GLOBAL DYNAMICS OF A SYSTEM COMINING FROM THE STUDY OF A STATIC STAR

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Abstract. We classify the global dynamics of a one-parameter family of planar quadratic polynomial differential systems which for some interval of values of its parameter describes the evolution of an static star. The characterization of their distinct topological phase portraits is done in the Poincaré disc. In this way we can describe the dynamics of these systems near infinity and to provide their global phase portrait.

1. Introduction and statement of the main results

The structure equations using geometrical units for a static star in general relativity are

$$\frac{dM}{dr} = 4\pi r^2 \rho \tag{1}$$

and

$$\frac{dP}{dr} = -\frac{(\rho + P)(M + 4\pi r^3 P)}{r^2(1 - 2M/r)}, \tag{2}$$

where $M(r)$ is the mass of the star inside a sphere of radius r from the center of mass satisfying $M(r) \geq 0$ for all $r \geq 0$, and $M(r) \to 0$ as $r \to 0$, r is the distance to the center of mass of this star, ρ is the density and $P(r)$ is the pressure inside the sphere of radius r such that $P(r) \to 0$ as $r \to 0$.

Now we consider the equations (1) and (2) for a static star in general relativity in the case of the existence of a homologous family of solutions which requires the existence of an state equation given by $P = (\gamma - 1)\rho$ with $1 < \gamma \leq 2$, see for details [4, 5]. For convenience we express equations (1) and (2) in function of the variables $x := M/r$ and $y := 4\pi r^2 \rho$, and setting $t = \log r$, the equations take the form

$$\dot{x} = y - x, \quad \dot{y} = \frac{y}{1 - 2x}\Big(2 - \frac{5\gamma - 4}{\gamma - 1}x - \gamma y\Big), \tag{3}$$

where the dot means derivative with respect to the variable t. Note that this system is not defined when $\gamma = 1$.

These equations where obtained from Collins in [5]. However the equations in [5] have a typo because they appear as

$$\dot{x} = y - x, \quad \dot{y} = \frac{y}{1 - 2y}\Big(2 - \frac{5\gamma - 4}{\gamma - 1}x - \gamma y\Big).$$

But this typo does not affect the results of [5] which are excellent. Due to the importance of equations (3) and their simplicity they have been studied by many authors from different points of view, see for instance [3, 9]. We note that the mentioned typo in paper [5] was not detected by all the readers, see for instance [2].

The objective of this paper is to study the global dynamics of the differential system (3) for all values of its parameter $\gamma \in \mathbb{R} \setminus \{1\}$ in the whole plane $\mathbb{R}^2$. Of course for its physical applications we are only interested in the values of $\gamma \in (1, 2]$ and in the positive quadrant of the plane $\mathbb{R}^2$.

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We write the differential system (3) as the quadratic polynomial differential system

$$\dot{x} = (y-x)(1-2x), \quad \dot{y} = y\Big(2 - \frac{5\gamma-4}{\gamma-1}x - \gamma y\Big), \tag{4}$$

doing the change of the independent variable $ds = (1-2x)dt$, where now the dot denotes derivative with respect the new independent variable s.

Note that the dynamics of system (3) is not defined on the straight line $x = 1/2$. However for system (4) the dynamics are defined in the whole plane $\mathbb{R}^2$. Since the qualitative theory of differential equaitons is very well developped for the polynomial differential systems we will work with system (4), obtaining all the phase portraits of system (4) in the whole plane $\mathbb{R}^2$. After removing from these phase portraits the straight line $x = 1/2$ we get the phase portraits of system (3).

In fact we shall present the distinct global phase portraits of system (4) when its parameter γ varies in $\mathbb{R}\setminus\{0\}$ in the Poincaré disc. In this way we can describe the dynamics of their orbits which come or go to the infinity of $\mathbb{R}^2$. For doing this we will use the Poincaré compactification.

Roughly speaking the Poincaré compactification of the polynomial differential system (4) consists in extending this system to an analytic system on a closed disc $\mathbb{D}^2$ of radius one, whose interior is identified with $\mathbb{R}^2$ and its boundary, the circle $\mathbb{S}^1$, plays the role of the infinity. This closed disc is called the Poincaré disc, because the technique for doing such an extension is due to Poincaré. For details on this compactification see [7, chapter 5] or the summary presented in subsection 2.1.

The main result of the paper is the following.

