# POLYNOMIAL DIFFERENTIAL SYSTEMS WITH INVARIANT ALGEBRAIC CURVES OF ARBITRARILY HIGH DEGREE

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Abstract. In 1891 Poincaré asked: *Given $m \geq 2$, is there a positive integer $M(m)$ such that if a polynomial differential system of degree $m$ has an invariant algebraic curve of degree $\geq M(m)$, then it has a rational first integral?* Brunella and Mendes repeated the same open question in 2000, and Lins-Neto in 2002. Between the years 2001 and 2003 three different families of quadratic polynomial differential systems provided a negative answer to this question. Here we characterize the phase portraits of these three families of quadratic systems, and a new open question appears: *Given $m \geq 2$, is there a positive integer $M(m)$ such that if a polynomial differential system of degree $m$ has an invariant algebraic curve of degree $\geq M(m)$, then it has a Darboux invariant?*

## 1. Introduction and statement of the main results

A *polynomial differential system* in $\mathbb{R}^2$ is a differential system of the form

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y), \tag{1}$$

where the dependent variables $x$ and $y$, and the independent one (the time) $t$ are real, and $P$ and $Q$ are polynomials in the variables $x$ and $y$ with coefficients in $\mathbb{R}$. The number $m = \max\{\deg P, \deg Q\}$ is the *degree* of the polynomial differential system (1).

Let $f = f(x,y) \in \mathbb{R}[x,y]$. i.e. $f$ is a polynomial. Then $f = 0$ is an *invariant algebraic curve* of the polynomial differential system (1) if $f$ satisfies the equation

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf,$$

for some polynomial $K \in \mathbb{R}[x,y]$, called the *cofactor* of $f = 0$.

The polynomial differential system (1) is *integrable* on an open subset $U$ of $\mathbb{R}^2$ if there exists a non-locally constant analytic function $H : U \to \mathbb{R}$, called a *first integral* of the system on $U$, which is constant on all solution curves $(x(t), y(t))$ of system (1) contained in $U$. Moreover, if $H$ is a rational function then $H$ is a *rational first integral.*

In the papers of Brunella and Mendes [2] and Lins Neto [8] the following open question about polynomial differential systems appeared:

2010 *Mathematics Subject Classification.* 34C0, 37C27.

*Key words and phrases.* Polynomial differential systems, invariant algebraic curve, rational first integral, Darboux invariant.