

Liouvillian integrability of three dimensional vector fields

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Abstract

We consider a three dimensional complex polynomial, or rational, vector field (equivalently, a two-form in three variables) which admits a Liouvillian first integral. We prove that there exists a first integral whose differential is the product of a rational 1-form with a Darboux function, or there exists a Darboux Jacobi multiplier. Moreover, we prove that Liouvillian integrability always implies the existence of a first integral that is obtained by two successive integrations from a one-forms with coefficients in a finite algebraic extension of the rational function field.

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1 Introduction and overview of results

The classical theory of integration in finite terms goes back to Liouville. For 20th century accounts we refer to the seminal works of Risch [23, 24] and Rosenlicht [25, 26]. Liouvillian functions play a special role in the integrability problem for functions, vector fields and differential forms. In an influential paper [27], Singer showed that the existence of a Liouvillian first integral of a two dimensional polynomial vector field is equivalent to the existence of a first integral whose differential is a closed rational 1-form. Moreover, such 1-forms are necessarily logarithmic differentials of *Darboux* functions, that is, functions of the form

$$\exp(g/f) \prod f_i^{a_i},$$

where the f_i , f and g are polynomials in the coordinate variables, and a_i are complex numbers.

Thus, by Singer's work in [27], Darbouxian integrability captures all closed form solutions of two dimensional systems. These solutions arise from rational functions via a finite sequence of adjoining integrals, exponentials and algebraic functions, see [27, 9]. This means that by the Darboux method one will obtain all Liouvillian first integrals.

It should be emphasized that Liouvillian integrability is not only of interest for its own sake but also relevant for applications. There is a number of publications that characterize the Liouvillian first integrals of certain planar families; pars pro toto we just mention Cairo et al. [6], Oliveira et al. [21]. We recall that the existence of a first integral has important consequences for the dynamics of a system; see for example García and Giné [15].

Several algorithmic procedures have been presented in the literature to obtain Liouvillian first integrals for two dimensional vector fields; for instance, some of them build on the classical Preller–Singer method [22]. See e.g. Avellar et al. [2], Chèze and Combot [8], Duarte and da Mota [13].

Singer's theorem [27] has been generalized in various ways, see for example Żołądek [29], Casale [5], and Zhang [28]. In particular, Żołądek in [29] presents a multi-dimensional version of Singer's theorem for rational 1-forms. Zhang [28] provides a generalization of Singer's Theorem to vector fields in n dimensions that admit Darbouxian Jacobi multipliers.

One should also mention work that characterizes Liouvillian first integrals of some families in three dimensions; see Ollagnier [19, 20], and some recent studies on integrability aspects of certain three dimensional systems; see Ferčec et al. [14], and also [18]. Concerning algorithms for the computation of Liouvillian first integrals in higher dimensions, see for instance Avellar et al. [3], Combot [11].

The objective of the present paper is to extend and modify Singer's theorem for complex polynomial or rational vector fields in three dimensions.

As a preliminary step we characterize rational closed one-forms in Theorem 1, and (re-)prove Singer's Theorem for rational one-forms in n variables in a purely algebraic manner; see Theorem 2.

With $K = \mathbb{C}(x, y, z)$, our first main result states that a polynomial vector field in three dimensions has a Liouvillian first integral only if one of the following holds: (i) There exists a first integral whose differential is the product of a Darboux function with a 1-form over K , or (ii) there exists an inverse Jacobi multiplier over K of Darboux type, see Theorem 3. (In case (i), similar to the planar setting, the search for Liouvillian integrals is reduced to the semi-algorithmic search for invariant algebraic surfaces and their associated exponential factors.)

Our second main result, see Theorem 4, implies that a three dimensional Liouville integrable system always admits a finite algebraic extension \tilde{K} of K with the following property: There exist 1-forms ω, α over \tilde{K} and an integral whose differential equals $\omega \cdot \exp(\int \alpha)$. This is a version of Singer's theorem for vector fields in three dimensions, with K being replaced by a finite algebraic extension.

2 Background

We start by recalling some basic definitions and facts from differential algebra. For more details see e.g. the monograph by Kolchin [17]. Moreover we will prove (or reprove) some preliminary results on Liouvillian integrability. Fields are always assumed to be of characteristic zero.

A *differential field* is a pair (K, Δ) where K is a field together with a finite set Δ of derivations of K . Thus for all $\partial \in \Delta$ and all $x, y \in K$ one has the identities

$$\partial(x + y) = \partial x + \partial y, \quad \partial(xy) = (\partial x)y + x(\partial y).$$

We will restrict attention to commutative differential fields, that is the derivations in Δ commute.

The *constants* of (K, Δ) are those elements $x \in K$ such that $\partial x = 0$ for all $\partial \in \Delta$, and the subfield of constants will be denoted by C_K .

A *differential extension* of (K, Δ) is a differential field $(\tilde{K}, \tilde{\Delta})$ where \tilde{K} is an extension field of K and each derivation $\tilde{\partial} \in \tilde{\Delta}$ restricts (uniquely) to an element $\partial \in \Delta$. Therefore, it is natural to write (\tilde{K}, Δ) .

We will be mostly interested in the rational function field $\mathbb{C}(x_1, \dots, x_n)$, with $\Delta = \{\partial/\partial x_1, \dots, \partial/\partial x_n\}$, and its extensions.¹ Moreover, we focus on Liouvillian extensions (see also Singer [27] for the notion):

Definition 1. An extension $L \supset K$ of differential fields is called a *Liouvillian extension* of K if $C_K = C_L$ and if there exists a tower of fields of the

¹One could replace \mathbb{C} by any algebraically closed field of characteristic zero.

form

$$K = K_0 \subset K_1 \subset \dots \subset K_m = L, \quad (1)$$

such that for each $i \in \{0, \dots, m-1\}$ we have one of the following:

- (i) $K_{i+1} = K_i(t_i)$, where $t_i \neq 0$ and $\partial t_i/t_i \in K_i$ for all $\partial \in \Delta$; thus t_i is an exponential of an integral of some element of K_i .
- (ii) $K_{i+1} = K_i(t_i)$, where $\partial t_i \in K_i$ for all $\partial \in \Delta$; thus t_i is an integral of an element of K_i .
- (iii) $K_{i+1} = K_i(t_i)$, where t_i is algebraic over K_i .²

We will make extensive use of differential forms, which generally are more convenient both for the statements and proofs of our results than vector fields. If L is a differential extension of $K = \mathbb{C}(x_1, \dots, x_n)$ then we denote by L' the space of differential 1-forms with coefficients in L . That is, every 1-form $\alpha \in L'$ can be written as $\alpha = \sum a_i dx_i$ with $a_i \in L$. Since the x_i are algebraically independent, we can treat the dx_i simply as inert placeholders for the calculations, and do not need to invoke the more general theory of differentials of a field.

Likewise, we will work with forms of higher degree, and freely use the familiar properties of the exterior derivative operator d and of wedge products. Recall that one calls a form β *closed* whenever $d\beta = 0$, and *exact* when $\beta = d\theta$ for some form θ .