Theorem 1. *The phase portraits of system* (4) *for* $\gamma \in \mathbb{R} \setminus \{0, 1\}$ *in the Poincaré disc are topologically equivalent to one of the* 13 *phase portraits of Figure* 1*.*

The proof of Theorem 1 is given in section 3.

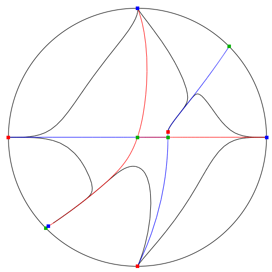
We note that really there are only 12 different topological phase portraits because the phase portraits (i) and (j) of Figure 1 are topologically equivalent, see a precise definition of topologicall equivalence in subsection 2.2. The unique difference between them is that a node in the first is a focus in the second. Note that the mentiooned node of (i) is very close to the infinity and difficult to distinguish. In the phase portrait (m) a saddle appears mixed with an unstable node and it is not possible to appreciate correctly all the separatrices and canonical regions in this case. We note that all the phase portraits of Figure 1 are quantitative phase portraits for some values of the parameter γ in the corresponding intervals given in Table 1.

Moreover the letters S and R which appear in each of the phase portraits of Figure 1 denote the number of separatrices and the number of canonical regions that each phase portrait has. For the definition of separatrix and canonical region see subsection 2.2.

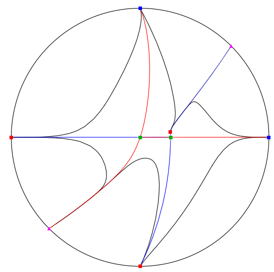
We can think on the Poincaré disc $\mathbb{D}^2$ as the disc of radius one centered at the origin of coordinates of the plane. The intersection of the x and y axes of the plane with such a disc are identified with the x and y axes of $\mathbb{R}^2$, so the positive quadrant Q of $\mathbb{R}^2$ is identified with the region $\mathbb{D}^2 \cap Q$ on the Poincaré disc.

Going back to the physical problem of the static star we must look at the regions $\mathbb{D}^2 \cap Q$ of the phase portraits (j) and (k) of Figure 1, and removing from them the corresponding identification of the straight line $x = 1/2$ in $\mathbb{D}^2$. Doing so we obtain the phase portraits in the positive quadrant Q of $\mathbb{R}^2$ for $\gamma \in (1, 2)$ and $\gamma = 2$, respectively. These two phase portraits are topologically equivalent in the positive quadrant Q of $\mathbb{R}^2$, and of course they coincide with the phase portrait already studied in Q by Misner and Zapolsky [9] and Collins [5].

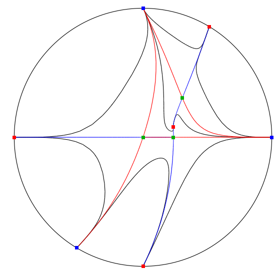
In the next section we summarize the basic results that we need for proving our Theorem 1.



