LIMIT CYCLES BIFURCATING FROM A ZERO-HOPF EQUILIBRIUM OF THE LOTKA-VOLTERRA SYSTEMS IN $\mathbb{R}^n$

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Abstract. It is known that the Lotka-Volterra systems in $\mathbb{R}^2$ cannot have a zero-Hopf bifurcation, and that the zero-Hopf bifurcations of the Lotka-Volterra systems in $\mathbb{R}^3$ have been investigated, therefore in this paper we study the limit cycles of the Lotka-Volterra systems in $\mathbb{R}^n$ with $n \geq 4$ which bifurcate from a positive zero-Hopf equilibrium point, using the averaging theory of first and second order.

1. Introduction and statement of the main results

Since the well-known Lotka-Volterra systems was proposed ([16, 22]), an increasing interest appeared for studying the dynamics of the n-dimensional Lotka-Volterra systems, and later on their extension to the n-dimensional Kolmogorov systems. This interest is mainly due to the fact that these differential systems are used widely to model different natural phenomena, see for example, biology ([18]), chemical reactions ([9]) and economics ([21]). The n-dimensional Lotka-Volterra systems are of the form

$$\dot{x}_i = x_i P_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n, \tag{1}$$

where $x_i \geq 0$ for $i = 1, \ldots, n$, and $P_i(x_1, \ldots, x_n)$ is a linear polynomial in the variables $(x_1, \ldots, x_n)$ for $i = 1, \ldots, n$. The periodic oscillation phenomena occur frequently in system (1) (see [5, 18, 19]). The Hopf bifurcation is one of the ways for producing periodic solutions (see [7, 10, 17]).

It was shown that the Lotka-Volterra systems (1) with $n = 2$ cannot have a limit cycle (see [1] or the survey [4]). The zero-Hopf bifurcation of system (1) with $n = 3$ was studied in [13] using the averaging theory, which was developed recently in [3, 6, 8, 11, 12, 20]. Moreover some authors investigated the zero-Hopf bifurcation for general polynomial differential systems (see [13] for 3-dimensional systems using the averaging theory of second order, and [14] for n-dimensional systems using the averaging theory of first order).

It is an interesting problem to study the zero-Hopf bifurcation of the Lotka-Volterra systems (1) in $\mathbb{R}^n$ with $n \geq 4$ using the averaging theory of first and second order. Assume that the n-dimensional Lotka-Volterra systems (1) has a positive equilibrium point $(e_1, \ldots, e_n)$ with $e_i > 0$ for $i = 1, \ldots, n$. Applying the rescaling $x_i \to x_i/e_i$ for $i = 1, \ldots, n$, without loss of generality we can assume

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