GLOBAL DYNAMICS OF THE ERMAKOV-PINNEY EQUATION

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ABSTRACT. The Ermakov-Pinney second order differential equation

$$\ddot{x} + \omega^2 x = \frac{h^2}{x^3},$$

has been studied from many different points of view. Here we study its global dynamics through the differential system of first order

$$\dot{x} = y, \qquad \dot{y} = -\omega^2 x + \frac{h^2}{x^3},$$

classifying all their phase portraits in the Poincaré disc.

1. Introduction and statement of the main results

Much attention has been paid to the Ermakov-Pinney differential equation

$$\ddot{x} + \omega^2 x = \frac{h^2}{x^3},$$

where the dot denotes derivative with respect to the time t. This differential equation was introduced in 1880 by Ermakov [3] studying the integrability of the second order ordinary differential equations presenting an oscillation. Seventy years later Pinney in [7] provided its explicit solution

$$x(t) = \sqrt{Au(t)^2 + 2Bu(t)v(t) + Cv(t)^2},$$

where the constants A, B and C satisfying $AC - B^2 = h^2/W$, being $u(t)$ and $v(t)$ two linearly independent solutions of the harmonic oscillator $\ddot{x} + \omega^2 x = 0$ and W is their Wronskian. But from these explicit solutions it is not easy to obtain the dynamics of its orbits. On the other hand, the Ermakov-Pinney differential equation admits the rescaling symmetry $x\partial_x + \frac{1}{2}y\partial_y$ and its hierachy is studied in [4]. Moreover it has the three element algebra of Lie point symmetries $sl(2, \mathfrak{R})$, see [6]. Some integrability of the Ermakov-Pinney equation has been studied in [5]. Now if you introduce in MathSciNet in anywhere the words "Ermakov-Pinney equation" it appears 61 papers related with this differential equation.

In any case it is unknown the dynamics of all the orbits of the Ermakov-Pinney equation. Thus we first transform this second order differential equation in the differential system of first order

$$\dot{x} = y, \qquad \dot{y} = -\omega^2 x + \frac{h^2}{x^3}. \tag{1}$$

Since the parameters ω and h appear in the differential system (1) through ω^2 and h^2 without loss of generality we can assume that $\omega \geq 0$ and $h \geq 0$.

Our objective is to describe all the phase portraits of the differential systems (1) in the Poincaré disc when its parameters ω and h vary. See an introduction of the Poincaré disc in section 2.2. Roughly speaking the Poincaré disc is the unit closed disc whose interior has been identified with the plane $\mathbb{R}^2$ and its boundary, the circle $\mathbb{S}^1$, with the infinity of $\mathbb{R}^2$.

A phase portrait of a differential system consists in describing the domain of definition of the differential system as union of all the orbits of the differential system. Thus from a phase portrait we can see where the orbits start and where they end (i.e. their α- and ω-limits),

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