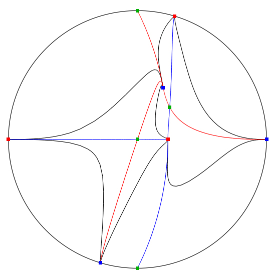
(a) $S = 25,\ R = 6.$



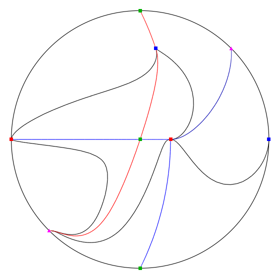
(b) $S = 23,\ R = 6.$



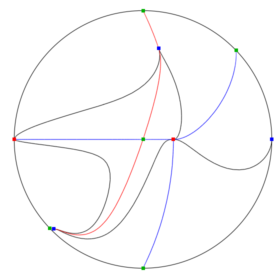
(c) $S = 29,\ R = 8.$



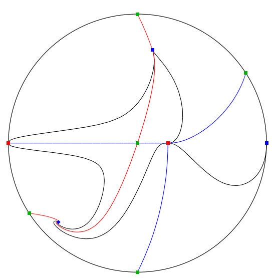
(d) $S = 26,\ R = 7.$



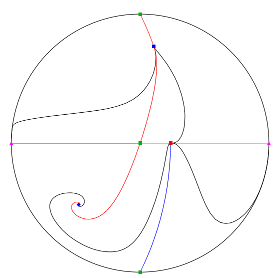
(e) $S = 22,\ R = 5.$



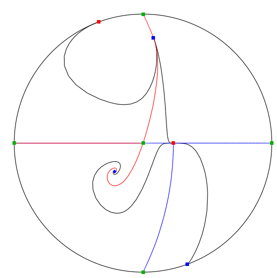
(f) $S = 24,\ R = 5.$



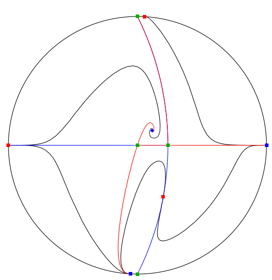
(g) $S = 24,\ R = 5.$



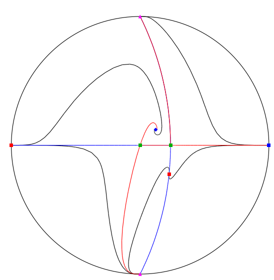
(h) $S = 19,\ R = 4.$



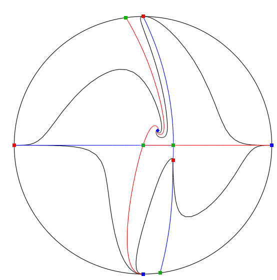
(i) $S = 23,\ R = 4.$



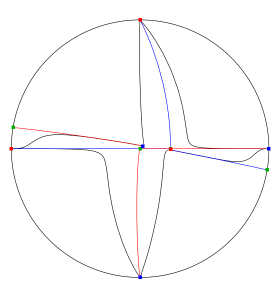
(j) $S = 24,\ R = 5.$



(k) $S = 20,\ R = 5.$



(l) $S = 25,\ R = 6.$



(m) $S = 26,\ R = 8.$

FIGURE 1. Phase portraits of system (4) on the Poincaré disc.

2. Preliminary results

2.1. **Poincaré compactification.** In order to classify the global dynamics of a polynomial differential system the first crucial step is to characterize their finite and infinite singular points in the Poincaré compactification [12]. The second main step for determining the global dynamics in the Poincaré disc of a polynomial differential system is the characterization of their separatrices. For the polynomial differential systems in the Poincaré disc it is known that the *separatrices* are the infinite orbits, the finite singular points, the separatrices of the hyperbolic sectors of the finite and infinite singular points, and the limit cycles. If Σ denotes the set of all separatrices in the Poincaré disc $\mathbb{D}^2$, Σ is a closed set and the components of $\mathbb{D}^2 \setminus \Sigma$ are called the canonical regions. We denote by S and R the number of separatrices and *canonical regions*, respectively.

We consider the set of all polynomial vector fields in $\mathbb{R}^2$ of the form

$$(\dot{x_1}, \dot{x_2}) = X(x_1, x_2) = (P(x_1, x_2), Q(x_1, x_2)), \tag{5}$$

where P and Q are real polynomials in the variables x_1 and x_2 of degrees d_1 and d_2, respectively. Take $d = \max\{d_1, d_2\}$.

Denote by $T_p\mathbb{S}^2$ be the tangent space to the 2-dimensional sphere $\mathbb{S}^2 = \{\mathbf{s} = (s_1, s_2, s_3) \in \mathbb{R}^3 : s_1^2 + s_2^2 + s_3^2 = 1\}$ at the point p. Assume that X is defined in the plane $T_{(0,0,1)}\mathbb{S}^2 = \mathbb{R}^2$. Consider the central projection $f\colon T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2$. This map defines two copies of X, one in the open northern hemisphere and the other in the open southern hemisphere. Denote by X' the vector field $Df \circ X$ defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. Clearly $\mathbb{S}^1$ is identified to the infinity of $\mathbb{R}^2$. If X is a planar polynomial vector field of degree d, then $p(X)$ is the only analytic extension of $y_3^{d-1}X'$ to $\mathbb{S}^2$. The vector field $p(X)$ is called the *Poincaré compactification* of the vector field X, for more details see [7, chapter 5].

