THE DYNAMICS OF THE LADDER SYSTEM

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Abstract. We consider the n-dimensional ladder system, that is the homogeneous differential systems of the form

$$\dot{x}_i = x_i \sum_{j=1}^{n} (i+1-j)x_j, \quad i = 1, \ldots, n$$

introduced by Imai and Hirata for studying the integrability of a new class of Lotka-Volterra systems. Here we describe the dynamics of these Lotka-Volterra systems in arbitrary dimension.

1. Introduction and statement of the main results

Imai and Hirata using Lie point symmetries found in [8] a new integrable family in the class of the Lotka–Volterra systems $\dot{x}_i = x_i \sum_{j=1}^{n} a_{ij} x_j$ for $i = 1, \ldots, n$, see also [10]. More precisely, they found the n-dimensional ladder system

$$\dot{x}_i = x_i \sum_{j=1}^{n} (i+1-j)x_j \text{ for } i = 1, \ldots, n. \tag{1}$$

When $n = 1$ system (1) is the well-known Riccati equation. When $n = 2$ the 2-dimensional ladder system is reducible to the scalar second-order Ermakov–Pinney equation (which is well-known in physics). For $n = 3$ the 3-dimensional ladder system is reducible to the scalar third-order equation of maximal symmetry. This is one of the main reasons that this system has been previously studied by many authors. In [1] system (1) was studied using the Painlevé method. In [2] the ladder system is generalised to hyperladder system, and in [11] the superintebraility of the ladder system is analyzed.

In this paper we focus on the dynamical aspects of system (1). The main results of our paper are the following (see again section 2 for the definitions of all the notions that appear in the main theorems).

The first theorem is precisely the statement of the complete integrability of system (1) with very easy rational first integrals. Set $S_n = \sum_{i=1}^{n} x_i$.

Theorem 1. *The ladder system* (1) *with $n \geq 2$ has the invariant algebraic hyperplane $S_n = 0$ with cofactor S_n. Moreover, it is completely integrable with the $n-1$ independent rational first integrals*

$$H_k = \frac{x_k S_n}{x_{k+1}} = \frac{x_k \left(\sum_{j=1}^{n} x_j\right)}{x_{k+1}}, \quad k = 1, \ldots, n-1.$$

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