Remark 1. If L is a differential extension of $K = \mathbb{C}(x_1, \dots, x_n)$, then we can restate conditions (i)–(iii) in Definition 1 by the following Types:

- (i) $K_{i+1} = K_i(t_i)$, where $t_i \neq 0$ and $dt_i = \delta_i t_i$ with some $\delta_i \in K'_i$ (necessarily $d\delta_i = 0$).
- (ii) $K_{i+1} = K_i(t_i)$, where $dt_i = \delta_i$ with $\delta_i \in K'_i$ (necessarily $d\delta_i = 0$).
- (iii) K_{i+1} is a finite algebraic extension of K_i .

We note that the condition $C_K = C_L$ on constants can always be met in our reasoning for extensions of the rational function field K (see Singer [27]); so we will not mention it explicitly in the following.

As mentioned in the Introduction, a key role will be played by *Darboux functions*. These are functions of the form

$$\phi = \exp(g/f) \prod f_i^{a_i}, \quad (2)$$

where the f_i and g and f are elements of $\mathbb{C}[x_1, \dots, x_n]$ and a_i are complex numbers. Given a Darboux function ϕ , its logarithmic differential, $d\phi/\phi$,

²By the primitive element theorem, this is equivalent to K_{i+1} being a finite algebraic extension of K_i .

is clearly a closed rational 1-form. Conversely, we shall show that every closed rational 1-form must be the logarithmic differential of some Darboux function.

Theorem 1. *Consider a 1-form $\alpha \in \mathbb{C}(x_1, \dots, x_n)'$. If α is closed, then there exist elements $g, f, f_i \in \mathbb{C}[x_1, \dots, x_n]$ and constants $a_i \in \mathbb{C}$ such that*

$$\alpha = d\left(\frac{g}{f}\right) + \sum a_i \frac{df_i}{f_i}.$$

Proof. We proceed by induction on n . The case $n = 1$ amounts to the well-known fact that the primitive of a rational function in x_1 has the form $r(x_1) + \sum a_i \log(x_1 - b_i)$ with $a_i, b_i \in \mathbb{C}$ and a rational function r . Now suppose that $n > 1$ and the theorem holds for $\mathbb{C}(x_1, \dots, x_{n-1})$. Let \bar{K} be a splitting field over $\mathbb{C}(x_1, \dots, x_{n-1})$ of a common denominator of the coefficients of α , and denote the distinct roots of this common denominator by $b_1, \dots, b_r \in \bar{K}$. Then we can write α as a partial fraction expansion in x_n over $\mathbb{C}(x_1, \dots, x_{n-1})$:

$$\alpha = \sum_{i=1}^r \sum_{j=1}^{n_i} \frac{a_{i,j}}{(x_n - b_i)^j} dx_n + \sum_{i=0}^N c_i x_n^i dx_n + \sum_{i=1}^r \sum_{j=1}^{m_i} \frac{\Omega_{i,j}}{(x_n - b_i)^j} + \sum_{i=0}^M x_n^i \omega_i,$$

where the $\Omega_{i,j}, \omega_i$ are elements of $\mathbb{C}(x_1, \dots, x_{n-1})'$, and $a_{i,j}$ and c_i are elements of $\mathbb{C}(x_1, \dots, x_{n-1})$.

By evaluating $d\alpha = 0$ and comparing coefficients in the partial fraction expansion we get the following for all $i, j \geq 0$, where it is understood that $a_{i,0} = 0$ and $\Omega_{i,0} = 0$:

$$dc_i - (i+1)\omega_{i+1} = 0, \quad (3)$$

$$da_{i,j+1} + ja_{i,j}db_i - j\Omega_{i,j} = 0, \quad (4)$$

$$d\omega_i = 0, \quad (5)$$

$$d\Omega_{i,j+1} + jdb_i \wedge \Omega_{i,j} = 0. \quad (6)$$

These may be seen as identities in $\mathbb{C}(x_1, \dots, x_{n-1})'$. In particular, $da_{i,1} = 0$, so $a_{i,1} \in \mathbb{C}$. From (5) $d\omega_0 = 0$ and hence by hypothesis we can write

$$\omega_0 = d\left(\frac{\tilde{g}}{\tilde{f}}\right) + \sum \tilde{a}_i \frac{d\tilde{f}_i}{\tilde{f}_i},$$

for some $\tilde{g}, \tilde{f}, \tilde{f}_i \in \mathbb{C}(x_1, \dots, x_{n-1})$ and $\tilde{a}_i \in \mathbb{C}$. Equations (3) – (6) allow us to write

$$\begin{aligned} \alpha - \omega_0 &= \sum_i a_{i,1} \frac{d(x_n - b_i)}{(x_n - b_i)} + \sum_{j>1} \sum_i d\left(\frac{a_{i,j}}{(x_n - b_i)^{j-1}} \left(\frac{-1}{j-1}\right)\right) \\ &\quad + \sum_i d\left(\frac{c_i x_n^{i+1}}{i+1}\right). \end{aligned} \quad (7)$$

Now let G be the Galois group of \bar{K} over $\mathbb{C}(x_1, \dots, x_{n-1})$. For any differential form μ over \bar{K} and $\sigma \in G$ we denote by $\sigma(\mu)$ the form obtained by letting σ act on its coefficients. Taking the trace of both sides of equation (7) and noting that σ and the exterior derivative commute, we have

$$\begin{aligned} \frac{1}{|G|} \sum_{\sigma \in G} \sigma(\alpha - \omega_0) &= \frac{1}{|G|} \sum_{\sigma \in G} \sum a_{i,1} \frac{d(x_n - \sigma(b_i))}{(x_n - \sigma(b_i))} \\ &+ \frac{1}{|G|} \sum_{\sigma \in G} \sum \sum d \left(\frac{\sigma(a_{i,j})}{(x_n - \sigma(b_i))^{j-1}} \binom{-1}{j-1} \right) \\ &+ \frac{1}{|G|} \sum_{\sigma \in G} \sum d \left(\frac{\sigma(c_i) x_n^{i+1}}{i+1} \right). \end{aligned} \tag{8}$$

Since G is the set of all automorphisms of \bar{K} fixing $\mathbb{C}(x_1, \dots, x_{n-1})$, the left hand side of this equation is equal to $\alpha - \omega_0$, and we obtain α in the desired form. \square

The following definition may seem unusual, but it turns out to be the most convenient for our setting. See also the following remark.

Definition 2. Given a 1-form $\omega \in \mathbb{C}(x_1, \dots, x_n)'$, we say that ω is *Liouvillian integrable* if there exists a 1-form $\alpha \in L'$ for some Liouvillian extension L of $\mathbb{C}(x_1, \dots, x_n)$ such that $d\omega = \alpha \wedge \omega$ and $d\alpha = 0$. More specifically, we will state that ω is Liouvillian integrable over L when the field of definition is relevant.

