LIMIT CYCLES OF A GENERALIZED MATHIEU DIFFERENTIAL SYSTEM

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ABSTRACT. We study the maximum number of limit cycles which bifurcate from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, when it is perturbed in the form

$$\dot{x} = y - \varepsilon(1 + \cos^l \theta)P(x, y), \quad \dot{y} = -x - \varepsilon(1 + \cos^m \theta)Q(x, y), \tag{1}$$

where $\varepsilon > 0$ is a small parameter, l and m are positive integers, $P(x, y)$ and $Q(x, y)$ are arbitrary polynomials of degree n, and $\theta = \arctan(y/x)$. As we shall see the differential system (1) is a generalization of the Mathieu differential equation. The tool for studying such limit cycles is the averaging theory.

1. Introduction and Statement of the Main Results

A *limit cycle* of a differential system is a periodic orbit having a neighborhood where it is the unique periodic orbit of the differential system. The notion of limit cycle was introduced in 1881 by Poincaré [10].

The study of the existence and number of limit cycles that a differential system in $\mathbb{R}^2$ can exhibit is one of the more difficult problems in the qualitative theory of the differential system in the plane. Thus in 1900 Hilbert [6] presented a list of 23 problems to the International Conference of Mathematicians in Paris, most of these problems were solved partially or completely, but the second part of the 16th problem remains unsolved up today. This problem ask about the existence of an upper bound for the maximal number of limit cycles that polynomial differential systems in $\mathbb{R}^2$ of a given degree can exhibit.

A source of producing limit cycles is by perturbing the periodic orbits of a center, see for instance the papers [3, 11] and the book of Christhoper and Li [5], and the hundreds of references quoted there.

The classical Mathieu's differential equation [9]) is

$$\ddot{x} + b\,(1 + \cos\theta)\,x = 0,$$

where b is real parameter, and the dots denote second derivative with respect to the time t. This equation was first discussed in 1868 by Mathieu while studying the problem of vibrations on an elliptical drumhead. Matthieu's equation has many applications in engineering [12, 14] and also in theoretical and experimental physics [2, 15].

Mathieu's equation can be written as the differential system

$$\dot{x} = y, \qquad \dot{y} = -b\,(1 + \cos\theta)\,x,$$

In [4] Chen and Llibre studied the limit cycles of the differential system

$$\dot{x} = y, \qquad \dot{y} = -x - \varepsilon\,(1 + \cos^m \theta)\,Q(x, y), \tag{2}$$

where $\varepsilon > 0$ is a small parameter and $Q\,(x, y)$ is an arbitrary polynomial of degree n.

In the present work we study the limit cycles of the following generalization of the differential system (2)

$$\dot{x} = y - \varepsilon\left(1 + \cos^l \theta\right)P(x, y), \qquad \dot{y} = -x - \varepsilon\,(1 + \cos^m \theta)\,Q(x, y), \tag{3}$$

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