COEXISTENCE OF UNCOUNTABLY MANY ATTRACTING SETS FOR SKEW-PRODUCTS ON THE CYLINDER

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Abstract. The aim of this paper is to show that the existence of attracting sets for quasiperiodically forced systems can be extended to appropriate skew-products on the cylinder, homotopic to the identity, in such a way that the general system will have (at least) one attracting set corresponding to every irrational rotation number ρ in the rotation interval of the base map. This attracting set is a copy of the attracting set of the system quasiperiodically forced by a (rigid) rotation of angle ρ. This shows the co-existence of uncountably many attracting sets, one for each irrational in the rotation interval of the basis map.

1. Introduction

We want to show that the existence of attracting sets for quasiperiodically forced systems can be extended to a class of skew-products on the cylinder which are homotopic to the identity. These systems have an attracting set corresponding to every irrational rotation number ρ in the rotation interval of the base map. This attracting set is a copy of the attracting set of the system quasiperiodically forced by a (rigid) rotation of angle ρ. In particular we show that the systems from our class can have uncountably many coexisting attracting sets (one for each irrational in the rotation interval of the base map).

To better explain the above and to state the main result of the paper we need to recall the basics of rotation theory on the circle, define what we understand by an attracting set, and to fix some notation.

In what follows the circle $\mathbb{R}/\mathbb{Z}$ will be denoted by $\mathbb{S}^1$. To simplify the notation, given $x \in \mathbb{R}$, we will identify $[x] \in \mathbb{S}^1$ with its representative in $[0, 1)$ (that is, with the fractional part of x, denoted by $\{\!\{x\}\!\}$).

It is well known that there exists a natural projection $e\colon \mathbb{R} \longrightarrow \mathbb{S}^1$ defined by $e(x) := \{\!\{x\}\!\}$ and that any continuous circle map f lifts to a continuous map $F\colon \mathbb{R} \longrightarrow \mathbb{R}$ (called a *lifting of f*) in such a way that $f \circ e = e \circ F$. If F is a lifting of f, then $F + n$ is also a lifting of f for every $n \in \mathbb{Z}$ and there exists $d \in \mathbb{Z}$ such that $F(x + 1) = F(x) + d$ for every $x \in \mathbb{R}$. Such integer d is called the *degree of f*.

In this paper we are only interested in continuous degree one circle maps. These are continuous maps such that $F(x + 1) = F(x) + 1$ for every $x \in \mathbb{R}$ and every lifting F. We denote by $\mathfrak{L}_1$ the set of all liftings of continuous circle maps of degree one.

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For each $F \in \mathfrak{L}_1$ and $x \in \mathbb{R}$ we define the *F-rotation number of x* as

$$\rho_F(x) := \limsup_{n\to\infty} \frac{F^n(x) - x}{n},$$

and the *rotation set of F* as:

$$\mathrm{Rot}(F) := \{\rho_F(x) : x \in \mathbb{R}\}.$$

From [9] we know that $\mathrm{Rot}(F)$ is a closed interval of $\mathbb{R}$.

To simplify the notation we will denote by $\mathrm{Rot}_I(F)$ the set of irrationals in $\mathrm{Rot}(F)$.

A circle map f is said to be *non-decreasing* whenever it has a non-decreasing lifting F.

We denote the cylinder by $\mathbb{S}^1 \times \mathcal{K}$ (where $\mathcal{K}$ is either a closed interval of $\mathbb{R}$ containing zero, $\mathbb{R}^+$ or $\mathbb{R}$ itself). We consider the class of skew-products on $\mathbb{S}^1 \times \mathcal{K}$ of the form:

$$\begin{pmatrix} \theta_{n+1} \\ x_{n+1} \end{pmatrix} = T \begin{pmatrix} \theta_n \\ x_n \end{pmatrix} \qquad \text{where} \qquad T \begin{pmatrix} \theta \\ x \end{pmatrix} = \begin{pmatrix} f(\theta) \\ p(x)q(\theta) \end{pmatrix}, \tag{1}$$

f is a continuous circle map of degree one with a lift F such that $\mathrm{Rot}_I(F)$ is non-empty, q is a continuous map from $\mathbb{S}^1$ to $\mathcal{K}$ and p is a continuous map from $\mathcal{K}$ to itself.

The function p plays an essential role in assuring that models of the above type have attracting sets (in a sense to be made precise below). Standard assumptions found in the literature on the function p are, $p(0) = 0$ and, for instance,

- p is strictly increasing and strictly concave when $\mathcal{K}$ is $\mathbb{R}^+$ (see [8, 7]) or strictly concave in the positives and strictly convex in the negatives when $\mathcal{K}$ is the whole $\mathbb{R}$ (see [5, 1]). Concrete examples or this map are $\tanh(x)$ in [8, 5] and x^α with $0 < \alpha < 1$ in [7].
- p is unimodal and $\mathcal{K} = [0, 1]$ or bimodal and $\mathcal{K} = [-1, 1]$ (see [3, 1]).

Also, some additional assumptions are demanded on the function q to assure the existence of attracting sets. On the other hand, in [8, 5, 1, 3] it is required that q vanishes at some point to assure that the attracting set is pinched, whereas in [7] the function q is assumed to be log-integrable with respect to an ergodic measure of the basis map.