Remark 2.

(a) We note:

- If there exists ϕ in some Liouvillian extension L of $\mathbb{C}(x_1, \dots, x_n)$ such that $d\phi \wedge \omega = 0$ (which would seem a more obvious definition), then $\omega = m d\phi$ for some $m \in L$ and $d\omega = \alpha \wedge \omega$ with $\alpha = dm/m$.
- Conversely, if the condition in Definition 2 holds then by Remark 1, part (i), there exists m in a Liouvillian extension L_1 of L such that $\alpha = -dm/m$, whence

$$d(m\omega) = dm \wedge \omega + m d\omega = m(-\alpha \wedge \omega + d\omega) = 0,$$

which implies $m\omega = d\phi$ for some ϕ in a Liouvillian extension $L_2 \supset L_1$, hence of $\mathbb{C}(x_1, \dots, x_n)$.

We call m an *inverse integrating factor* for ω .

- (b) In particular, the condition in Definition 2 implies that $d\omega \wedge \omega = 0$, so that ω is completely integrable in the usual sense (cf. e.g. Camacho and Lins Neto [7], Appendix §3).

The following theorem is well-known for forms (and vector fields) in two variables, see [27]. Here, we give an algebraic proof for the general case.

Theorem 2 (Singer's Theorem for 1-forms). *Let ω be a rational 1-form over $\mathbb{C}(x_1, \dots, x_n)$. Then ω is Liouvillian integrable if and only if there exists a closed 1-form $\alpha \in \mathbb{C}(x_1, \dots, x_n)'$ such that $d\omega = \alpha \wedge \omega$.*

Proof. We proceed by induction on the tower of fields (1), with $K = \mathbb{C}(x_1, \dots, x_n)$. Let K_{i+1} be a Liouvillian extension of K_i , of one of the types (i)–(iii) in Definition 1, and consider a closed 1-form $\alpha \in K_{i+1}'$ such that $d\omega = \alpha \wedge \omega$. We have to show that there exists $\tilde{\alpha} \in K_i'$ such that $d\omega = \tilde{\alpha} \wedge \omega$ with $d\tilde{\alpha} = 0$. We discuss the types from Remark 1 separately.

Type (i). We can suppose that $t_i = t$ is transcendental over K_i , else this falls into type (iii). Then (by Lemma 4) write α as a formal Laurent series in decreasing powers of t ,

$$\alpha = \alpha_r t^r + \alpha_{r-1} t^{r-1} + \dots, \quad \alpha_r \in K_i', \quad \alpha_r \neq 0. \quad (9)$$

Equating powers of t^0 in $\alpha \wedge \omega = d\omega$ and $d\alpha = 0$, we see that

$$d\omega = \alpha_0 \wedge \omega, \quad d\alpha_0 = 0.$$

Therefore, we can choose $\tilde{\alpha} = \alpha_0 \in K_i$.

Type (ii). As above, we suppose that $t_i = t$ is transcendental over K_i , and write α in the form (9). From $d\alpha = 0$ we deduce that $d\alpha_r = 0$. Furthermore, from $d\omega = \alpha \wedge \omega$, we obtain three cases depending on r :

- If $r > 0$, then $\alpha_r \wedge \omega = 0$. In this case, there exists $h \in K_i$ such that $\alpha_r = h\omega$, thus we get $d\omega = -\frac{dh}{h} \wedge \omega$. We may take $\tilde{\alpha} = -\frac{dh}{h}$.
- If $r = 0$, we have $d\omega = \alpha_0 \wedge \omega$ and we may take $\tilde{\alpha} = \alpha_0$.
- If $r < 0$, we see $d\omega = 0$ and we may take $\tilde{\alpha} = 0$.

Type (iii). There is no loss of generality in assuming that the extension is Galois, with Galois group G of order N . Take traces of both sides of $d\omega = \alpha \wedge \omega$, and of $d\alpha = 0$, respectively, to obtain

$$d\omega = \left(\frac{1}{N} \sum_{\sigma \in G} \sigma(\alpha) \right) \wedge \omega, \quad d \left(\frac{1}{N} \sum_{\sigma \in G} \sigma(\alpha) \right) = 0.$$

Thus we can choose $\tilde{\alpha} = \frac{1}{N} \sum_{\sigma \in G} \sigma(\alpha) \in K_i$.

□

Remark 3. Combining Theorem 2 and Theorem 1, we see that a 1-form ω is Liouvillian integrable if and only if it admits a Darboux integrating factor.

3 Three dimensional vector fields with Liouvillian first integrals

In this section we will consider three-dimensional rational vector fields

$$\mathcal{X} = P \frac{\partial}{\partial x} + Q \frac{\partial}{\partial y} + R \frac{\partial}{\partial z} \quad (10)$$

in \mathbb{C}^3 ; equivalently we will look at the corresponding 2-forms

$$\Omega = P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy. \quad (11)$$

Definition 3. A non-constant element, ϕ , of a Liouvillian extension of $\mathbb{C}(x, y, z)$ is called a *Liouvillian first integral* of the vector field \mathcal{X} if it satisfies $\mathcal{X}\phi = 0$ or, equivalently, $d\phi \wedge \Omega = 0$.

Remark 4. In view of Remark 2, this property is equivalent to the existence of some Liouvillian extension L of K and one-forms $\omega \neq 0$, α in L' such that

$$\omega \wedge \Omega = 0, \quad d\omega = \alpha \wedge \omega, \quad d\alpha = 0. \quad (12)$$

In this case we will briefly say that Ω is *Liouvillian integrable* and we will say specifically that Ω is Liouvillian integrable over L whenever the field of definition is relevant.

3.1 Extending Singer's theorem, first version

We seek to generalize Singer's theorem (see Theorem 2 in the previous section) to vector fields (or 2-forms) in dimension three. Our first main result is the following:

Theorem 3 (First extension of Singer's theorem for 2-forms in three dimensions). *Let $K = \mathbb{C}(x, y, z)$, and let Ω be the 2-form (11) over K . If there exists a Liouvillian first integral of Ω , then one of the following holds:*

(I) *There exist 1-forms $\omega, \alpha \in K'$ such that*

$$\omega \neq 0, \quad \omega \wedge \Omega = 0, \quad \alpha \wedge \omega = d\omega, \quad d\alpha = 0.$$

So, using the notion from Definition 2, Ω is Liouvillian integrable over K .

(II) *There exists a 1-form $\beta \in K'$ such that $\beta \wedge \Omega = d\Omega$ with $d\beta = 0$. So, Ω admits an inverse Jacobi multiplier³ of Darboux type over $K = \mathbb{C}(x, y, z)$.*

Remark 5.

³For the notion of inverse Jacobi multiplier see Berrone and Giacomini [4]. Note that we permit nonzero constant functions as multipliers.

