# PERIODIC ORBITS AND NON-EXISTENCE OF $C^1$ FIRST INTEGRALS FOR ANALYTIC DIFFERENTIAL SYSTEMS EXHIBITING A ZERO-HOPF BIFURCATION IN $\mathbb{R}^4$

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Abstract. In this paper we investigate the zero-Hopf bifurcation of a four dimensional analytic differential system. We prove that at most five periodic orbits bifurcate from the zero-Hopf equilibrium using the averaging theory of first order and give a specific example to illustrate this conclusion. Moreover we prove the non-existence of $C^1$ first integrals in a neighbourhood of these periodic orbits.

## 1. Introduction and statement of the main result

In an $n$-dimensional analytic autonomous system, there always exist analytic first integrals near a regular point (and there are $n-1$ functionally independent analytic first integrals). However, in general, this is not the case in a neighbourhood of an equilibrium point. The existence of analytic first integrals near them depends on the resonance of the eigenvalues of the linearized system at equilibrium. For details, see the book [11].

Recently Yagasaki [9, 10] studied the analytic non-integrability of three-dimensional analytic differential systems at a zero-Hopf equilibrium. A *zero-Hopf* equilibrium of an $n$-dimensional autonomous differential system is an equilibrium that has a pair of purely imaginary eigenvalues and the rest are all zero eigenvalues.

Motivated by the works of Yagasaki in [9, 10] Llibre and Zhang [6] studied the dynamics in a neighbourhood of a zero-Hopf equilibrium in $\mathbb{R}^3$. They obtained sufficient conditions for the existence of two periodic orbits bifurcating from the zero-Hopf equilibrium and proved the non-existence of $C^1$ first integrals in a neighbourhood of these periodic orbits.

In 2021 Llibre and Tian [4] studied a four-dimensional hyperchaotic system depending on six parameters. They characterized the values of the parameters for which their equilibria are zero-Hopf equilibria and obtain the sufficient conditions for the existence of four periodic orbits bifurcating from these zero-Hopf equilibria.

Inspired by the works [4, 6], we consider the following analytic differential system in $\mathbb{R}^4$ which has a zero-Hopf bifurcation at the equilibrium localized at the origin of

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