On the Poincaré sphere $\mathbb{S}^2$ we use the following six local charts to do the calculations, which are given by $U_i = \{\mathbf{s} \in \mathbb{S}^2 : \ s_i > 0\}$ and $V_i = \{\mathbf{s} \in \mathbb{S}^2 : \ s_i < 0\}$, for $i = 1, 2, 3$, with the corresponding diffeomorphisms

$$\varphi_i : \ U_i \to \mathbb{R}^2, \qquad \psi_i : \ V_i \to \mathbb{R}^2, \tag{6}$$

defined by $\varphi_i(\mathbf{s}) = -\psi_i(\mathbf{s}) = (s_m/s_i, s_n/s_i) = (u, v)$ for $m < n$ and $m, n \neq i$. Thus (u, v) will play different roles in the distinct local charts. The expressions of the vector field $p(X)$ are

$$(\dot{u}, \dot{v}) = \left(v^d \left(Q\left(\frac{1}{v}, \frac{u}{v}\right) - uP\left(\frac{1}{v}, \frac{u}{v}\right)\right), -v^{d+1}P\left(\frac{1}{v}, \frac{u}{v}\right)\right) \qquad \text{in } U_1,$$

$$(\dot{u}, \dot{v}) = \left(v^d \left(P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right)\right), -v^{d+1}Q\left(\frac{u}{v}, \frac{1}{v}\right)\right) \qquad \text{in } U_2,$$

$$(\dot{u}, \dot{v}) = (P(u, v), Q(u, v)) \qquad \text{in } U_3.$$

We note that the expressions of the vector field $p(X)$ in the local chart (V_i, ψ_i) is equal to the expression in the local chart (U_i, ϕ_i) multiplied by $(-1)^{d-1}$ for $i = 1, 2, 3$.

The orthogonal projection under $\pi(y_1, y_2, y_3) = (y_1, y_2)$ of the closed northern hemisphere of $\mathbb{S}^2$ onto the plane $s_3 = 0$ is a closed disc $\mathbb{D}^2$ of radius one centered at the origin of coordinates called the *Poincaré disc*. Since a copy of the vector field X on the plane $\mathbb{R}^2$ is in the open northern hemisphere of $\mathbb{S}^2$, the interior of the Poincaré disc $\mathbb{D}^2$ is identified with $\mathbb{R}^2$ and the boundary of $\mathbb{D}^2$, the equator $\mathbb{S}^1$ of $\mathbb{S}^2$, is identified with the infinity of $\mathbb{R}^2$. Consequently the phase portrait of the vector field X extended to the infinity corresponds to the projection of the phase portrait of the vector field $p(X)$ on the Poincaré disc $\mathbb{D}^2$.

The singular points of $p(X)$ in the Poincaré disc lying on $\mathbb{S}^1$ are the *infinite singular points* of the corresponding vector field X. The singular points of $p(X)$ in the interior of the Poincaré disc, i.e. on $\mathbb{S}^2 \setminus \mathbb{S}^1$, are the *finite singular points*. We note that in the local charts U_1, U_2, V_1 and V_2 the infinite singular points have their coordinate $v = 0$.

For a polynomial vector field (5) if $s \in \mathbb{S}^1$ is an infinite singular point, then $-s \in \mathbb{S}^1$ is another infinite singular point. Thus the number of infinite singular points is even and the local phase portrait of one is that of the other multiplied by $(-1)^{d+1}$.

2.2. **Separatrix skeleton.** Given a flow $(\mathbb{D}^2, \phi)$ by the *separatrix skeleton* we mean the union of all the separatries of the flow together with one orbit from each one of the canonical regions. Let C_1 and C_2 be the separatrix skeletons of the flows $(\mathbb{D}^2, \phi_1)$ and $(\mathbb{D}^2, \phi_2)$ respectively. We say that C_1 and C_2 are *topologically equivalent* if there exists a homeomorphism $h : \mathbb{D}^2 \to \mathbb{D}^2$ which sends orbits to orbits preserving or reversing the direction of all orbits. From Markus [8], Neumann [10] and Peixoto [11] it follows the next theorem which shows that is enough to describe the separatrix skeleton in order to determine the topological equivalence class of a differential system in the Poincaré disc $\mathbb{D}^2$.

Theorem 2 (Markus–Neumann–Peixoto Theorem)**.** *Assume that $(\mathbb{D}^2, \phi_1)$ and $(\mathbb{D}^2, \phi_2)$ are two continuous flows with only isolated singular points. Then these flows are topologically equivalent if and only if their separatrix skeletons are equivalent.*

3. Proof of Theorem 1

We separate the proof in three subsections.

3.1. **Finite singular points.** We note that the two components of the quadratic polynomial differential system (4) for $\gamma = 0$ has the common factor $1-2x$, so for this value of the parameter γ the system has the straight line $x = 1/2$ filled of singular points. So doing the change of the independent variable $d\tau = (1-2x)ds$ the differential system becomes a linear differential system and we do consider this easy case.

