1. Oblateness of a massive primary body is a source of perturbed force [32].
2. Oblateness of a smaller primary with the radiation of a bigger primary are sources of perturbed forces [33].
3. Lake of sphericity of the three participated is a source of perturbed forces as well as the primaries' radiations [34].
4. Effect of zonal harmonic coefficients (J_2 and J_4) of the massive body are considered [35].
5. First primary is oblate while the second is a source of radiation considered [36].
6. Effect of zonal harmonic coefficients (J_2 and J_4) of both primaries bodies are considered [37].
7. Both primaries are taken as triaxial rigid bodies [38].
8. Asteroids belt effect is considered as a source of a perturbation force [39].
9. Quantum corrections are considered as a source of perturbation forces [40].

The perturbed models for two and three bodies with their own first integral are not limited to the stated in the previous two lists which are mentioned as examples.

Appendix A. Theorems 2, 3 and 4

Theorem 2. *Assume that the restricted three-body problem is perturbed through very small perturbed force and that the perturbed system has a first integral. Also we assume that $1/(1 - nj^3) \notin \mathbb{Z}$ where $j^3 \neq 1/n$ and \mathbb{Z} is the set of integer numbers. Then the periodic orbits of first type with angular momentum j can be continued to the perturbed circular restricted three-body problem.*

Theorem 3. *Assume that the restricted three-body problem is perturbed through a small perturbed force and that the perturbed system has a first integral. If $\tau_0 \neq (l+1/2)\pi/n$ with $l \in \mathbb{Z}^+$ where \mathbb{Z}^+ is the set of positive integers numbers and $\epsilon > 0$ is very small, then the perturbed restricted problem has periodic orbits, which tend to the periodic orbits of first type with τ_0 -period when ϵ tends to zero.*

Theorem 4. *Assume that I and J are prime integers, $K \in \mathbb{Z}$ where \mathbb{Z} is the set of integers numbers and $\tau = 2\pi I/|K|$. Then the elliptical solution with τ -period for the rotating Kepler problem satisfying*

$$g(0) = -\pi, \quad l(0) = \pi, \quad L^3(0) = I/J,$$

and such that it does not cross the location of the smaller primary can be continued to the perturbed circular restricted three-body problem when $\epsilon > 0$ is small and its motion equations (9) (in Reference [2]) come from equations (6) (in Reference [2]) satisfying the following invariant symmetry by $(x, y, t) \rightarrow (x, -y, -t)$.

Author contributions

Formal analysis: E.I.A., J.L.G.G and J.L.; Investigation: E.I.A., J.L.G.G; Methodology: E.I.A., J.L.G.G and J.L.; Project administration: J.L.; Software: E.I.A., J.L.G.G and J.L.; Validation: E.I.A., J.L.G.G and J.L.; Visualization: E.I.A., J.L.G.G and J.L.; Writing—original draft: E.I.A., and J.L.; Writing—review & editing: E.I.A and J.L.; Approval of the version of the manuscript to be published: E.I.A., J.L.G.G and J.L.;

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Data availability

The study does not report any data.

Declarations

Conflict of interest

The authors declare no conflict of interest

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