- (a) Roughly speaking, condition I means there is a first integral of the form $\phi = \int \frac{\omega}{e^{\int \alpha}}$. Note that $e^{\int \alpha}$ is of Darboux type by Theorem 1. In the special case when $\alpha = 0$, there is a first integral of the form $\int \omega$.
- (b) In the same way, condition II means that Ω admits an inverse Jacobi multiplier of the form $e^{\int \beta}$, with $\beta \in K'$, i.e. of Darboux type.

Before turning to the proof of Theorem 3, we state two lemmas, the first of which is straightforward. The second shows that the existence of two independent Liouvillian first integrals implies the existence of a Liouvillian inverse Jacobi multiplier. (We note that this is a special case of Zhang [28], Theorem 1.2.)

Lemma 1. *Let L be a differential extension of $K = \mathbb{C}(x, y, z)$, moreover $0 \neq \ell \in L$, $0 \neq \omega \in L'$ and $\alpha \in L'$ such that $d\omega = \alpha \wedge \omega$, $d\alpha = 0$. Then*

$$d\left(\frac{\omega}{\ell}\right) = \left(\alpha - \frac{d\ell}{\ell}\right) \wedge \frac{\omega}{\ell}, \quad d\left(\alpha - \frac{d\ell}{\ell}\right) = 0. \quad (13)$$

Lemma 2. *Let $K = \mathbb{C}(x, y, z)$, let Ω be the 2-form (11) over K , and let L be a differential extension of K . Assume that there exist linearly independent $\omega_1, \omega_2 \in L'$ (thus $\omega_1 \wedge \omega_2 \neq 0$) and moreover $\alpha_1, \alpha_2 \in L'$, such that*

$$\omega_i \wedge \Omega = 0, \quad d\omega_i = \alpha_i \wedge \omega_i \quad \text{and} \quad d\alpha_i = 0, \quad i = 1, 2.$$

Then the following hold.

- (a) There is $\ell \in L^*$ so that

$$\omega_1 \wedge \omega_2 = \ell \Omega, \quad (14)$$

hence $\frac{1}{\ell} \omega_1 \wedge \omega_2 = \Omega$, and $d\left(\frac{1}{\ell} \omega_1\right) = \tilde{\alpha}_1 \wedge \frac{1}{\ell} \omega_1$ with $\tilde{\alpha}_1 = \alpha_1 - \frac{d\ell}{\ell}$, $d\tilde{\alpha}_1 = 0$. Replacing ω_1 by $\tilde{\omega}_1 := \frac{\omega_1}{\ell}$, one thus has $\tilde{\omega}_1 \wedge \omega_2 = \Omega$.

- (b) Given that $\omega_1 \wedge \omega_2 = \ell \Omega$:

- With $\beta := \alpha_1 + \alpha_2 - \frac{d\ell}{\ell}$ one gets

$$\beta \wedge \Omega = d\Omega, \quad d\beta = 0;$$

and whenever $d\Omega \neq 0$ then $\beta \neq 0$.

- In case $\beta \neq 0$ but $d\Omega = 0$, $\int \beta$ is a Liouvillian first integral of (11).
- In case $\beta = 0$ and $d\Omega = 0$, obviously the constant 1 is an inverse Jacobi multiplier of (11).

Proof. The space of all two-forms over L has dimension three, and from $\omega_1 \wedge \Omega = \omega_2 \wedge \Omega = 0$, one sees that Ω lies in a one-dimensional subspace which also contains $\omega_1 \wedge \omega_2$. Thus there exists $\ell \in L$, $\ell \neq 0$, such that

$\omega_1 \wedge \omega_2 = \ell \Omega$, and with Lemma 1 part (a) is proven. Applying the exterior derivative to (14) gives

$$d(\omega_1 \wedge \omega_2) = d\ell \wedge \Omega + \ell d\Omega. \quad (15)$$

On the other hand

$$\begin{aligned} d(\omega_1 \wedge \omega_2) &= d\omega_1 \wedge \omega_2 - \omega_1 \wedge d\omega_2 \\ &= \alpha_1 \wedge \omega_1 \wedge \omega_2 - \omega_1 \wedge \alpha_2 \wedge \omega_2 \\ &= (\alpha_1 + \alpha_2) \wedge (\omega_1 \wedge \omega_2) \\ &= (\alpha_1 + \alpha_2) \wedge \ell \Omega \end{aligned}$$

by (14). Combining this with (15) yields

$$\left(\alpha_1 + \alpha_2 - \frac{d\ell}{\ell} \right) \wedge \Omega = d\Omega, \quad (16)$$

which shows the first statement of part (b). The remaining assertions are obvious. \square

Proof of Theorem 3. Since II (with $\beta = 0$) is trivially true when $d\Omega = 0$, we will assume $d\Omega \neq 0$ in the following. We proceed by induction on the tower of fields $K = K_0 \subset K_1 \subset \dots \subset K_n$. Let I_i and II_i denote Conditions I and II with K' replaced by K'_i . Clearly, the existence of a Liouvillian first integral ϕ implies that we are in case I_n for some n , by taking $\omega = d\phi$, $\alpha = 0$.

Claim 1. If condition II_{i+1} holds then condition II_i or condition I_i holds.

Proof. Let $K_{i+1} \supset K_i$ be one of the types (i)–(iii) listed in Remark 1. Assume that $\beta \in K'_{i+1}$ such that $\beta \wedge \Omega = d\Omega$ with $d\beta = 0$. We will show that there exists $\tilde{\beta} \in K'_i$ such that $\tilde{\beta} \wedge \Omega = d\Omega$ with $d\tilde{\beta} = 0$, or there exists $0 \neq \tilde{\omega} \in K'_i$ such that $\tilde{\omega} \wedge \Omega = 0$ and $d\tilde{\omega} = 0$, which is a special instance of case I_n (with $\tilde{\alpha} = 0$).

If $K_{i+1} = K_i(t)$ with t transcendental over K_i (thus the extension is of Type (i) or (ii)), then consider β as a formal Laurent series⁴ in decreasing powers of t :

$$\beta = \beta_\ell t^\ell + \dots, \quad \beta_k \in K'_i, \quad k = \ell, \ell - 1, \dots \quad \beta_\ell \neq 0. \quad (17)$$

Since $\beta \wedge \Omega = d\Omega$ and t is transcendental, we see

$$\beta_i \wedge \Omega = \begin{cases} d\Omega & i = 0, \\ 0 & i \neq 0. \end{cases} \quad (18)$$

⁴See Appendix, Lemma 4.

Type **(i)**. We may assume that t is transcendental; otherwise see type (iii). Since β is closed, we obtain (with $dt = \delta t$, $\delta \in K'_i$)

$$\sum_{i=-\infty}^{\ell} t^i (d\beta_i + i\delta \wedge \beta_i) = 0,$$

in particular $d\beta_0 = 0$, and $\beta_0 \wedge \Omega = d\Omega$. So we can choose $\tilde{\beta} = \beta_0$.