An easy computation shows that the finite singular points of the polynomial differential system (4) are

$$p_1 = (0,0), \quad p_2 = \left(\frac{1}{2}, \frac{1}{2-2\gamma}\right), \quad p_3 = \left(\frac{2\gamma-2}{\gamma^2+4\gamma-4}, \frac{2\gamma-2}{\gamma^2+4\gamma-4}\right), \quad p_4 = \left(\frac{1}{2}, 0\right).$$

The singular point p_1, p_2 and p_3 exist for all $\gamma \in \mathbb{R} \setminus \{0,1\}$, but the singular point p_3 only exist if $\gamma \in \mathbb{R} \setminus \{2(\sqrt{2}-1), 0, 1, 2(\sqrt{2}+1)\}$.

We can determine all the local phase portraits of these finite singular points using the Hartman–Grobman theorem (see for instance [7, Theorem 2.15]) because as we shall see all these singular points are hyperbolic when $\gamma \in \mathbb{R} \setminus \{0,1\}$. Recall that a singular point p is *hyperbolic* if the real part of the eigenvalues of the linear part of the differential system evaluated at p are not zero.

The eigenvalues of the singular point p_1 are -1 and 2, so p_1 is a saddle.

The eigenvalues of the singular point p_2 are $\gamma/(\gamma-1)$ and $\gamma/(2\gamma-2)$, so p_2 is an unstable node if $\gamma < 0$, a stable node if $\gamma \in (0,1)$, and an unstable node if $\gamma > 1$.

The eigenvalues of the singular point p_3 are

$$\frac{\gamma\left(2-3\gamma \pm \sqrt{\gamma^2-44\gamma+36}\right)}{2\left(\gamma^2+4\gamma-4\right)}.$$

Therefore p_3 is a stable node if $\gamma < -2(\sqrt{2}+1)$, a saddle if $\gamma \in (-2(\sqrt{2}+1), 2(\sqrt{2}-1)) \setminus \{0\}$, a stable node if $\gamma \in (2(\sqrt{2}-1), 2(11-4\sqrt{7}]$, a stable focus if $\gamma \in (2(11-4\sqrt{7}), 2(11+4\sqrt{7})) \setminus \{1\}$, and a stable node if $\gamma >= 2(11+4\sqrt{7})$.

The eigenvalues of the singular point p_4 are 1 and $\gamma/(2-2\gamma)$. Hence p_4 is a saddle if $\gamma < 0$, an unstable node if $\gamma \in (0,1)$, and a saddle if $\gamma > 1$.

3.2. **Infinite singular points.** From subsection 2.1 the polynomial differential system (4) in the local chart U_1 writes

$$\dot{u} = \frac{u(7\gamma - 6 + (2 - 3\gamma + \gamma^2)u - 3(\gamma - 1)v + (\gamma - 1)uv)}{1 - \gamma}, \quad \dot{v} = (1 - u)(v - 2)v.$$

In this local chart the system has the infinite singular points

$$S_1 = (0, 0), \qquad S_2 = \left(\frac{7\gamma - 6}{(\gamma - 1)(\gamma - 2)}, 0\right).$$

The eigenvalues of the singular point S_1 are -2 and $(6 - 7\gamma)/(\gamma - 1)$. So S_1 is hyperbolic except when $\gamma = 6/7$ which has one eigenvalue zero and consequently is semi-hyperbolic, thus being its local phase portrait determined by [7, Theorem 2.19]. Therefore we obtain that S_1 is a stable node if $\gamma < 6/7$ and γ is not zero, a semi-hyperbolic saddle-node if $\gamma = 6/7$, a saddle if $\gamma \in (6/7, 1)$, and a stable node if $\gamma > 1$.

The eigenvalues of the singular point S_2 are $(6 - 7\gamma)/(\gamma - 1)$ and $-2\left(\gamma^2 + 4\gamma - 4\right)/((\gamma - 2)(\gamma - 1))$. Therefore S_2 is a saddle if $\gamma < -2(\sqrt{2} + 1)$, a semi-hyperbolic saddle-node if $\gamma = -2(\sqrt{2}+1)$, an unstable node if $\gamma \in (-2(\sqrt{2}1), 2(\sqrt{2}+1)) \setminus \{0\}$, a semi-hyperbolic saddle-node if $\gamma = 2(\sqrt{2} - 1)$, a saddle if $\gamma \in (2(\sqrt{2} - 1), 6/7)$, a stable node if $\gamma \in (6/7, 1)$, a unstable node if $\gamma \in (1, 2)$, and a saddle if $\gamma > 2$.

