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PERIODIC POINTS OF $\mathcal{C}^1$ MAPS AND THE ASYMPTOTIC LEFSCHETZ NUMBER

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We study the set of periodic points and the set of periods of different classes of $\mathcal{C}^1$ self maps of a compact manifold. We give sufficient conditions in order that these sets be infinite. Our main tools are the Lefschetz fixed point theory and homology.

1. Introduction and Statement of Main Results

Let M be a compact, connected manifold of dimension n and let f: $M \to M$ be a continuous map of M. A point $x \in M$ is called *periodic* if there exists some $m \in \mathbb{N}$ such that $f^m(x) = x$. We call the least such m the *period* of x under f. If the period of x under f is 1 then we call x a *fixed point* of f. Let $\mathrm{Fix}(f)$ denote the set of all fixed points of f and let $\mathrm{Per}(f)$ denote the set of all periods of f. Finally we denote the set of all periodic points of f by $\mathrm{PP}(f)$.

We are interested in studying the set $\mathrm{Per}(f)$. One good way to do this is via the Lefschetz fixed point theory which relates the action induced by the map on the homology groups of the manifold and the fixed points of the map. For $k = 0, 1, \ldots, n$, let f_{*k}: $H_k(M; \mathbb{Q}) \to H_k(M; \mathbb{Q})$ be the endomorphism induced by f on the kth rational homology group of M. Let f_* be the block matrix which has the matrices f_{*k} on the diagonal for $k = 0, 1, \ldots, n$ and zeroes elsewhere. Let $(f_*)^e$ (respectively $(f_*)^o$) be the block matrix which has the matrices f_{*k} for k even (respectively k odd) on the diagonal and zeroes elsewhere. Given a matrix A, define its *spectral radius* $\sigma(A)$ to be equal to the maximum of the absolute values of its eigenvalues. Let $\mathrm{Tr}(A)$ denote the trace of A. The *Lefschetz number* $L(f)$ of f is defined to be:

$$L(f) = \sum_{k=0}^{n} (-1)^k \mathrm{Tr}(f_{*k}) = \mathrm{Tr}((f_*)^e) - \mathrm{Tr}((f_*)^o)\,.$$

If $L(f) \neq 0$ then f has a fixed point by the Lefschetz fixed point theorem [Brown, 1971].

In our work we shall assume that f is $\mathcal{C}^1$. We shall also assume that f: $M \to \mathrm{Int}(M)$ is *transversal*, i.e., that if $m \in \mathbb{N}$ and $x \in \mathrm{Fix}(f^m)$ then $\det(I - df^m(x)) \neq 0$. So 1 is not an eigenvalue of $df^m(x)$. Notice that if f is transversal then the graph of f^m intersects the diagonal $\{(y, y) : y \in M\}$ transversally at each point (x, x) such that $x \in \mathrm{Fix}(f^m)$.

The main results of this paper are the following:

Theorem 1. *Let M be a compact manifold, and suppose that f: $M \to \mathrm{Int}(M)$ is a $\mathcal{C}^1$ map and is transversal. Further, assume that the limit $\lim_{m\to\infty} |\mathrm{Tr}(F_*^m)|^{1/m}$ exists for $F_* \in \{(f_*)^e, (f_*)^o\}$ and that $\sigma((f_*)^e) \neq \sigma((f_*)^o)$. If there is an eigenvalue of f_* different from a root of unity and zero then there*