In what follows we will denote the closure of a set X by $\mathrm{Cl}(X)$. Also we say that a set A is f-invariant whenever $f(A) = A$.

We look for *attracting sets* of System (1). They are defined as follows. Let μ be an ergodic measure of f and let $\mathscr{U}$ be a μ-measurable f-invariant set such that $\mu(\mathscr{U}) = 1$. Let $\varphi\colon \mathscr{U} \longrightarrow \mathcal{K}$ be a multivalued μ-measurable map whose graph is T-invariant on $\mathscr{U}$ (i.e. $T(\mathrm{graph}(\varphi)) = \mathrm{graph}(\varphi)$). The closure of $\mathrm{graph}(\varphi)$ will be called an *attracting set with support $\mathscr{U}$ and generated by φ* whenever

$$\lim_{n\to\infty} \|T^n(\theta, x) - T^n(\theta, z(x))\| = 0$$

for every $\theta \in \mathscr{U}$ and x in a subset of $\mathcal{K}$ of positive Lebesgue measure, and some $z(x) \in \varphi(\theta)$ (in particular, $\omega_T(\theta, x) \subset \omega_T(\theta, \varphi(\theta))$).

For every $\rho \in \mathbb{R}$ we denote the rotation by angle ρ by $\varPhi_\rho(\theta) := \theta + \rho \pmod 1$.

The main result of the paper is the following theorem which shows that the attracting sets of System (1) can be obtained from the attracting sets of quasiperiodically forced systems which can be considered to be subsystems of System (1).

Theorem A. *Consider a system of the form* (1). *To every $\rho \in \mathrm{Rot}_I(F)$ we can associate a measurable f-invariant set $\mathscr{U}_\rho \subset \mathbb{S}^1$, a continuous non-decreasing circle map of degree one h_ρ, and an f-ergodic measure μ_ρ such that $\mu_\rho(\mathscr{U}_\rho) = 1$,*

$\rho_F(e^{-1}(\mathrm{Cl}(\mathscr{U}_\rho))) = \rho$, $h_\rho\big|_{\mathscr{U}_\rho} : \mathscr{U}_\rho \longrightarrow h_\rho(\mathscr{U}_\rho)$ is a homeomorphism and $h_\rho(\mathscr{U}_\rho)$ is a dense Φ_ρ-invariant set. Additionally, the sets $\mathrm{Cl}(\mathscr{U}_\rho)$ are pairwise disjoint.

Assume that, for every $\rho \in \mathrm{Rot}_I(F)$, the system

$$(2)\qquad \begin{pmatrix}\theta_{n+1}\\ x_{n+1}\end{pmatrix} = S_\rho \begin{pmatrix}\theta_n\\ x_n\end{pmatrix} \qquad \text{where} \qquad S_\rho\begin{pmatrix}\theta\\ x\end{pmatrix} = \begin{pmatrix}\Phi_\rho(\theta)\\ p(x)q(h_\rho^{-1}(\theta))\end{pmatrix},$$

has an attracting set with support $h_\rho(\mathscr{U}_\rho)$ which is the closure of the graph of a multivalued map $\varphi_\rho \colon h_\rho(\mathscr{U}_\rho) \longrightarrow \mathcal{K}$. Then, the closure of the graph of $\varphi_\rho \circ h_\rho$ is an attracting set of T with support $\mathscr{U}_\rho$. Thus, whenever $\mathrm{Rot}(F)$ is non-degenerate, T has uncountably many attracting sets coexisting dynamically.

Remark 1.1. The map $q \circ h_\rho^{-1}$ is continuous in $h_\rho(\mathscr{U}_\rho)$ which is dense in $\mathbb{S}^1$. Hence, if $q \circ h_\rho^{-1}$ is discontinuous, it has only jump discontinuities in the complement of $h_\rho(\mathscr{U}_\rho)$.

Next we derive some consequences from Theorem A organized in a sequence of remarks.

Remark 1.2 (On the invariant measures of T)**.** The measure μ_ρ lifted to the closure of the graph of $\varphi_\rho \circ h_\rho$ is an invariant measure of T. Moreover, this measure is ergodic if and only if the closure of the graph of $\varphi_\rho \circ h_\rho$ is a minimal set of T. In particular this is a criterion to decide the undecomposability of the attracting set into several smaller attracting sets.

Remark 1.3 (On the number of pieces of an attracting set)**.** Theorem A tells us that the attracting sets of systems of the form (1) are graphs of multivalued maps from f-invariant subsets $\mathscr{U} \subset \mathbb{S}^1$ to the fibres. As it has been shown in [1] for the case when f is an irrational rotation and

- $\mathcal{K} = \mathbb{R}$, p is strictly increasing in $\mathbb{R}$ and strictly concave in $\mathbb{R}^+$ and strictly convex in $\mathbb{R}^-$; or
- p is bimodal and $\mathcal{K} = [-1, 1]$,

there are two possibilities for such an attracting set: either its closure is a minimal attractor or it splits into two different minimal attractors and each of these attractors is the closure of the graph of a map from $\mathscr{U}$ to the fibres. This dichotomy is inherited by the attracting sets of the systems of the form (1) with f an arbitrary continuous function of degree one that are obtained by transporting the attracting sets obtained in the case of rotations as stated in Theorem A.

