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ORIENTATION-PRESERVING SELF-HOMEOMORPHISMS OF THE SURFACE OF GENUS TWO HAVE POINTS OF PERIOD AT MOST TWO

WARREN DICKS AND JAUME LLIBRE

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ABSTRACT. We show that for any orientation-preserving self-homeomorphism α of the double torus Σ_2 there exists a point p of Σ_2 such that $\alpha(\alpha(p)) = p$. This answers a question raised by Jakob Nielsen in 1942.

1. Background

Throughout this article, R will denote a commutative ring, and g a positive integer. We shall write Σ_g to denote the closed, connected, orientable surface of genus g, and $\mathbb{Z}_g$ to denote the ring $\mathbb{Z}/g\mathbb{Z}$.

Nielsen [10] (cf. [7]) showed that, for any $g \geq 2$, there exists an orientation-preserving self-homeomorphism α of Σ_g such that $\alpha, \alpha^2, \dots, \alpha^{2g-3}$ are all fixed-point free, that is, have no fixed-points. He showed further that, for any orientation-preserving self-homeomorphism α of Σ_g, at least one of $\alpha, \alpha^2, \dots, \alpha^{2g-3}, \alpha^{2g-2}$ has a fixed-point if $g \geq 3$, and at least one of $\alpha, \alpha^2, \alpha^3$ has a fixed-point if $g = 2$.

Since $2g - 2 = 2$ for $g = 2$, this left open the question of exactly what happens in the case $g = 2$, and he commented that it seemed difficult to him [10, Section 4] (cf. [7]). The problem has not been forgotten; for example, in a recent paper in which he obtains results analogous to Nielsen's for orientation-reversing self-homeomorphisms and for non-orientable surfaces, Wang [13] mentions that it is still open. In the next section we solve this problem, rounding off Nielsen's above-mentioned results by showing that, for $g = 2$, at least one of α, α^2 has a fixed-point, or, equivalently, α^2 has a fixed-point.

This result has its origins in classical topology, has connections with dynamical systems, and has a proof which is mainly algebraic. Let us review the basic information that will be used.

Recall that the fundamental group of Σ_g has a one-relator presentation

$$\pi_1(\Sigma_g) = \langle x_i, y_i \, (1 \leq i \leq g) \mid \prod_{i=1}^{g} [x_i, y_i] = 1 \rangle,$$

where $[x, y]$ denotes $xyx^{-1}y^{-1}$. Recall also that the first homology R-module of Σ_g, denoted $\mathrm{H}_1(\Sigma_g, R)$, is the R-module obtained by abelianizing the fundamental group and then tensoring over $\mathbb{Z}$ with R. There is a natural group homomorphism

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