DYNAMICS OF THE PAINLEVÉ-INCE EQUATION

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ABSTRACT. The Painlevé-Ince differential equation $y'' + 3yy' + y^3 = 0$ has been studied from many points of view. Here we complete its study providing its phase portrait in the Poincaré disc.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

The Painlevé–Ince differential equation

$$\frac{d^2y}{dt^2} + 3y\frac{dy}{dt} + y^3 = 0, \tag{1}$$

has been studied for several authors due to its interesting properties:

(i) It has eight Lie point symmetries with the Lie Algebra $sl(2, R)$ consequently through a point transformation it is linearisable, see [11].
(ii) It has a Riccati hierarchy based on the Riccati differential equation with the operator $\frac{d}{dy} + y$, see [5].
(iii) It satisfies the Painlevé property, see [10].
(iv) Its left Painlevé series together with its well known right Painlevé series have been studied in [6, 7, 8].
(v) Its mixed Painlevé series together with their geometric interpretations were studied in [2].

Extensions of the Painlevé–Ince differential equation (1) can be found in [9].

It is easy to check that the general solution of the Painlevé–Ince differential equation (1) is

$$y(t) = \frac{2a(b+t)}{3 + ab^2 + 2abt + at^2},$$

where a and b are arbitrary constants. From this general solution we can determine the constants a and b for each particular solution $y(t)$ such that $y(t_0) = y_0$ and $y'(t_0) = y_0'$. But to know these explicit solutions, it is not easy to determine the qualitative properties of these solutions, i.e. where they are born, where they die tells, if they define homoclinic orbits, or heteroclinic orbits, ...

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