Observe that when q is negative, the orbits keep alternating between $\mathbb{R}^+$ and $\mathbb{R}^-$. Thus, typically, there will be an attracting set which is a 2-periodic orbit of function graphs. When q is positive, System (1) can be split into two (one restricted to $\mathbb{R}^+$ and the other one restricted to $\mathbb{R}^-$). Consequently we get two attracting sets intersecting at $x \equiv 0$.

A particular example of the first case (that is, when the attracting set is a 2-periodic orbit of function graphs — q negative) is the following (see Figure 1): consider System (1) with $f = \Phi_{\frac{\sqrt{5}-1}{2}}$, $\mathcal{K} = \mathbb{R}$,

$$p(x) = \begin{cases} \tanh(x) & \text{if } x \geq 0\\ \frac{\tanh(x-2)+\tanh(2)}{1-\tanh(2)^2} & \text{if } x \leq 0 \end{cases} \quad \text{and } q(\theta) = \begin{cases} 7(\cos(2\pi\theta) - 2) & \text{if } \theta \in [0, \frac{1}{2})\\ 5(\cos(2\pi\theta) - 4) & \text{otherwise.} \end{cases}$$

Also, an example of the case when the attracting set has two minimal components intersecting at $x \equiv 0$ (q positive) can be obtained from System (1) by taking

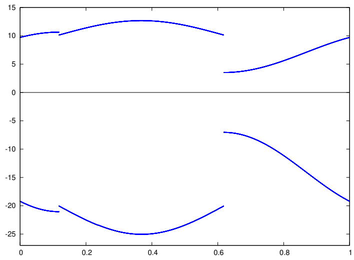

FIGURE 1. The unique attracting set for System (1) when p is monotone and q is negative.

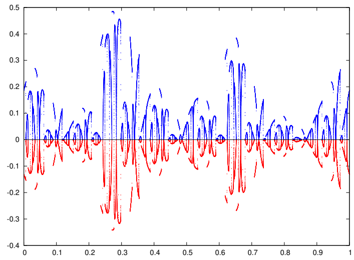

FIGURE 2. The two attracting sets (one in red and the other one in blue) for System (1) when p is bimodal and q is positive.

$f = \Phi_{\frac{\sqrt{5}-1}{2}}$, $\mathcal{K} = [-1, 1]$,

$$p(x) = \begin{cases} x(1-x) & \text{if } x \geq 0 \\ \frac{1}{2}x(x+1)(x+2) & \text{if } x \leq 0 \end{cases} \text{ and } q(\theta) = \begin{cases} 2.1|\cos(2\pi\theta)| & \text{if } \theta \in [0, \frac{1}{2}) \\ 2.5|\sin(2\pi\theta)| & \text{otherwise.} \end{cases}$$

In Figure 2 we show the two attracting sets for this system (one in red and the other one in blue).

Remark 1.4 (On the strangeness of the attracting sets)**.** An attracting set with support $\mathscr{U}$ and generated by φ will be called *strange* if $\varphi\big|_{\mathscr{U}}$ is discontinuous in a dense subset of $\mathscr{U}$.

Assume that the attracting set with support $h_\rho(\mathscr{U}_\rho)$ generated by φ_ρ is strange. Then, there exists a set D_ρ, dense in $h_\rho(\mathscr{U}_\rho)$, such that φ_ρ is discontinuous at every point of D_ρ. Since $h_\rho\big|_{\mathscr{U}_\rho}$ is a homeomorphism, $\varphi_\rho \circ h_\rho$ is discontinuous in $\left(h_\rho\big|_{\mathscr{U}_\rho}\right)^{-1}(D_\rho)$ and this set is dense in $\mathscr{U}_\rho$. Therefore the attracting set generated by $\varphi_\rho \circ h_\rho$ is also strange.

Remark 1.5 (On the Lyapunov exponents)**.** From the proof of Theorem A it follows that Systems (1) and (2) are semiconjugate by the map (h_ρ, Id). Hence, the vertical Lyapunov exponents of Systems (1) and (2) coincide on the attracting sets. In the complement of the attracting sets, when the Lyapunov exponents exist, the attractiveness implies that both of them are non-positive.

Examples of skew-products which satisfy the hypothesis required for System (2) can be found in [1]. They are extensions of known results from [8, 7, 3] to the case where the function q may have jump discontinuities in an invariant zero-Lebesgue measured subset of $\mathbb{S}^1$.

To prove Theorem A we need some more knowledge of rotation theory on the circle and its relation with water functions and to study the dynamics of non-decreasing degree one circle maps without periodic points. This will be done in the next section. Finally, in Section 3 we will prove Theorem A.

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2. WATER FUNCTIONS AND NON-DECREASING DEGREE ONE CIRCLE MAPS

We will start this section with a survey on rotation theory in the circle and water functions (see [2, Section 3.7]).

We begin by introducing some notation. In what follows $\mathfrak{L}_1^+$ will denote the class of non-decreasing maps of $\mathfrak{L}_1$.

If f is a continuous circle map then $\operatorname{Const}(f)$ denotes the open set of all points $\theta \in \mathbb{S}^1$ for which there exists a neighbourhood $U \ni \theta$ such that $f\big|_U$ is constant. Analogously, if F is a lifting of f, then $\operatorname{Const}(F)$ denotes the open set of all points $x \in \mathbb{R}$ for which there exists a neighbourhood $U \ni x$ such that $F\big|_U$ is constant. Observe that $\operatorname{Const}(F) = e^{-1}(\operatorname{Const}(f))$.

