PHASE PORTRAITS OF A GENERALIZED LIÉNARD SYSTEM WITH HOMOGENEOUS NONLINEARITIES

MONTSERRAT CORBERA AND CLAUDIA VALLS

Abstract. We classify the phase portraits in the Poincaré disc of the differential equations of the form $x' = -y + a\,xy^{n-1}$, $\dot{y} = x + b\,yx^{n-1}$ and $n \geq 2$. This system is a certain generalization of the classical Liénard system.

1. Introduction and statement of the main results

The Liénard equation given by

$$\ddot{x} + f(x)\dot{x} + g(x) = 0,$$

or equivalently the Liénard system

$$\dot{x} = y, \qquad \dot{y} = -g(x) - yf(x), \tag{1}$$

where $f(x)$ and $g(x)$ are analytic functions appears in many applications and it is probably one of the most studied model, see for instance [14, 26]. For this model there are many results on the existence and uniqueness of periodic orbits as well as lower bounds on the maximum number of bifurcating limit cycles (see [2, 3, 5, 7, 9, 11, 13, 16, 20, 21, 22, 23, 26, 27, 28, 29] and the references therein). Due to its importance several generalizations of Liénard equations have been proposed and different dynamical aspects have been studied. See [6, 4, 8, 10, 15, 19, 25], to cite just a few.

In this paper we consider the following generalization of the classical Liénard system (1)

$$\dot{x} = -y + a\,xy^{n-1}, \quad \dot{y} = x + b\,yx^{n-1}, \tag{2}$$

with $n > 1$ and $a, b \in \mathbb{R}$ not simultaneously 0. Without loss of generality we can assume that $a \neq 0$ since otherwise making the change of variables $(x, y) = (Y, -X)$ we can always achieve it, so system (2) becomes

$$\dot{x} = -y + axy^{n-1}, \quad \dot{y} = x + byx^{n-1}, \tag{3}$$

with $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$. With the change of variables and parameterization of time

$$X = a^{1/(n-1)}x, \quad Y = a^{1/(n-1)}y, \quad T = t$$

if either $a > 0$ or $a < 0$ and n is even, or

$$X = -(-a)^{1/(n-1)}x, \quad Y = (-a)^{1/(n-1)}y, \quad T = -t$$

if $a < 0$ and n is odd, we transform system (3) into system

$$x' = -y + xy^{n-1}, \quad y' = x + \alpha yx^{n-1}, \tag{4}$$

where $\alpha \in \mathbb{R}$, the prime means derivative with respect to the new time T and where we have renamed the new variables(X, Y) as the old ones (x, y) and the new time T as the old one t.

When $\alpha = 1$ system (4) is a particular case of the generalization of the Liénard system studied in [17]. In that paper it is proved when $\alpha = 1$ that the origin of (4) is a center if and only if n is even. Later on in [18] the authors proved that when n is odd system (4) has a center at the origin if and only if $\alpha = -1$ and that when $n = 2$ system (4) has a center at the origin for all $\alpha \in \mathbb{R}$.

2010 *Mathematics Subject Classification.* Primary 34C05. Secondary 34A34, 34C07.
Key words and phrases. Polynomial vector field, Liénard system, phase portrait, Poincaré disc.