Type **(ii)**. As above, we may assume that $K_{i+1} = K_i(t)$ with t transcendental, and (by Lemma 4) we have the Laurent series expansion (17). Since $\beta \wedge \Omega = d\Omega$, equating highest powers of t yields three possibilities:

1) When $\ell > 0$, we have $\beta_\ell \wedge \Omega = 0$ by (18), and with $dt = \delta \in K'_i$ one finds

$$0 = t^\ell d\beta_\ell + t^{\ell-1} (\ell \delta \wedge \beta_\ell + d\beta_{\ell-1}) + \cdots,$$

omitting terms of lower degree. This implies $d\beta_\ell = 0$, and we have case I with $\tilde{\omega} = \beta_\ell$ and $\tilde{\alpha} = 0$.

2) When $\ell = 0$, we see $\beta_0 \wedge \Omega = d\Omega$, and $d\beta_0 = 0$ from $d\beta = 0$. In this case take $\tilde{\beta} = \beta_0$.

3) When $\ell < 0$, then $d\Omega = 0$ by (18), contrary to the blanket assumption $d\Omega \neq 0$.

Type **(iii)**. In this case, without loss of generality, we assume that $K_{i+1} \supset K_i$ is a Galois extension. Denote by G its Galois group, and by N its order. For $\sigma \in G$ and any differential form μ we denote by $\sigma(\mu)$ the form obtained by applying σ to its coefficients.

From $\beta \wedge \Omega = d\Omega$ and $d\beta = 0$, we see

$$\sigma(\beta) \wedge \Omega = d\Omega \text{ and } d(\sigma(\beta)) = 0$$

for all $\sigma \in G$, and therefore

$$d\Omega = \left(\frac{1}{N} \sum_{\sigma \in G} \sigma(\beta) \right) \wedge \Omega \quad \text{and} \quad \frac{1}{N} d \left(\sum_{\sigma \in G} \sigma(\beta) \right) = 0.$$

We can therefore choose

$$\tilde{\beta} = \frac{1}{N} \sum_{\sigma \in G} \sigma(\beta) \in K'_i.$$

This completes the proof of Claim 1.

Claim 2. If I_{i+1} holds then one of I_i or II_i must also hold. Moreover, I_{i+1} always implies I_i unless Ω admits two independent Liouvillian first integrals.

Proof. Let $\omega \neq 0$, $\alpha \in K'_{i+1}$ such that $\omega \wedge \Omega = 0$, $\alpha \wedge \omega = d\omega$, $d\alpha = 0$. Then we need to show that for all three types (i)–(iii) of Remark 1, there exists $\tilde{\omega} \neq 0$, $\tilde{\alpha} \in K'_i$ such that $\tilde{\omega} \wedge \Omega = 0$, $\tilde{\alpha} \wedge \tilde{\omega} = d\tilde{\omega}$, $d\tilde{\alpha} = 0$, or there exists $\tilde{\beta} \in K'_i$ such that $\tilde{\beta} \wedge \Omega = d\Omega$ with $d\tilde{\beta} = 0$.

When $K_{i+1} = K_i(t)$, and t is transcendental over K_i (thus the extension is of type (i) or (ii)), then (by Lemma 4) we can write ω , α as formal Laurent series in decreasing powers of t , thus

$$\omega = \omega_r t^r + \omega_{r-1} t^{r-1} \dots, \quad \omega_k \in K'_i (k \leq r), \quad \omega_r \neq 0, \quad (19)$$

and

$$\alpha = 0 \quad \text{or} \quad \alpha = \alpha_s t^s + \alpha_{s-1} t^{s-1} \dots, \quad \alpha_k \in K'_i (k \leq s), \quad \alpha_s \neq 0. \quad (20)$$

By hypothesis, for the transcendental case we have $\omega \wedge \Omega = 0$, hence $\omega_k \wedge \Omega = 0$ for all k .

Type (i). Since $\alpha \wedge \omega = d\omega$, one gets (with $dt = \delta t$)

$$\sum_{i=-\infty}^s \alpha_i t^i \wedge \sum_{i=-\infty}^r \omega_i t^i = \sum_{i=-\infty}^r (d\omega_i + i \delta \wedge \omega_i) t^i \quad (21)$$

(with left hand side understood to be zero when $\alpha = 0$). Comparing highest powers of t yields the following three cases:

- 1) When $\alpha = 0$ or $s < 0$ (thus the highest degree on the left hand side is $< r$), we just have $d(\omega_r t^r) = d\omega_r + r \delta \wedge \omega_r = 0$. In this case choose $\tilde{\alpha} = -r \delta$ (with $d\tilde{\alpha} = 0$) and $\tilde{\omega} = \omega_r$.
- 2) When $s = 0$, we see $\alpha_0 \wedge \omega_r = d\omega_r + r \delta \wedge \omega_r$. In this case take $\tilde{\alpha} = \alpha_0 - r \delta$ and $\tilde{\omega} = \omega_r$. From $d\alpha_0 = 0$ and $d\delta = 0$, is it clear that $d\tilde{\alpha} = 0$.
- 3) When $s > 0$, we get $\alpha_s \wedge \omega_r = 0$ and therefore $\alpha_s = h \omega_r$ for some $h \in K_i$. Since $\omega_r \wedge \Omega = 0$, then $\alpha_s \wedge \Omega = 0$. Moreover $d(\alpha_s t^s) = (d\alpha_s + s \delta \wedge \alpha_s) t^s = 0$ from $d\alpha = 0$. So we may choose $\tilde{\alpha} = -s \delta$ (with $d\tilde{\alpha} = 0$) and $\tilde{\omega} = \alpha_s$.

Type (ii). Here we have again $K_{i+1} = K_i(t)$ with t transcendental, $dt = \delta$, and Laurent expansions for ω and α as in (19), (20). Since $\omega \wedge \Omega = 0$, then $\omega_k \wedge \Omega = 0$ for all k . From the assumption $\alpha \wedge \omega = d\omega$, we obtain three cases by comparing highest powers of t :

- 1) When $s > 0$, we get $\alpha_s \wedge \omega_r = 0$ and hence $\alpha_s = h \cdot \omega_r$ for some $h \in K_i$. Since $\omega_r \wedge \Omega = 0$, then $\alpha_s \wedge \Omega = 0$. From $d(\alpha_s t^s) = (d\alpha_s)t^s + s\alpha_s \wedge \delta t^{s-1}$ and $d\alpha = 0$ one sees $d\alpha_s = 0$. In this case take $\tilde{\alpha} = 0$ and $\tilde{\omega} = \alpha_s$.
- 2) When $s = 0$, we see $\alpha_0 \wedge \omega_r = d\omega_r$, and $d\alpha_0 = 0$ from $d\alpha = 0$. In this case we choose $\tilde{\alpha} = \alpha_0$ and $\tilde{\omega} = \omega_r$.
- 3) When $\alpha = 0$ or $s < 0$, then $d\omega_r = 0$. Take $\tilde{\alpha} = 0$ and $\tilde{\omega} = \omega_r$.

