PERIODIC ORBITS AND NON-EXISTENCE OF C^1 FIRST INTEGRALS FOR ANALYTIC DIFFERENTIAL SYSTEMS EXHIBITING A ZERO-HOPF BIFURCATION IN $\mathbb{R}^3$

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Abstract. In this paper we study the zero-Hopf bifurcation of three dimensional analytic differential systems. We get sufficient conditions for the existence of two periodic orbits via the averaging theory. We prove the non-existence of C^1 first integrals in a neighborhood of these periodic orbits, by using the relation between the characteristic multipliers of the variational equation of the original system along a periodic orbit and the eigenvalues of the equilibrium of the averaging system, which is established here for the first time.

1. Introduction and statement of the main results

In the analytic differential system having an equilibrium the existence of analytic or meromorphic first integrals in a neighborhood of this equilibrium depends on the resonance of the eigenvalues of the linearized system at the equilibrium, see for instance [4, 10, 11, 7, 16, 17]. The existence of analytic or meromorphic first integrals in a neighborhood of a regular orbit depends on commutation of the connected component containing the identity element of the differential Galois group of the variational equation of the system along the regular orbit, see for example [1, 2, 9, 12].

Recently the local integrability of three dimensional analytic differential systems at the zero-Hopf equilibrium has been studied by Yagasaki [14, 15]. An equilibrium point of a differential system in $\mathbb{R}^3$ such that the eigenvalues of the Jacobian matrix of the system at the equilibrium are 0 and ωi and $-\omega i$ with $\omega > 0$ is called a *zero-Hopf equilibrium point*.

In 2023 Yagasaki [14] studied the analytic non-integrability of the differential system

$$
\begin{aligned}
x' &= -\omega y + axz - byz,\\
y' &= \omega x + bxz + ayz,\\
z' &= c(x^2 + y^2) + dz^2,
\end{aligned}
\tag{1}
$$

at the equilibrium point localized at the origin of the coordinate system. System (1) is the second order cutoff of the Poincaré-Dulac normal form of an analytic

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