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The dynamics of the relativistic Kepler problem

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A R T I C L E I N F O

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A B S T R A C T

We deal with the Hamiltonian system (HS) associated to the Hamiltonian in polar coordinates $H = \frac{1}{2}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) - \frac{1}{r} - \frac{\epsilon}{2r^2}$, where ϵ is a small parameter. This Hamiltonain comes from the correction given by the special relativity to the motion of the two-body problem, or by the first order correction to the two-body problem coming from the general relativity. This Hamiltonian system is completely integrable with the angular momentum C and the Hamiltonian H. We have two objectives.

First we describe the global dynamics of the Hamiltonian system (HS) in the following sense. Let S_h and S_c are the subset of the phase space where $H = h$ and $C = c$, respectively. Since C and H are first integrals, the sets S_c, S_h and $S_{hc} = S_h \cap S_c$ are invariant by the action of the flow of the Hamiltonian system (HS). We determine the global dynamics on those sets when the values of h and c vary.

Second recently Tudoran (2017) provided a criterion which detects when a non-degenerate equilibrium point of a completely integrable system is Lyapunov stable. Every equilibrium point q of the completely integrable Hamiltonian system (HS) is degenerate and has zero angular momentum, so the mentioned criterion cannot be applied to it. But we will show that this criterion is also satisfied when it is applied to the Hamiltonian system (HS) restricted to zero angular momentum.

1. Introduction

In celestial mechanics, the Kepler problem is a special dynamical system coming from the two-body problem. In this model, two objects move under their mutual Newtonian gravitational force [1]. This attractive force varies in size as the inverse square of the separation distance between them and it is proportional to the product of the masses of the two bodies. This system is used to find the position and the velocity vectors of the two bodies at specified time. Using the laws of classical mechanics, the solution can be provided as a Kepler orbit through finding the six orbital elements. This problem is called the Kepler problem in honor of the German astronomer Johannes Kepler, after he proposed Kepler's laws of the motion of the planets and illustrated the kinds of forces which can provide orbits obeying those laws [2].

The Kepler problem appears in various fields, some of these are beyond the physics, which have been studied by Kepler himself. This problem is important in celestial mechanics because the Newtonian gravity obeys the law of inverse square distance between two bodies. Thus for example, the motion of two stars around each other, the planets moving surrounding the Sun, a satellite orbiting a planet, and many other examples of orbital motion. The Kepler problem has also a significant relevance in the study of the motion of two charged particles, because Coulomb's law of electrostatics follows the law of inverse square distance too. For Example the hydrogen atom, muonium and positronium, which have played serious roles as models of dynamical systems to test physical theories and measuring constants of nature, see [3–5].

The importance of the Kepler problem is not only related to its varied applications in different fields of science, but rather to its use in developing some mathematical methods. Thus, this problem has been used to develop new methods in classical mechanics, like Hamiltonian mechanics, the Hamilton-Jacobi equation, Lagrangian mechanics and action-angle coordinates. Furthermore, the Kepler and simple harmonic oscillator problems are two of the most fundamental problems in classical mechanics. They are also the only integrable dynamical systems

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