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Limit Cycles of Discontinuous Piecewise Differential Hamiltonian Systems Separated by a Straight Line

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Abstract: In this article, we study the maximum number of limit cycles of discontinuous piecewise differential systems, formed by two Hamiltonians systems separated by a straight line. We consider three cases, when both Hamiltonians systems in each side of the discontinuity line have simultaneously degree one, two or three. We obtain that in these three cases, this maximum number is zero, one and three, respectively. Moreover, we prove that there are discontinuous piecewise differential systems realizing these maximum number of limit cycles. Note that we have solved the extension of the 16th Hilbert problem about the maximum number of limit cycles that these three classes of discontinuous piecewise differential systems separated by one straight line and formed by two Hamiltonian systems with a degree either one, two, or three, which such systems can exhibit.

Keywords: discontinuous piecewise differential systems; limit cycle; Hamiltonian systems

MSC: 34C05; 37G15

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1. Introduction and Statement of the Main Results

Discontinuous dynamical systems find pervasive applications across various domains, from electrical circuits and mechanical systems to biological processes, see [1–3]. These systems frequently exhibit abrupt changes, discontinuities, or switching phenomena, leading to sudden state transitions or impacts. For further insights, one may refer to the comprehensive works of Simpson [4], di Bernardo et al. [5], and the survey by Makarenkov and Lamb [6]. The Filippov convention stands out as a robust framework for modeling and analyzing such systems, providing a nuanced understanding of their dynamics.

The identification of a periodic orbit isolated within the set of all periodic orbits of a system is termed a "limit cycle". Understanding the existence or absence of these limit cycles is pivotal in studying the dynamics of differential systems. Consequently, numerous authors have delved into the examination of limit cycles in discontinuous piecewise linear differential systems separated by a straight line, as evidenced by the literature in this field, see ref. [7] for more details without being exhaustive for a piecewise model on the Savanna ecosystem, ref. [8] for a piecewise model on canard limit cycles, ref. [9] for the index theory of the piecewise models, ref. [10] for the global dynamics of some piecewise models, ref. [11] for a center problem in piecewise models, ref. [12] for showing a switching phenomenon of a limit cycle under Filippov construction of the Hamiltonian system, refs. [13,14] for some properties of the piecewise linear models, and refs. [15–21] for the study of the limit cycles of different piecewise models.

In this paper, we study the limit cycles for the class of discontinuous piecewise differential systems separated by a straight line and formed by two Hamiltonian systems of degree either one, two, or three. Without loss of generality, we can consider that the straight line of discontinuity is $x = 0$ and that the vector field associated with these discontinuous piecewise differential systems is

$$Z(x,y) = \begin{cases} Z_1(x,y), & \text{if } x \le 0, \\ Z_2(x,y), & \text{if } x \ge 0, \end{cases} \tag{1}$$

where Z_i is the vector field of the Hamiltonian system

$$\dot{x} = \frac{\partial}{\partial y} H_i(x,y), \quad \dot{y} = -\frac{\partial}{\partial x} H_i(x,y),$$

with Hamiltonian $H_i(x,y)$ for $i \in \{1,2\}$. The behavior of the piecewise differential system on the line of discontinuity $x = 0$ is defined following Filippov's rules, see [22]. Usually such discontinuous piecewise differential systems are called Filippov's systems.

Our main result is the following one.

Theorem 1. *Consider the discontinuous piecewise differential system* (1) *formed by two arbitrary Hamiltonians $H_1(x,y)$ and $H_2(x,y)$ of degree*

(a) 2, *then system* (1) *has no limit cycles.*
(b) 3, *then system* (1) *has at most one limit cycle.*
(c) 4, *then system* (1) *has at most three limit cycles.*

Moreover, there are differential systems (1) *formed by two convenient Hamiltonians $H_1(x,y)$ and $H_2(x,y)$ of the corresponding degree realizing the upper bounds on the number of limit cycles of statements (a) and (b).*

Theorem 1 is proved in Section 2.

2. Proof of Theorem 1

Proof of statement (a) of Theorem 1. Consider two arbitrary Hamiltonians of degree two as follows

$$H_1(x,y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2,$$
$$H_2(x,y) = b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2.$$

These Hamiltonians generate the next Hamiltonian systems of degree one

$$\dot{x} = a_2 + a_4 x + 2a_5 y, \quad \dot{y} = -a_1 - 2a_3 x - a_4 y, \tag{2}$$

$$\dot{x} = b_2 + b_4 x + 2b_5 y, \quad \dot{y} = -b_1 - 2b_3 x - b_4 y. \tag{3}$$

Of course, $H_1(x,y)$ and $H_2(x,y)$ are first integrals of systems (2) and (3), respectively. Now, we look for the limit cycles that intersect the straight line $x = 0$ at the points $(0, y_1)$ and $(0, y_2)$ with $y_1 \neq y_2$. To complete this, we analyze how many solutions have the following polynomial system.

$$\begin{aligned} e_1(y_1,y_2) &:= H_1(0,y_1) - H_1(0,y_2) = 0, \\ e_2(y_1,y_2) &:= H_2(0,y_1) - H_2(0,y_2) = 0. \end{aligned} \tag{4}$$

