PHASE PORTRAITS OF THE EQUATION $\ddot{x} + ax\dot{x} + bx^3 = 0$

JAUME LLIBRE[1] AND CLAUDIA VALLS*[2]

Abstract. The second-order differential equation $\ddot{x} + ax\dot{x} + bx^3 = 0$ with $a, b \in \mathbb{R}$ has been studied by several authors mainly due to its applications. Here, for the first time, we classify all its phase portraits in function of its parameters a and b. This classification is done in the Poincaré disc in order to control the orbits which scape or come from infinity. We prove that there are exactly six topologically different phase portraits in the Poincaré disc of the first order differential system associated by the second-order differential equation. Additionally we show that this system is always integrable providing explicitly its first integrals.

1. Introduction and statement of the main results

In the book of Ince [7] or in the work of Painleve [12] appeared the second order ordinary differential equation

$$\ddot{x} + ax\dot{x} + bx^3 = 0,$$

with $a, b \in \mathbb{R}$. This differential equation is equivalent to the differential system of first order

$$\dot{x} = y, \quad \dot{y} = -axy - bx^3. \tag{1}$$

This differential system arises in many areas of mathematics such as the analysis of fusion of pellets [3], the theory of univalent functions [4], the stability of gaseous spheres [8], in the operator of the Yang-Baxter equations [5, 7, 9], in the description of the motion of a free particle in a space of constant curvature [15],...

This differential system possesses the Painlevé property and it has been studied by many authors due to its simple form, it is a Lienard differential equation and it possesses the algebra $sl(3, \mathbb{R})$ of Lie point symmetries. More concretely it is quite nonlinear and belongs to the class of equations of the form $\ddot{x} = x^3 f(\dot{x}/x^2)$ which posses the two Lie point symmetries ∂_t and $t\partial_t - x\partial_x$ for a general function f and also it has the eight Lie point symmetry characteristic of the representative second-order ordinary differential

2010 *Mathematics Subject Classification.* Primary 34A05. Secondary 34C05, 37C10.
Key words and phrases. second-order differential equation, Poincaré compactification, global phase portraits.