For every $F \in \mathfrak{L}_1$ there exists a circle map F^e defined by

$$F^e := e \circ F \circ (e\big|_{[0,1)})^{-1}$$

(observe that $e\big|_{[0,1)}$ is a homeomorphism). Clearly, F^e is continuous and has F as a lifting. The map F^e is called the *projection of F to* $\mathbb{S}^1$. Observe that whenever F is a lifting of f, then $F^e = f$.

Given $F \in \mathfrak{L}_1$ we define the *lower* and *upper* liftings as follows:

$$F_l(x) := \inf\{F(y) : y \geq x\}, \text{and}$$
$$F_u(x) := \sup\{F(y) : y \leq x\}.$$

Clearly, F_l and F_u belong to $\mathfrak{L}_1^+$, $F_l \leq F \leq F_u$ and, if F is non-decreasing then $F = F_l = F_u$. Moreover, if $F, G \in \mathfrak{L}_1$ are such that $F \leq G$, then $F_l \leq G_l$ and $F_u \leq G_u$.

Now, given $F \in \mathfrak{L}_1$ and $0 \leq \alpha \leq \|F - F_l\|_\infty$, we define the *water function of level* α as

$$F_\alpha := (\min\{F, F_l + \alpha\})_u$$

(observe that, from all said above, $\min\{F, F_l + \alpha\} \in \mathfrak{L}_1$ and, hence, F_α is well defined). The projection to $\mathbb{S}^1$ of the lifting F_α, F_α^e, will be denoted by f_α.

From the definition and the properties of the upper and lower liftings it follows that $F_\alpha \in \mathfrak{L}_1^+$ for every α, $F_0 = F_l$, $F_{\|F-F_l\|_\infty} = F_u$ and $F_\alpha \leq F_{\alpha'}$ whenever $\alpha \leq \alpha'$. Moreover, for every α, F_α coincides with F in the complement of $\operatorname{Const}(F_\alpha)$. Then, since $e(\operatorname{Const}(F_\alpha)) = \operatorname{Const}(f_\alpha)$, it follows that f_α coincides with f in the complement of $\operatorname{Const}(f_\alpha)$.

If $F \in \mathfrak{L}_1^+$, then

$$\rho_F(x) = \lim_{n\to\infty} \frac{F^n(x) - x}{n},$$

and the limit is independent of $x \in \mathbb{R}$. This number is denoted as $\rho(F)$ and called the *rotation number of* F. It is well known that the map $F \longmapsto \rho(F)$ from $\mathfrak{L}_1^+$ to $\mathbb{R}$ is continuous and non-decreasing. On the other hand, for every $F \in \mathfrak{L}_1$, $\operatorname{Rot}(F) = [\rho(F_l), \rho(F_u)]$.

From all said above it follows that the map $\alpha \longmapsto \rho(F_\alpha)$ from $[0, \|F - F_l\|_\infty]$ to $\operatorname{Rot}(F)$ is continuous, onto and non-decreasing.

Now we dynamically study the maps from $\mathfrak{L}_1^+$ with irrational rotation number. One of the important facts that we will use in this study is the semiconjugacy of circle maps with degenerate rotation interval to a rotation. Let f be a continuous circle map with lifting $F \in \mathfrak{L}_1^+$ such that $\rho(F)$ is irrational. From [6] we know that f is semiconjugate to the irrational rotation $\Phi_{\rho(F)}$ by a *non-decreasing* map h_f: $h_f \circ f = \Phi_{\rho(F)} \circ h_f$.

The result we look for is the following.

Proposition 2.1. *Let f be a continuous circle map with lifting $F \in \mathfrak{L}_1^+$ such that $\rho(F)$ is irrational. Then f has a measurable invariant set $\mathscr{U} \subset \mathbb{S}^1$ and a unique ergodic measure μ such that $\mu(\mathscr{U}) = 1$, $\operatorname{Cl}(\mathscr{U})$ is disjoint from* $\operatorname{Const}(f)$, *$h_f\big|_{\mathscr{U}}$ is a homeomorphism and $h_f(\mathscr{U})$ is a dense $\Phi_{\rho(F)}$-invariant set. If f is not a homeomorphism, then $\mathscr{U}$ is nowhere dense in $\mathbb{S}^1$.*

To prove Proposition 2.1 we will use the results by Auslander and Katznelson [4] on continuous circle maps without periodic points. To relate both situations observe that if f is a continuous circle map of degree one with lifting $F \in \mathfrak{L}_1^+$ having irrational rotation number then, from [2, Lemma 3.7.2] it follows that f has no periodic points. On the other hand, if f has no periodic points then, from [2, Sections 3.4–6 and Lemma 3.7.2] it follows that f must have degree one and, if F is a lifting of f, then $\operatorname{Rot}(F)$ is degenerate to an irrational.

