# CHARACTERIZATION OF THE TREE CYCLES WITH MINIMUM POSITIVE ENTROPY FOR ANY PERIOD

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Abstract. Consider, for any integer $n \geq 3$, the set $\mathrm{Pos}_n$ of all $n$-periodic tree patterns with positive topological entropy and the set $\mathrm{Irr}_n \subset \mathrm{Pos}_n$ of all $n$-periodic irreducible tree patterns. The aim of this paper is to determine the elements of minimum entropy in the families $\mathrm{Pos}_n$, $\mathrm{Irr}_n$ and $\mathrm{Pos}_n \setminus \mathrm{Irr}_n$. Let $\lambda_n$ be the unique real root of the polynomial $x^n - 2x - 1$ in $(1, +\infty)$. We explicitly construct an irreducible $n$-periodic tree pattern $\mathcal{Q}_n$ whose entropy is $\log(\lambda_n)$. We prove that this entropy is minimum in $\mathrm{Pos}_n$. Since the pattern $\mathcal{Q}_n$ is irreducible, $\mathcal{Q}_n$ also minimizes the entropy in the family $\mathrm{Irr}_n$. We also prove that the minimum positive entropy in the set $\mathrm{Pos}_n \setminus \mathrm{Irr}_n$ (which is nonempty only for composite integers $n \geq 6$) is $\log(\lambda_{n/p})/p$, where $p$ is the least prime factor of $n$.

## 1. Introduction

The field of Combinatorial Dynamics has its roots in the striking Sharkovskii's Theorem [31], in the sense that the theory grew up as a succession of progressive refinements and generalizations of the ideas contained in the original proof of that result. The core of the theory is the notion of *combinatorial type* or *pattern*.

Consider a class $\mathcal{X}$ of topological spaces (closed intervals of the real line, trees, graphs and compact surfaces are classic examples) and the family $\mathcal{F}_\mathcal{X}$ of all maps $\{f\colon X \longrightarrow X : X \in \mathcal{X}\}$ satisfying a given property (continuous maps, homeomorphisms, etc). Any of such maps gives rise, by iteration, to a discrete dynamical system. Assume now that we have a map $f\colon X \longrightarrow X$ in $\mathcal{F}_\mathcal{X}$ which is known to have a periodic orbit $P$. The *pattern of* $P$ is the equivalence class $\mathcal{P}$ of all maps $g\colon Y \longrightarrow Y$ in $\mathcal{F}_\mathcal{X}$ having an invariant set $Q \subset Y$ that, at a combinatorial level, behaves like $P$. In this case, we say that every map $g$ in the class *exhibits* the pattern $\mathcal{P}$. Of course we have to precise in which sense a periodic orbit *behaves as* $P$. So, we have to decide which feature of $P$ has to be preserved inside the equivalence class $\mathcal{P}$. The period of $P$, just a natural number, is a first possibility (Sharkovskii's Theorem), but a richer option arises from imposing that

(a) the relative positions of the points of $Q$ inside $Y$ are the same as the relative positions of $P$ inside $X$
(b) the way these positions are permuted under the action of $g$ coincides with the way $f$ acts on the points of $P$.

An example is given by the family $\mathcal{F}_\mathcal{M}$ of surface homeomorphisms. The pattern (or *braid type*) of a cycle $P$ of a map $f\colon M \longrightarrow M$ from $\mathcal{F}_\mathcal{M}$, where $M$ is a surface, is defined by the isotopy class, up to conjugacy, of $f\big|_{M\setminus P}$ [19, 27].

When $\mathcal{F}_\mathcal{X}$ is the family of continuous maps of closed intervals, the points of an orbit $P$ of a map in $\mathcal{F}_\mathcal{X}$ are totally ordered and the pattern of $P$ can be simply

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