PARALLEL VECTOR FIELDS AND GLOBAL INJECTIVITY IN TWO DIMENSIONS

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Abstract. Let U be simply connected open subset of $\mathbb{R}^2$, and let $f : U \to \mathbb{R}^2$ be a local diffeomorphism. In this paper we study the global injectivity of f using the planar vector fields of type annular, radial or strip. Our main result allows to unify proofs of classical results on global injectivity, as the Hadamard global invertibility theorem and the condition that the levels sets of one of the coordinates of f are connected.

1. Introduction and statement of the main results

In this paper we provide a relationship between parallel vector fields and global injectivity of local diffeomorphisms in dimension two.

Let $U \subset \mathbb{R}^2$. We say that a vector field $\mathcal{X} : U \to \mathbb{R}^2$ is *parallel* when it is topologically equivalent to one of the vector fields:

(a) $\mathcal{X} = (-y, x)$ in $\mathbb{R}^2 \setminus \{(0,0)\}$.
(r) $\mathcal{X} = (x, y)$ in $\mathbb{R}^2 \setminus \{(0,0)\}$.
(s) $\mathcal{X} = (1, 0)$ in $\mathbb{R}^2$.

A parallel vector field is called *annular*, *radial* or *strip* when it is topologically equivalent to the vector field (a), (r) or (s), respectively.

If a vector field $\mathcal{X} : U \to \mathbb{R}^2$ is such that $\mathcal{X}|_{U\setminus\{z\}}$, for some $z \in U$, is annular or radial, we simply say that $\mathcal{X}$ is *annular* or *radial surrounding* z, respectively. This clearly imposes that U is simply connected.

On the other hand, let $f = (f_1, f_2) : U \to \mathbb{R}^2$ be a C^k, $k \geq 1$, local diffeomorphism, i.e., such that its Jacobian determinant

$$Jf(z) = \det Df(z) = f_{1x}(z)f_{2y}(z) - f_{1y}(z)f_{2x}(z), \quad z \in U,$$

is nowhere zero. By the inverse function theorem f is a local diffeomorphism, but may fail to be a global one, as for instance $f(x, y) = (e^x \cos y, e^x \sin y)$ defined in $\mathbb{R}^2$, which is neither injective nor onto. It is an old problem to find additional conditions in order to "globalize" the inverse function theorem, i.e., in order to guarantee that f is a global diffeomorphism, or at least globally injective (and so a global diffeomorphism onto its image), see [18, 21].

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