Given a circle map f we will denote the *forward* and *backward* orbit of a point $\theta \in \mathbb{S}^1$ by $\operatorname{Orb}_f(\theta)$ and $\operatorname{Orb}^-_f(\theta)$ respectively:

$$\operatorname{Orb}_f(\theta) := \{f^n(\theta) : n \in \mathbb{Z}^+\},$$
$$\operatorname{Orb}^-_f(\theta) := \bigcup_{n\in\mathbb{Z}^+} f^{-n}(\theta).$$

The the next theorem is a version of [4, Theorem 2]. The unique ergodicity of f is proved at the very end of [4]. The fact that $\mathscr{P}$ is uncountable follows from the arguments from the third paragraph of [4, Page 380].

Theorem 2.2. *Let f be a continuous circle map without periodic points. Then f has a unique minimal set $\mathscr{P}$ and $\omega_f(\theta) = \mathscr{P}$ for all $\theta \in \mathbb{S}^1$. Moreover, f has a unique ergodic measure (and, consequently, its support is precisely $\mathscr{P}$). The set $\mathscr{P}$ is uncountable and, if f is not a homeomorphism, it is nowhere dense in $\mathbb{S}^1$. Additionally, for any $\theta \in \mathscr{P}$, $1 \le \operatorname{Card}(f^{-1}(\theta)\cap\mathscr{P}) \le 2$ and the set of $\theta \in \mathscr{P}$ such that $\operatorname{Card}(f^{-1}(\theta)\cap\mathscr{P}) = 2$ is countable. Moreover, for every $\theta \in \mathscr{P}$, there is at most one $\theta' \in \operatorname{Orb}_f(\theta) \cup \left(\operatorname{Orb}^-_f(\theta)\cap\mathscr{P}\right)$ such that $\operatorname{Card}(f^{-1}(\theta')\cap\mathscr{P}) = 2$.*

Remark 2.3. The set $\mathscr{P}$ is disjoint from $\operatorname{Const}(f)$. Otherwise, there exists a point $\theta \in U \cap \mathscr{P}$ where U is a connected component of $\operatorname{Const}(f)$. Also, by the minimality of $\mathscr{P}$, there exists $n \in \mathbb{N}$ such that $f^n(\theta) \in U$. Thus, since $f\big|_U$ is constant, $f(\theta) = f^{n+1}(\theta)$ is periodic; a contradiction.

Remark 2.4. For every $\theta \in \mathscr{P}$ we have $\mu(\theta) = 0$, where μ is the ergodic measure from Theorem 2.2. To show it we assume that there exists $\theta \in \mathscr{P}$ such that $\mu(\theta) > 0$ and set $A_0 = \{\theta\}$ and $A_i = f^{-1}(A_{i-1}) \cap \mathscr{P}$ for $i = 1, 2, \dots$. Since μ is f-invariant and has support $\mathscr{P}$, it follows by induction that $\mu(A_i) = \mu(\{\theta\}) > 0$ for every $i = 1, 2, \dots$. Suppose now that there exists $\nu \in \mathbb{S}^1$ and $l > n \ge 0$ such that $\nu \in A_l \cap A_n$. Then,

$$\theta = f^l(\nu) = f^{l-n}(f^n(\nu)) = f^{l-n}(\theta)$$

and θ is periodic; a contradiction. This shows that the sets A_i are pairwise disjoint. Hence, $\mu\left(\bigcup_{n\ge0} A_i\right) = \sum_{n=0}^{\infty} \mu(A_i)$ is infinite, which contradicts the fact that μ is a finite measure.

Now we are ready to prove Proposition 2.1.

In what follows we will denote the boundary (that is, the endpoints when X is homeomorphic to an interval of the real line) of a set X by $\operatorname{Bd}(X)$.

Proof of Proposition 2.1. As we have said before, f has no periodic points by [2, Lemma 3.7.2]. So, we can use Theorem 2.2. We get that f has a unique ergodic measure and the support of this measure is $\mathscr{P}$.

Next we construct the set $\mathscr{U}$ by removing some undesirable orbits from $\mathscr{P}$. To do it we denote by $\mathscr{S}$ the family of all connected components of $\mathbb{S}^1 \setminus \mathscr{P}$ and we set

$$\begin{aligned}
\widetilde{\mathscr{P}} &:= \bigcup_{C\in\mathscr{S}} \operatorname{Bd}(C),\\
\mathscr{U}^c &:= \left(\bigcup_{\theta\in\widetilde{\mathscr{P}}} \left(\bigcup_{n=0}^{\infty} \operatorname{Orb}^-_f(f^n(\theta))\right)\right) \cap \mathscr{P} \text{ and}\\
\mathscr{U} &:= \mathscr{P} \setminus \mathscr{U}^c.
\end{aligned}$$

From the above definitions it follows that $f(\mathscr{U}^c) \subset \mathscr{U}^c$ and $f^{-1}(\theta) \cap \mathscr{P} \subset \mathscr{U}^c$ for every $\theta \in \mathscr{U}^c$. Consequently, $f(\mathscr{U}) = \mathscr{U}$ because $f(\mathscr{P}) = \mathscr{P}$.

Since $\mathscr{P}$ is closed the set $\widetilde{\mathscr{P}}$ is at most countable and, hence, $\mathscr{U}^c$ is also at most countable by Theorem 2.2. Therefore, $\mathscr{U}^c$ is measurable and, by Remark 2.4, $\mu(\mathscr{U}^c) = 0$. Also, the set $\mathscr{P}$ consists of uncountably many orbits of f. Since $\mathscr{P}$ is measurable and $\mu(\mathscr{P}) = 1$ it follows that $\mathscr{U} = \mathscr{P} \setminus \mathscr{U}^c$ is measurable and $\mu(\mathscr{U}) = 1$.

