SMALL-AMPLITUDE PERIODIC SOLUTIONS IN THE JERK EQUATION OF ARBITRARY DEGREE

JAUME LLIBRE[1] AND XIANBO SUN[2]

Abstract. A zero-Hopf singularity for a 3-dimensional differential system is an singularity for which the Jacobian matrix of the differential system evaluated at it has eigenvalues zero and $\pm\omega i$ with $\omega \neq 0$. In this paper we study the periodic orbits bifurcating from the zero-Hopf singularity localized at the origin of coordinates of the general nth-degree jerk equation $\dddot{x} - \phi(x, \dot{x}, \ddot{x}) = 0$, where $\phi(*, *, *)$ is a nth-degree polynomial in three variables, i.e. we study the zero-Hopf bifurcations of such differential systems. We obtain the sharp upper bounds on the number of limit cycles that can bifurcate from this zero-Hopf bifurcation using the averaging theory of first and second order. After we apply our results to characterize the small-amplitude periodic traveling waves in generalized non-integrable Kawahara equation. To do this we study a jerk equation on a normally hyperbolic critical manifold.

1. Introduction

The Newton equation $\ddot{x} = \psi(x)$, Liénard equation, $\ddot{x} = \psi(x) + \chi(x)\dot{x}$, and the jerk equation

$$\dddot{x} = \phi(x, \dot{x}, \ddot{x}) \tag{1}$$

are three basic models describing the motion of a single particle, where $\psi(*)$, $\chi(*)$ and $\phi(*, *, *)$ are certain functions, the dot is the time derivative, x, $\dot{x}$, $\ddot{x}$ and $\dddot{x}$ represent the displacement, velocity, acceleration and the changing rate of acceleration (also called jolt or jerk in physics).

The Newton equation is a conservative model, the total energy of the particle is equal to the plus of Kinetic energy and potential energy. It can admit continuous-amplitude periodic motion, in other words Newton equation admits a family of periodic solutions for any function $\phi(x)$ with a positive derivative near a central position. It is feasible in describing the periodic phenomenon in ideal environments. When the dissipative factor $\chi(x)\dot{x}$ is considered, one obtains the Liénard system, which is also known as a nonlinear oscillator. It has many applications in mechanic, electronic, chemistry, construction industry (for wind-resistant design of tall buiding), etc. After adding the dissipation, the integrability of Liénard system is lost and the family of persisting periodic solutions may be broken, only a finite number of periodic solutions persist as limit cycles. For the Liénard equation, researchers in mathematics communities study the integrability problem [1], center problem [2], monotonicity of period function [4], and number of limit cycles [5]. Smale [6] proposed to study the small-amplitude limit cycles of polynomial Liénard system with $\psi(x) = x$ as a weak version of Hilbert's 16th problem [7]. This problem is still open for $\deg \chi(x) > 4$, see [8]. When $\chi(x)$ has a very small factor ε, the number of zeros of the related Abelian integral were systemically investigated for the cubic Liénard systems, see [9, 10] and references therein, which is related to the weak Hilbert's problem proposed by Anorld [7]. There exist much to do for studying the weak

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