Type (iii). Without loss of generality, assume that K_{i+1} is a Galois extension of K_i , with Galois group G of order N . We have two cases: Either $\sigma(\omega) \wedge \omega = 0$ for all $\sigma \in G$ or there exists $\tau \in G$ such that $\tau(\omega) \wedge \omega \neq 0$.

- 1) Let us first assume that $\sigma(\omega) \wedge \omega = 0$ for all $\sigma \in G$. Choose $\eta, \theta \in K'_i$ such that $\eta \wedge \Omega = \theta \wedge \Omega = 0$, $\eta \wedge \theta \neq 0$. Then there exist $k, \ell \in K_{i+1}$ such that $\omega = k\eta + \ell\theta$. With Lemma 1 one sees that

$$\tilde{\omega} := \frac{\omega}{k} = \eta + \tilde{\ell}\theta$$

satisfies $\tilde{\omega} \wedge \Omega = 0$, and $d\tilde{\omega} = \tilde{\alpha} \wedge \tilde{\omega}$, $d\tilde{\alpha} = 0$ for some $\tilde{\alpha} \in K'_{i+1}$. Moreover $\sigma(\tilde{\omega}) \wedge \tilde{\omega} = 0$ for all $\sigma \in G$. But $\sigma(\tilde{\omega}) \wedge \tilde{\omega} = (\ell - \sigma(\ell)) \eta \wedge \theta$, so $\sigma(\tilde{\ell}) = \tilde{\ell}$ for all $\sigma \in G$, and $\tilde{\omega} \in K'_i$. Finally, forming the trace of $d\tilde{\omega} = \tilde{\alpha} \wedge \tilde{\omega}$ shows that one may take $\tilde{\alpha} \in K'_i$.

- 2) Assume now that $\tau(\omega) \wedge \omega \neq 0$ for some $\tau \in G$. In this case there exist two independent Liouvillian first integrals, with differentials ω and $\tau(\omega)$. By Lemma 2, there exists $\beta \in K'_{i+1}$ such that $\beta \wedge \Omega = d\Omega$, $d\beta = 0$, and $\beta \neq 0$ by the blanket assumption $d\Omega \neq 0$, hence condition II_{i+1} is satisfied. Thus, with Claim 1 we find that II_i holds.

This finishes the proof of Claim 2. Combining Claim 1 and Claim 2, the theorem is proven. \square

The proof of Claim 2 (specifically, case 2 for Type (iii)) makes passage from I_{i+1} to II_i necessary only when $K_{i+1} \supset K_i$ is algebraic, and Ω admits two independent Liouvillian first integrals. We restate this observation:

Corollary 1. *Let $K = \mathbb{C}(x, y, z)$, and let Ω be the 2-form (11) over K . If Ω admits a Liouvillian first integral, but not two independent Liouvillian first integrals, then (according to Definition 2) Ω is Liouvillian integrable over K .*

Likewise, Claim 1 and its proof directly imply a noteworthy property of 2-forms that admit a Liouvillian inverse Jacobi multiplier:

Corollary 2. *Let $K = \mathbb{C}(x, y, z)$ and let Ω be the 2-form (11) over K . Assume that Ω does not admit a Liouvillian first integral, but there exists a Liouvillian extension L of K and a 1-form $\beta \in L'$ such that $\beta \wedge \Omega = d\Omega$ with $d\beta = 0$. Then there exists a 1-form $\bar{\beta} \in K'$ such that $\bar{\beta} \wedge \Omega = d\Omega$ with $d\bar{\beta} = 0$.*

3.2 Extending Singer's theorem, second version

Our proof of Claim 2 in Theorem 3 does not imply the existence of a first integral of the 2-form (11) that is defined over $K = \mathbb{C}(x, y, z)$. The obstacle in the argument appears with an algebraic extension $K_{i+1} \supset K_i$ in the tower of field extensions, when there exist two independent first integrals. We now will show that in any case there exists a first integral that is defined over a finite algebraic extension of $K_0 = K$. Thus one might say that a weaker version of Singer's theorem holds, with K replaced by a finite algebraic extension.

Our argument is based on Lemma 5 in the Appendix. We first note an auxiliary result. Recall (here and later on) that finite algebraic extensions of Liouvillian extensions are always Liouvillian.

Lemma 3. *Let L_0 be a Liouvillian extension of $K = \mathbb{C}(x, y, z)$, t transcendental over L_0 , and q algebraic over $L_0(t)$, thus $L := L_0(t, q)$ Liouvillian over L_0 . Moreover let $\omega, \alpha \in L'$ such that $\omega \neq 0$, $\omega \wedge \Omega = 0$, $d\omega = \alpha \wedge \omega$, $d\alpha = 0$.*

Then there exists a finite algebraic extension \tilde{L}_0 of L_0 , and $\tilde{\omega}, \tilde{\alpha} \in \tilde{L}'_0$ such that $\tilde{\omega} \neq 0$, $\tilde{\omega} \wedge \Omega = 0$, $d\tilde{\omega} = \tilde{\alpha} \wedge \tilde{\omega}$, $d\tilde{\alpha} = 0$. Briefly, Ω is Liouvillian integrable over \tilde{L}_0 .

Proof. By Lemma 5, in the appendix, there exists a finite extension $\tilde{L}_0 \supset L_0$ so that we may write ω, α as formal Laurent series in decreasing powers of $\tau = t^{1/m}$ with some positive integer m , thus

$$\omega = \omega_r \tau^r + \omega_{r-1} \tau^{r-1} \dots, \quad \omega_k \in \tilde{L}'_0 \ (k \leq r), \quad \omega_r \neq 0, \quad (22)$$

and either $\alpha = 0$ or

$$\alpha = \alpha_s \tau^s + \alpha_{s-1} \tau^{s-1} \dots, \quad \alpha_k \in \tilde{L}'_0 \ (k \leq s), \quad \alpha_s \neq 0. \quad (23)$$

With t transcendental, we have $\omega \wedge \Omega = 0$, hence $\omega_k \wedge \Omega = 0$ for all k .

We now follow the pattern of the proof of Theorem 3, Claim 2.