Notice that, since $\mathscr{P}$ is minimal, $\mathscr{P} = \operatorname{Cl}(\operatorname{Orb}(\theta)) \subset \operatorname{Cl}(\mathscr{U}) \subset \mathscr{P}$ for every $\theta \in \mathscr{U}$. Hence, $\operatorname{Cl}(\mathscr{U}) = \mathscr{P}$.

The fact that $\operatorname{Cl}(\mathscr{U})$ is disjoint from $\operatorname{Const}(f)$ follows from Remark 2.3. Also, if f is not a homeomorphism, then $\operatorname{Cl}(\mathscr{U})$ is nowhere dense in $\mathbb{S}^1$ by Theorem 2.2.

Since f and $\Phi_{\rho(F)}$ are semiconjugate by h_f, $\Phi_{\rho(F)}(h_f(\mathscr{U})) = h_f(f(\mathscr{U})) = h_f(\mathscr{U})$. Consequently, since $\rho(F)$ is irrational, $\mathbb{S}^1 = \operatorname{Cl}\left(\operatorname{Orb}_{\Phi_{\rho(F)}}(\theta)\right) \subset \operatorname{Cl}(h_f(\mathscr{U}))$ for every $\theta \in h_f(\mathscr{U})$ and, hence, $h_f(\mathscr{U})$ is dense in $\mathbb{S}^1$.

Now we will show that $h_f\big|_{\mathscr{U}} \colon \mathscr{U} \longrightarrow h_f(\mathscr{U})$ is a homeomorphism. Since h_f is continuous and $h_f\big|_{\mathscr{U}} \colon \mathscr{U} \longrightarrow h_f(\mathscr{U})$ is clearly onto, it is enough to show that the map $h_f\big|_{\mathscr{U}}$ is one-to-one and closed.

First we prove that $h_f\big|_{\mathscr{U}}$ is one-to-one. Otherwise, Since h_f is non-decreasing, there exists an open interval $J \subset \mathbb{S}^1$ such that $\operatorname{Bd}(J) \subset \mathscr{U}$, $\operatorname{Card}(h_f(\operatorname{Bd}(J)) = 1$ and $h_f\big|_{\operatorname{Cl}(J)}$ is constant. If $\mathscr{P} \cap J = \emptyset$, then $J \in \mathscr{S}$ and so $\operatorname{Bd}(J) \subset \widetilde{\mathscr{P}} \subset \mathbb{S}^1 \setminus \mathscr{U}$. Thus, there exists a point $\theta \in \mathscr{P} \cap J$. By the minimality of $\mathscr{P}$, there exists $n \in \mathbb{N}$ such that $f^n(\theta) \in J$. Therefore,

$$h_f(\theta) = h_f(f^n(\theta)) = \Phi^n_{\rho(F)}(h_f(\theta));$$

a contradiction with the fact that $\rho(F)$ is irrational.

To show that $h_f\big|_{\mathscr{U}}$ is a closed map we will prove that $h_f(C \cap \mathscr{U})$ is a closed subset of $h_f(\mathscr{U})$ for every $C \subset \mathbb{S}^1$ closed. Since h_f is continuous and C is compact in $\mathbb{S}^1$, $h_f(C)$ is closed (compact) in $\mathbb{S}^1$. Hence, if we show that $h_f(C \cap \mathscr{U}) = h_f(C) \cap h_f(\mathscr{U})$ we are done. The inclusion $h_f(C \cap \mathscr{U}) \subset h_f(C) \cap h_f(\mathscr{U})$ holds trivially and we will prove the other inclusion by showing that if $\theta \in h_f(C) \cap h_f(\mathscr{U})$ then $h_f^{-1}(\theta) \subset C \cap \mathscr{U}$. Since h_f is continuous and non-decreasing, $h_f^{-1}(\theta)$ is either a single point or a non-degenerate closed arc of $\mathbb{S}^1$. In the first case, since $\theta \in h_f(C) \cap h_f(\mathscr{U})$, the only element of $h_f^{-1}(\theta)$ must belong to $C \cap \mathscr{U}$ as we wanted.

Now assume that $h_f^{-1}(\theta)$ is a non-degenerate closed arc of $\mathbb{S}^1$ and let J denote the interior of $h_f^{-1}(\theta)$. If there exists $\xi \in J \cap \mathscr{P}$, as before, the minimality of $\mathscr{P}$ implies that there exists $n \in \mathbb{N}$ such that $f^n(\xi) \in J$ and $h_f(\xi) = h_f(f^n(\xi)) = \Phi^n_{\rho(F)}(h_f(\xi))$; which contradicts the irrationality of $\rho(F)$. Consequently, J is contained in a connected component of $\mathbb{S}^1 \setminus \mathscr{P}$ and, hence,

$$\emptyset \neq h_f^{-1}(\theta) \cap \mathscr{U} \subset h_f^{-1}(\theta) \cap \mathscr{P} \subset \operatorname{Bd}(J) \cap \mathscr{P} \subset \widetilde{\mathscr{P}} \cap \mathscr{P} \subset \mathscr{U}^c;$$

a contradiction because $\mathscr{U} \cap \mathscr{U}^c = \emptyset$. □

Remark 2.5. It is not difficult to show that in the situation of the above proposition $f^{-1}(\mathscr{U}) = \mathscr{U}$ and $f^{-1}\big|_{\mathscr{U}}\colon \mathscr{U} \longrightarrow \mathscr{U}$ is a homeomorphism.

