THE HILBERT NUMBER OF SOME CLASSES OF DIFFERENTIAL EQUATIONS ON THE TORUS

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ABSTRACT. In this paper we characterize the maximum number of limit cycles that the following two classes of differential equations

$$\frac{de^{i\varphi}}{d\theta} = a_0(\theta) + a_1(\theta)e^{i\varphi} + \dots + a_n(\theta)e^{in\varphi},$$

and

$$\frac{de^{i\varphi}}{d\theta} = \frac{a(\theta)}{a_0(\theta) + a_1(\theta)e^{i\varphi} + \dots + a_n(\theta)e^{in\varphi}},$$

on the torus $\mathbb{S}^1 \times \mathbb{S}^1 = \left\{ \left(e^{i\alpha}, e^{i\beta}\right) | \alpha, \beta \in \mathbb{R} \right\}$ can have in function of the positive integer n, where the functions $a(\theta)$ and $a_i(\theta)$ for $i = 0, 1, \dots, n$ are continuous 2π-periodic functions.

1. Introduction and statements of the main results

A geometric torus is a surface of revolution created by rotating a circle in three-dimensional space around an axis of rotation that is coplanar with the circle. If there is no contact between the axis of rotation and the circle, the generated surface has a ring shape and is called *ring torus*. In topology, a ring torus is homeomorphic to the Cartesian product of two circles: $\mathbb{S}^1_a \times \mathbb{S}^1_b$. A one way to embed this space into Euclidean space is a ring torus but another way is the Cartesian product of the embedding of $\mathbb{S}^1$ in the plane with itself. This produces a geometric object called the *Clifford torus*. It resides in $\mathbb{R}^4$ because each of $\mathbb{S}^1_a$ and $\mathbb{S}^1_b$ exist in their own independent embedding spaces $\mathbb{R}^2_a$ and $\mathbb{R}^2_b$, respectively, then the resulting product space will be $\mathbb{R}^4$. Therefore, the Clifford torus can be seen as residing inside the complex coordinate space $\mathbb{C}^2$, since $\mathbb{C}^2$ is topologically equivalent to $\mathbb{R}^4$. It is also common to consider the Clifford torus as an embedded torus in $\mathbb{C}^2$. In two copies of $\mathbb{C}$, we have the following unit circles:

$$\mathbb{S}^1 = \{e^{i\alpha} | \alpha \in \mathbb{R}\}, \qquad \mathbb{S}^1 = \{e^{i\beta} | \beta \in \mathbb{R}\}$$

Now the points of the Clifford torus appears as

$$\mathbb{S}^1 \times \mathbb{S}^1 = \left\{ \left(e^{i\alpha}, e^{i\beta}\right) | \alpha, \beta \in \mathbb{R} \right\}.$$

As we know the Hilbert 16th problem for the planar real polynomial differential systems is about the existence of a finite upper bound for the maximum number of limit cycles in function of the degree of the polynomial differential system, see [4, 5, 6, 10]. Periodic orbits in polynomial planar differential systems can be isolated or belong to an annulus of periodic orbits. In the isolated case they are called *limit cycles*. Usually, the maximum of the number of limit cycles is denoted by $H(n)$, and is called the *Hilbert number*. Then here we extend this problem to the torus. Here the maximum of the number of limit cycles on the torus is denoted by $\mathbb{H}(n)$.

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