In order to complete the study of the infinite singular points we must study if the origin of the local chart U_2 is an infinite singular point, because it is the unique point at infinity together with the origin of the local chart V_2, which is not covered by the local charts U_1 and V_1.

Again from subsection 2.1 the polynomial differential system (4) in the local chart U_2 writes

$$\dot{u} = \frac{((\gamma - 2)(\gamma - 1)u + (\gamma - 1)v + (7\gamma - 6)u^2 - 3(\gamma - 1)uv}{\gamma - 1},$$
$$\dot{v} = \frac{v(\gamma^2 - \gamma + (5\gamma - 4)u - 2(\gamma - 1)v)}{\gamma - 1}.$$

The eigenvalues of the origin O of U_2 are γ and $\gamma - 2$. Hence O is a stable node if $\gamma < 0$, a saddle if $\gamma \in (0, 1)$, a saddle if $\gamma \in (1, 2)$, a semi-hyperbolic saddle-node if $\gamma = 2$, and an unstable node if $\gamma > 2$.

Of course we also have the diametrally opposite infinite singular points in the local charts V_1 and V_2, and since the degree of system (4) is two, their orientation of the orbits at those infinite singular points is the contrary to the ones of the local charts U_1 and U_2.

3.3. **Separatrices and phase portraits in the Poincaré disc.** According with Theorem 2 in order to obtain the global phase portraits of the polynomial differential system (4) we must draw their separatrix skeleton depending on the parameter γ.

Since all the finite and infinite singular points are separatrices of the polynomial differential system (4) we have unified them in Table 1, where we denote by S, N^u, N^s, F^s and SN a saddle, an unstable node, a stable node, a stable focus and a saddle-node, respectively.

Additionally to the singular points the other separatrices are the limit cycles and the separatrices of the hyperbolic sectors of the finite and infinite singular points.

Bautin in [1] (see also [6]) proved that the quadratic polynomial differential system having two invariant straight lines has no limit cycles. Since our quadratic polynomial differential system (4) has the two invariant straight lines $x = 1/2$ and $y = 0$, it has no limit cycles.

In summary, we only need to determine the behaviour of the separatrices of the hyperbolic sectors of the finite and infinite singular points, i.e. where they born and where they die. Doing so we will have all the separatrix skeleton adding one orbit in each canonical region, and consequently we will have the global phase portraits of system (4). But taking into account the two invariant straight lines and the local phase portraits at all the finite and infinite singular

γ	p_1	p_2	p_3	p_4	S_1	S_2	O
$\gamma < -2(\sqrt{2}+1)$	S	N^u	N^s	S	N^s	S	N^s
$\gamma = -2(\sqrt{2}+1)$	S	N^u		S	N^s	SN	N^s
$\gamma \in (-2(\sqrt{2}+1), 0)$	S	N^u	S	S	N^s	N^u	N^s
$\gamma \in (0, 2(\sqrt{2}-1))$	S	N^s	S	N^u	N^s	N^u	S
$\gamma = 2(\sqrt{2}-1)$	S	N^s		N^u	N^s	SN	S
$\gamma \in (2(\sqrt{2}-1), 2(11-4\sqrt{7}))$	S	N^s	N^s	N^u	N^s	S	S
$\gamma \in (2(11-4\sqrt{7}), 6/7)$	S	N^s	F^s	N^u	N^s	S	S
$\gamma = 6/7$	S	N^s	F^s	N^u	SN		S
$\gamma \in (6/7, 1)$	S	N^s	F^s	N^u	S	N^s	S
$\gamma \in (1, 2)$	S	N^u	F^s	S	N^s	N^u	S
$\gamma = 2$	S	N^u	F^s	S	N^s		SN
$\gamma \in (2, 2(11+4\sqrt{7}))$	S	N^u	F^s	S	N^s	S	N^u
$\gamma \geq 2(11+4\sqrt{7})$	S	N^u	N^s	S	N^s	S	N^u

Table 1. The finite and infinite singular points of system (4).

points, the place where born and die all the separatrices of the hyperbolic sectors is determined in a unique way for every one of the 13 cases in function of the parameter γ described in Table 1. In this way we obtain the 13 phase portraits in the Poincaré disc of Figure 1. As we have mentioned in the introduction section the phase portraits (i) and (j) of Figure 1 are topologically equivalent because the unique difference between them is that a node in the first one is a focus in the second. This completes the proof of Theorem 1.

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