3. Proof of Theorem A

Since $\alpha \longmapsto \rho(F_\alpha)$ is a continuous map from $[0, \|F - F_l\|_\infty]$ onto $\operatorname{Rot}(F)$, to every $\rho \in \operatorname{Rot}_I(F)$ we can associate an $\alpha_\rho \in [0, \|F - F_l\|_\infty]$ such that $\rho(F_{\alpha_\rho}) = \rho$.

Then, for every $\rho \in \operatorname{Rot}_I(F)$, we denote respectively by $\mathscr{U}_\rho$, h_ρ and μ_ρ the set $\mathscr{U}$, the map $h_{f_{\alpha_\rho}}$ and the measure μ given by Proposition 2.1 for the map f_{α_ρ} with lifting F_{α_ρ}. Then we have that $\mathscr{U}_\rho$ is measurable and f_{α_ρ}-invariant, $\mu_\rho(\mathscr{U}_\rho) = 1$, $h_\rho\big|_{\mathscr{U}_\rho}\colon \mathscr{U}_\rho \longrightarrow h_\rho(\mathscr{U}_\rho)$ is a homeomorphism and $h_\rho(\mathscr{U}_\rho)$ is a dense Φ_ρ-invariant set.

Since f_{α_ρ} coincides with f in the complement of $\operatorname{Const}(f_{\alpha_\rho})$ and $\operatorname{Cl}(\mathscr{U}_\rho)$ is disjoint from $\operatorname{Const}(f_{\alpha_\rho})$,

$$f\big|_{\operatorname{Cl}(\mathscr{U}_\rho)} = f_{\alpha_\rho}\big|_{\operatorname{Cl}(\mathscr{U}_\rho)}. \tag{3}$$

Hence, $f(\mathscr{U}_\rho) = \mathscr{U}_\rho$ because $\mathscr{U}_\rho$ is f_{α_ρ}-invariant. Also, since F_{α_ρ} is non-decreasing,

$$\rho_F(e^{-1}(\operatorname{Cl}(\mathscr{U}_\rho))) = \rho_{F_{\alpha_\rho}}(e^{-1}(\operatorname{Cl}(\mathscr{U}_\rho))) = \rho(F_{\alpha_\rho}) = \rho.$$

Consequently, if $\rho, \rho' \in \operatorname{Rot}_I(F)$ and $\rho \neq \rho'$, then $\operatorname{Cl}(\mathscr{U}_\rho) \cap \operatorname{Cl}(\mathscr{U}_{\rho'}) = \emptyset$ since otherwise, for every $\theta \in e^{-1}(\operatorname{Cl}(\mathscr{U}_\rho) \cap \operatorname{Cl}(\mathscr{U}_{\rho'}))$, $\rho = \rho_F(\theta) = \rho'$; a contradiction.

Now we will prove that μ_ρ is an ergodic measure of f. First we will show that μ_ρ is f-invariant. From (3) we get

$$f^{-1}(A) \cap \mathscr{U} = f_{\alpha_\rho}^{-1}(A) \cap \mathscr{U} \tag{4}$$

for every $A \subset \mathbb{S}^1$. Hence, since $\mu_\rho(\mathscr{U}) = 1$,

$$\mu_\rho(f_{\alpha_\rho}^{-1}(A)) = \mu_\rho(f_{\alpha_\rho}^{-1}(A) \cap \mathscr{U}) = \mu_\rho(f^{-1}(A) \cap \mathscr{U}) = \mu_\rho(f^{-1}(A))$$

for every measurable set $A \subset \mathbb{S}^1$. Consequently, μ_ρ is f-invariant because it is f_{α_ρ}-invariant.

To prove the f-ergodicity of μ_ρ we will show that $\mu_\rho(A) \in \{0, 1\}$ for every measurable set $A \subset \mathbb{S}^1$ such that $\mu_\rho(f^{-1}(A) \triangle A) = 0$ (where $\triangle$ denotes the symmetric difference). By (4) we have

$$\begin{aligned}(f^{-1}(A) \triangle A) \cap \mathscr{U} &= (f^{-1}(A) \cap \mathscr{U}) \triangle (A \cap \mathscr{U}) = (f_{\alpha_\rho}^{-1}(A) \cap \mathscr{U}) \triangle (A \cap \mathscr{U}) \\ &= (f_{\alpha_\rho}^{-1}(A) \triangle A) \cap \mathscr{U}.\end{aligned}$$

Consequently,

$$\mu_\rho(f_{\alpha_\rho}^{-1}(A) \triangle A) = \mu_\rho(f^{-1}(A) \triangle A) = 0$$

and, since μ_ρ is f_{α_ρ}-ergodic, $\mu_\rho(A) \in \{0, 1\}$.