- Type (i). Let $dt = t\delta$ with $d\delta = 0$, hence

$$d\tau = \frac{1}{m} \tau \delta.$$

We thus obtain the highest degree terms

$$d\omega = \tau^r \left(\frac{r}{m} \delta \wedge \omega_r + d\omega_r \right) + \dots, \quad d\alpha = \tau^s \left(\frac{s}{m} \delta \wedge \alpha_s + d\alpha_s \right) + \dots$$

unless $\alpha = 0$. Comparing both sides of $\alpha \wedge \omega = d\omega$ yields the following three cases:

- When $\alpha = 0$ or $s < 0$ (thus the highest degree on the left hand side is $< r$), we just have $d\omega_r + \frac{r}{m} \delta \wedge \omega_r = 0$. In this case choose $\tilde{\alpha} = -\frac{r}{m} \delta$ (with $d\tilde{\alpha} = 0$) and $\tilde{\omega} = \omega_r$.
 - When $s = 0$, we see $\alpha_0 \wedge \omega_r = d\omega_r + \frac{r}{m} \delta \wedge \omega_r$. In this case take $\tilde{\alpha} = \alpha_0 - \frac{r}{m} \delta$ (noting $d\alpha_0 = 0$) and $\tilde{\omega} = \omega_r$.
 - When $s > 0$, we get $\alpha_s \wedge \omega_r = 0$ and therefore $\alpha_s = h\omega_r$ for some $h \in \tilde{L}_0$. Since $\omega_r \wedge \Omega = 0$, then $\alpha_s \wedge \Omega = 0$. Moreover $d\alpha_s + \frac{s}{m} \delta \wedge \alpha_s = 0$ from $d\alpha = 0$. So we may choose $\tilde{\alpha} = -\frac{s}{m} \delta$ (with $d\tilde{\alpha} = 0$) and $\tilde{\omega} = \alpha_s$.
- Type (ii). Here we have t transcendental over \tilde{L}_0 , $dt = \delta$, with $d\delta = 0$. Therefore $d\tau = \frac{1}{m} \tau^{1-m} \delta$, hence

$$d(\tau^r \omega_r) = \tau^r d\omega_r + \frac{r}{m} \tau^{r-m} \delta \wedge \omega_r,$$

which shows that the leading term of $d\omega$ is just $\tau^r d\omega_r$. Likewise, the leading term of $d\alpha$ equals $\tau^s d\alpha_s$ unless $\alpha = 0$. Comparing the leading terms of $\alpha \wedge \omega = d\omega$, we obtain three cases:

- When $s > 0$, we get $\alpha_s \wedge \omega_r = 0$ and hence $\alpha_s = h \cdot \omega_r$ for some $h \in \tilde{L}_0$. Since $\omega_r \wedge \Omega = 0$, then $\alpha_s \wedge \Omega = 0$. From $d\alpha = 0$ one sees $d\alpha_s = 0$. In this case take $\tilde{\alpha} = 0$ and $\tilde{\omega} = \alpha_s$.
- When $s = 0$, we see $\alpha_0 \wedge \omega_r = d\omega_r$, and $d\alpha_0 = 0$ from $d\alpha = 0$. In this case we choose $\tilde{\alpha} = \alpha_0$ and $\tilde{\omega} = \omega_r$.
- When $\alpha = 0$ or $s < 0$, then $d\omega_r = 0$. Take $\tilde{\alpha} = 0$ and $\tilde{\omega} = \omega_r$.

□

The following is now a direct consequence of Lemma 3.

Theorem 4 (Second extension of Singer's theorem for 2-forms in three dimensions). *Let $K = \mathbb{C}(x, y, z)$, and let Ω be the 2-form (11) over K . If there exists a Liouvillian first integral of Ω , then there exists a finite algebraic extension \tilde{K} of K such that Ω is Liouvillian integrable over \tilde{K} .*

Proof. Consider a tower

$$K = K_0 \subset K_1 \subset \dots \subset K_m = L,$$

as in Definition 1 (or Remark 1), and assume that for some $i > 1$ one has a finite algebraic extension $K_{i+1} \supset K_i$, and $\omega, \alpha \in K'_{i+1}$ subject to the conditions in (12). With no loss of generality, $K_i \supset K_{i-1}$ is then transcendental, and Lemma 3 shows that there exists a finite algebraic extension \tilde{K}_{i-1} of K_{i-1} , and $\tilde{\omega}, \tilde{\alpha} \in K'_{i-1}$ as required in (12). Thus all transcendental extensions can be eliminated by descent. □

With this result we have reached the conclusion of the present paper: If a rational 2-form in three variables admits a Liouvillian first integral, then it admits a first integral that is obtained, via (12), from integrating 1-forms defined over a finite algebraic extension of K . The cases where the further reduction to K cannot be made are exceptional and a detailed examination of them will be the subject of forthcoming work.

4 Appendix

4.1 Laurent and Puiseux expansions

Here we collect some pertinent facts about power series expansions. Both Lemma 4 and Lemma 5 might be considered standard. But we include them (with proof sketches), for easy reference, and because they are crucial in our arguments.

Lemma 4. *Let L_0 be a field, $L = L_0(t)$ with t transcendental over L_0 , and $r \in L$ nonzero. Then there exist an integer m and $c_j \in L_0$, $j \geq 0$, so that for any integer $\ell \geq 0$ there exists $r_\ell \in L$ with $r_\ell(0) \neq 0$ such that*

$$t^m r = c_0 + tc_1 + \cdots + t^\ell c_\ell + t^{\ell+1} r_\ell.$$

Moreover, the assertion also holds with t replaced by t^{-1} . Mutatis mutandis, these statements also hold for elements of any finite dimensional vector space over L .

Proof. There is an integer m such that

$$t^m r = \frac{a_0 + ta_1 + \cdots}{b_0 + tb_1 + \cdots} \text{ with } a_0 \neq 0, b_0 \neq 0.$$

To determine the c_j , proceed recursively, starting with $c_0 = a_0/b_0$ and

$$r_0 - \frac{a_0}{b_0} = \frac{a_0 + ta_1 + \cdots - a_0/b_0(b_0 + tb_1 + \cdots)}{b_0 + tb_1 + \cdots} = t r_1.$$

The recursion step works by applying the same argument to r_ℓ .

The last assertion is immediate from $L_0(t) = L_0(t^{-1})$. □

The following lemma is a consequence of the Newton-Puiseux theorem; see Abhyankar [1], Lecture 12. We cannot directly use the theorem as stated in [1] for algebraically closed base field, but we will closely trace Abhyankar's proof.

Lemma 5. *Let L_0 be field of characteristic zero, t transcendental over L_0 , moreover let q be algebraic over $L_0(t)$, and $L = L_0(t, q)$. Then there exist a*

finite algebraic extension \tilde{L}_0 of L_0 and a positive integer m , such that every element of L admits a representation

$$\sum_{i=N}^{\infty} a_i \tau^i; \quad \tau = t^{1/m},$$

with all $a_i \in \tilde{L}_0((t))$. Moreover, this statement also holds for all elements of any finite dimensional vector space over L , and analogous statements hold with τ replaced by τ^{-1} .

Proof. It suffices to prove the statement for q , since $L = L_0(t)[q]$, and with q , every polynomial in q with coefficients in $L_0(t)$ will have a representation in $\tilde{L}_0((\tau))$ as asserted.

Let $Q(t, y) \in L_0(t)[y]$ denote the minimal polynomial of q over $L_0(t)$;

$$Q = y^n + c_1 y^{n-1} + \cdots + c_n,$$

with all $c_j \in L_0(t) \subset L_0((t))$. Due to Lemma 4 we may assume that $n > 1$. We will show the existence of a finite extension \hat{L}_0 of L_0 such that Q is reducible over $\hat{L}_0((\tau))$. The following arguments (due to Abhyankar) do not rely on rationality of the c_j , or irreducibility of Q .

