GENERALIZATION OF THE LAPLACE-RUNGE-LENZ CONSERVATION LAW

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Abstract. Our main result is related with the solution of the following inverse problem of dynamics.

Let $\mathbb{E}^N$ be the N dimensional Euclidean space with coordinates $\mathbf{x} = (x_1, \ldots, x_N)$. For a particle with configuration space $\mathbb{E}^N$ which moves under the action of the potential field of force $U_{\mathbf{x}} = \left(\dfrac{\partial U}{\partial x_1}, \ldots, \dfrac{\partial U}{\partial x_N}\right)$, and for a given homogeneous function f we determine the potential U and the function Φ in such a way that

$$\Phi\, f_{\mathbf{x}} - \Big(<\dot{\mathbf{x}}, \dot{\mathbf{x}}> \mathbf{x} - <\dot{\mathbf{x}}, \mathbf{x}> \dot{\mathbf{x}} \Big) = \mathbf{0},$$

is a conservation law, where $\dot{\mathbf{x}} = \dfrac{d\mathbf{x}}{dt}$, t is the time and $<\ ,\ >$ is the inner product.

In particular if $N = 3$, $f = \sqrt{x^2+y^2+z^2} + <\mathbf{b}, \mathbf{x}>$, $\Phi = 1$ and $U = 1/\sqrt{x^2+y^2+z^2}$, then we obtain the classical conservation law of the the Kepler problem where $\mathbf{b}$ is the Laplace-Runge-Lenz vector.

1. Introduction and statement of the main results

We introduce the following notations. Let $\mathbb{E}^N$ be the three dimensional Euclidean space with coordinates $\mathbf{x} = (x_1, x_2, \ldots, x_N)$. By $<\ ,\ >$ we denote the inner product in $\mathbb{E}^N$, and by $f_{\mathbf{x}} = \left(\dfrac{\partial f}{\partial x_1}, \dfrac{\partial f}{\partial x_2}, \ldots, \dfrac{\partial f}{\partial x_N}\right) := (f_{x_1}, f_{x_2}, \ldots, f_{x_N})$, $r = \sqrt{x_1^2 + \ldots + x_N^2}$, and

$$\{f_1, \ldots, f_N\} = \begin{vmatrix} \dfrac{\partial f_1}{\partial x_1} & \ldots & \dfrac{\partial f_1}{\partial x_N} \\ \vdots & \ldots & \vdots \\ \vdots & \ldots & \vdots \\ \dfrac{\partial f_N}{\partial x_1} & \ldots & \dfrac{\partial f_N}{\partial x_N} \end{vmatrix}.$$

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