Now we assume that, for every $\rho \in \operatorname{Rot}_I(F)$, the system

$$\begin{pmatrix}\theta_{n+1}\\ x_{n+1}\end{pmatrix} = S_\rho \begin{pmatrix}\theta_n\\ x_n\end{pmatrix} \qquad \text{where} \qquad S_\rho \begin{pmatrix}\theta\\ x\end{pmatrix} = \begin{pmatrix}\Phi_\rho(\theta)\\ p(x)q(h_\rho^{-1}(\theta))\end{pmatrix}, \tag{5}$$

has an attracting set with support $h_\rho(\mathscr{U}_\rho)$ which is the closure of the graph of a multivalued map $\varphi_\rho\colon h_\rho(\mathscr{U}_\rho) \longrightarrow \mathcal{K}$.

We will prove that the closure of the graph of $\widetilde{\varphi}_\rho\colon \mathscr{U}_\rho \longrightarrow \mathcal{K}$, where $\widetilde{\varphi}_\rho := \varphi_\rho \circ h_\rho$, is an attracting set of T with support $\mathscr{U}_\rho$. To this end we will use the map $H := (h_\rho, \operatorname{Id})$ which is a homeomorphism from $\mathscr{U}_\rho \times \mathcal{K}$ to $h_\rho(\mathscr{U}_\rho) \times \mathcal{K}$.

Recall that h_ρ is a semiconjugacy between f_{α_ρ} and Φ_ρ. Thus,

$$h_\rho \circ f_{\alpha_\rho} = \Phi_\rho \circ h_\rho$$

and hence, from (3),

$$h_\rho(f(\theta)) = h_\rho(f_{\alpha_\rho}(\theta)) = \Phi_{\rho(F)}(h_\rho(\theta)) \tag{6}$$

for every $\theta \in \mathscr{U}_\rho$. Therefore,

$$\begin{aligned} H(T(\theta, z)) &= (h_\rho(f(\theta)), p(z)q(\theta)) = \big(\Phi_\rho(h_\rho(\theta)), p(z)q(h_\rho^{-1}(h_\rho(\theta)))\big) \\ &= S_\rho(H(\theta, z)), \end{aligned}$$

for every $\theta \in \mathscr{U}_\rho$ and $z \in \mathcal{K}$. Thus, for each $n \geq 0$ and $z \in \mathcal{K}$,

$$\pi_x\left(S_\rho^n(h_\rho(\theta), x)\right) = \pi_x\left(S_\rho^n(H(\theta, x))\right) = \pi_x\left(H(T^n(\theta, x))\right) = \pi_x\left(T^n(\theta, x)\right) \tag{7}$$

where $\pi_x \colon \mathbb{S}^1 \times \mathcal{K} \longrightarrow \mathcal{K}$ denotes the projection with respect to the second component.

First we will show that the graph of $\widetilde{\varphi}_\rho$ is T-invariant. That is, by (6), we have to prove

$$\pi_x\left(T(\theta, x)\right) \in \widetilde{\varphi}_\rho(f(\theta)) = \varphi_\rho(\Phi_{\rho(F)}(h_\rho(\theta)))$$

for every $\theta \in \mathscr{U}_\rho$ and $x \in \widetilde{\varphi}_\rho(\theta)$. By assumption, the graph of φ_ρ is S_ρ-invariant: $\pi_x\left(S_\rho(h_\rho(\theta), x)\right) \in \varphi_\rho(\Phi_{\rho(F)}(h_\rho(\theta)))$ for every $\theta \in \mathscr{U}_\rho$ and $x \in \varphi_\rho(h_\rho(\theta)) = \widetilde{\varphi}_\rho(\theta)$. From (7), we get $\pi_x\left(S_\rho(h_\rho(\theta), x)\right) = \pi_x\left(T(\theta, x)\right)$, which proves the T-invariance of the graph of $\widetilde{\varphi}_\rho$.

Now we have to prove that the closure of the graph of $\widetilde{\varphi}_\rho$ is an attracting set of T with support $\mathscr{U}_\rho$. Observe that, for any skew product R on the cylinder $\mathbb{S}^1 \times \mathcal{K}$ and any $n \geq 0$, $\theta \in \mathbb{S}^1$ and $x, z \in \mathcal{K}$,

$$\|R^n(\theta, x) - R^n(\theta, z)\| = |\pi_x\left(R^n(\theta, x)\right) - \pi_x\left(R^n(\theta, z)\right)|.$$

Therefore, we have to show that

$$\lim_{n\to\infty}\left(\pi_x\left(T^n(\theta, x)\right) - \pi_x\left(T^n(\theta, z(x))\right)\right) = 0 \tag{8}$$

for every $\theta \in \mathscr{U}_\rho$ and x in a subset of $\mathcal{K}$ of positive Lebesgue measure, and some $z(x) \in \widetilde{\varphi}_\rho(\theta)$. By assumption we know that

$$\lim_{n\to\infty}\left(\pi_x\left(S_\rho^n(h_\rho(\theta), x)\right) - \pi_x\left(S_\rho^n(h_\rho(\theta), z(x))\right)\right) = 0$$

for every $\theta \in \mathscr{U}_\rho$ and x in a subset of $\mathcal{K}$ of positive Lebesgue measure, and some $z(x) \in \varphi_\rho\left(h_\rho(\theta)\right) = \widetilde{\varphi}_\rho(\theta)$. From (7) this is equivalent to (8). This ends the proof of the theorem.

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