In case $Q = y^n$ reducibility is obvious. Otherwise, following Abhyankar's proof there exists a rational number d and a positive integer m , so that with $\tau = t^{1/m}$ one has

$$Q(t, t^d(y + c_1/n)) =: \hat{Q}(\tau, y) = y^n + \sum_{j=1}^n \hat{c}_j(\tau) y^{n-j}$$

with all $\hat{c}_j \in L_0[[\tau]]$, and $\hat{c}_1 = 0$, some $\hat{c}_j(0) \neq 0$. Note the correspondence between Q and \hat{Q} .

Now set $\hat{Q}_0 := \hat{Q}(0, y)$, and let \hat{L}_0 be its splitting field over L_0 . By the argument in [1], p. 93,

$$\hat{Q}_0 = \hat{P}_{0,1} \cdot \hat{P}_{0,2},$$

with relatively prime $\hat{P}_{0,i} \in \hat{L}_0[y]$. With Hensel's lemma (as stated in [1], p. 90) one gets

$$\hat{Q} = \hat{P}_1 \cdot \hat{P}_2$$

with relatively prime $\hat{P}_i \in \hat{L}_0[[\tau]][y]$. By the correspondence between Q and \hat{Q} one arrives at

$$Q = P_1 \cdot P_2; \quad P_i \in \hat{L}_0((\tau))[y].$$

Proceeding by induction on the degree (possibly requiring further field extensions and increase of m) one obtains a finite field extension \tilde{L}_0 and a decomposition

$$Q(t, y) = \prod (y - \eta_j)$$

as a product of linear factors, with the $\eta_j \in \tilde{L}_0((t^{1/m}))$. Now $Q(t, q) = 0$ shows that $q = \eta_k$ for some k . To prove the assertion for decreasing powers of τ , start with $s = t^{-1}$ and repeat the argument over $L_0(s)$.

The generalization to finite dimensional vector spaces over L is straightforward. \square

References

- [1] S.S. Abhyankar. *Algebraic geometry for scientists and engineers*. American Mathematical Society, Providence, RI (1990).
- [2] J. Avellar and L. G. S. Duarte and S. E. S. Duarte and L. A. C. P. da Mota. Determining Liouvillian first integrals for dynamical systems in the plane. *Comput. Phys. Commun.*, 177, 584–596, 2007.
- [3] J. Avellar and M.S. Cardoso and L.G.S. Duarte and L.A.C.P. da Mota. Dealing with rational second order ordinary differential equations where both Darboux and Lie find it difficult: The S-function method *Comput. Phys. Commun.*, 234, 302–314, 2019.
- [4] L.R. Berrone and H. Giacomini. Inverse Jacobi multipliers. *Rend. Circ. Mat. Palermo*, 52, 77–130, 2003.
- [5] G. Casale. Liouvillian first integrals of differential equations. *Banach Center Publ.*, 94, 153–161, 2011.
- [6] L. Cairó and H. Giacomini and L. Llibre. Liouvillian first integrals for the planar Lotka–Volterra system. *Rend. Circ. Mat. Palermo*, 52, 389–418, 2003.
- [7] C. Camacho and A. Lins Neto. *Geometric Theory of Foliations*. Springer, 1985.
- [8] G. Chèze and T. Combot. Symbolic Computations of First Integrals for Polynomial Vector Fields. *Found. Comput. Math.*, 20, 681–752, 2020.
- [9] C. Christopher. Liouvillian first integrals of second order polynomial differential equations. *Electron. J. Differ. Equ.*, 1–7, 1999.
- [10] C. Christopher, J. Llibre, C. Pantazi, S. Walcher. On planar polynomial vector fields with elementary first integrals. *J. Differ. Equ.*, 267, 4572–4588, 2019.
- [11] T. Combot. Symbolic integration of hyperexponential 1-forms. *Proc. Int. Symp. Symb. Algebr. Comput. ISSAC*, 98–105, 2019.

- [12] L.G.S. Duarte and L.A.C.P. da Mota. 3D polynomial dynamical systems with elementary first integrals. *J. Phys. A: Math. Theor.*, 43, 17 pp. 065204, 2010.
- [13] L.G.S. Duarte and L.A.C.P. da Mota. An efficient method for computing Liouvillian first integrals of planar polynomial vector fields. *J. Differ. Equ.*, 300, 356–385, 2021.
- [14] B. Ferčec, V.G. Romanovski, Y. Tang and L. Zhang. Integrability and bifurcation of a three-dimensional circuit differential system. *Discrete Contin. Dyn. Syst.- B*, 27, 4573–4588, 2022.
- [15] I.A. García and Jaume Giné, Center problem with characteristic directions and inverse integrating factors, *Commun. Nonlinear Sci. Numer. Simul.*, 108, 106276, 2022.
- [16] K. O. Geddes and S.R. Czapor and G. Labahn. *Algorithms for computer algebra*. Boston, MA: Kluwer Academic Publishers, 1992.
- [17] E.R. Kolchin. *Differential algebra and algebraic groups*. Academic Press, New York-London, 1973.
- [18] J. Llibre and C. Pantazi. Integrability and dynamics of a simplified class B Laser system. *Chaos*, 33, 103119, 2023.
- [19] J. M. Ollagnier. Liouvillian first integrals of homogeneous polynomial 3-dimensional vector fields. *Colloq. Math.*, 70, 195–217, 1996.
- [20] J. M. Ollagnier. Liouvillian integration of the Lotka-Volterra systems. *Qual. Theory Dyn. Syst.*, 2, 307–358, 2001.
- [21] R. Oliveira, D. Schlomiuk and A.M. Travaglini. Geometry and integrability of quadratic systems with invariant hyperbolas. *Electron. J. Qual. Theory Differ. Equ.*, 6, 1–56, 2021.
- [22] M.J. Prelle and M.F. Singer. Elementary first integrals of differential equations. *Trans. Amer. Math. Soc.*, 279, 619–636, 1983.
- [23] R. H. Risch. The problem of integration in finite terms. *Trans. Amer. Math. Soc.*, 139, 167—189, 1969.
- [24] R. H. Risch. The solution of the problem of integration in finite terms. *Bull. Amer. Math. Soc.*, 76, 605—608, 1970.
- [25] M. Rosenlicht. On the explicit solvability of certain transcendental equations. *Inst. Hautes Études Sci. Publ. Math.*, 36, 15–22, 1969.
- [26] M. Rosenlicht. Integration in finite terms. *Am. Math. Mon.*, 79, 963—972, 1972.

- [27] M. Singer. Liouvillian first integrals of differential equations. *Trans. Amer. Math. Soc.*, 333(2), 673–688, 1992.
- [28] X. Zhang. Liouvillian integrability of polynomial differential systems. *Trans. Amer. Math. Soc.*, 368(1), 607–620, 2016.
- [29] H. Żołądek. The extended monodromy group and Liouvillian first integrals. *J. Dyn. Control Syst.*, 4, 1–28, 1998.