Solving system (4) is equivalent to finding the solutions of the system

$$\begin{aligned} E_1(y_1,y_2) &:= \frac{e_1(y_1,y_2)}{(y_1 - y_2)} = 0 \Rightarrow a_2 + a_5 y_1 + a_5 y_2 = 0, \\ E_2(y_1,y_2) &:= \frac{e_2(y_1,y_2)}{(y_1 - y_2)} = 0 \Rightarrow b_2 + b_5 y_1 + b_5 y_2 = 0. \end{aligned} \tag{5}$$

Since the straight lines $E_1(y_1,y_2) = 0$ and $E_2(y_1,y_2) = 0$ are parallel, it follows that system (5) has either no solutions with respect to the variables y_1 and y_2, or infinitely many solutions. In both cases, the discontinuous piecewise differential system can not have limit cycles. □

For the proof of statement (b) of Theorem 1, we shall use the next well-known result; for the proof, see, for instance, [23].

Theorem 2 (Bézout Theorem)**.** *Let f and g be two polynomials in $\mathbb{R}[x, y]$ of degrees n and m, respectively. Then, if the set $V(f, g) := \{(x, y) \in \mathbb{R}^2 : f(x, y) = g(x, y) = 0\}$ has finitely many solutions, then it has at most nm points.*

Proof of statement (b) of Theorem 1. Consider the following two arbitrary Hamiltonians of degree three:

$$\begin{aligned} H_1(x, y) &= a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + a_6 x^3 + a_7 x^2 y + a_8 xy^2 + a_9 y^3, \\ H_2(x, y) &= b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2 + b_6 x^3 + b_7 x^2 y + b_8 xy^2 + b_9 y^3. \end{aligned}$$

These Hamiltonians generate the Hamiltonian systems

$$\begin{aligned} \dot{x} &= a_2 + a_4 x + 2a_5 y + a_7 x^2 + 2a_8 xy + 3a_9 y^2, \\ \dot{y} &= -a_1 - 2a_3 x - a_4 y - 3a_6 x^2 - 2a_7 xy - a_8 y^2, \end{aligned} \tag{6}$$

$$\begin{aligned} \dot{x} &= b_2 + b_4 x + 2b_5 y + b_7 x^2 + 2b_8 xy + 3b_9 y^2, \\ \dot{y} &= -b_1 - 2b_3 x - b_4 y - 3b_6 x^2 - 2b_7 xy - b_8 y^2, \end{aligned} \tag{7}$$

Again $H_1(x, y)$ and $H_2(x, y)$ are first integrals of systems (6) and (7), respectively. Now, we look for the limit cycles that intersect the straight line $x = 0$ at the points $(0, y_1)$ and $(0, y_2)$, with $y_1 \neq y_2$. So we must analyze how many solutions the system has.

$$\begin{aligned} e_1(y_1, y_2) &:= H_1(0, y_1) - H_1(0, y_2) = 0, \\ e_2(y_1, y_2) &:= H_2(0, y_1) - H_2(0, y_2) = 0. \end{aligned} \tag{8}$$

Solving system (8) is equivalent to finding the solutions of the system

$$\begin{aligned} E_1(y_1, y_2) &:= a_2 + a_5(y_1 + y_2) + a_9(y_1^2 + y_1 y_2 + y_2^2) = 0, \\ E_2(y_1, y_2) &:= b_2 + b_5(y_1 + y_2) + b_9(y_1^2 + y_1 y_2 + y_2^2) = 0, \end{aligned}$$

where $E_i(y_1, y_2) = e_i(y_1, y_2)/(y_1 - y_2)$. Notice that

$$\begin{aligned} E_{12}(y_1, y_2) = & \; b_9 E_1(y_1, y_2) - a_9 E_2(y_1, y_2) \\ = & \; b_9 a_2 - a_9 b_2 + (b_9 a_5 - a_9 b_5)(y_1 + y_2). \end{aligned}$$

Using the Bézout Theorem, the upper bound for the maximum number of solutions of system $E_1(y_1, y_2) = 0$ and $E_{12}(y_1, y_2) = 0$ is 2, when this system has finitely many solutions. Note that by the symmetry of these polynomial equations, if (y_1, y_2) is a solution, then (y_2, y_1) is also a solution, but these two solutions provide the same periodic orbit. Then, this family of discontinuous piecewise differential systems has at most one limit cycle. This upper bound is reached as can be seen in Example 1 of Section 3. □

Proof of statement (c) of Theorem 1. Consider two arbitrary Hamiltonians of degree four,

$$\begin{aligned} H_1(x, y) = & \; a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + a_6 x^3 + a_7 x^2 y + a_8 xy^2 \\ & + a_9 y^3 + a_{10} x^4 + a_{11} x^3 y + a_{12} x^2 y^2 + a_{13} xy^3 + a_{14} y^4, \\ H_2(x, y) = & \; b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2 + b_6 x^3 + b_7 x^2 y + b_8 xy^2 \\ & + b_9 y^3 a_{10} x^4 + b_{11} x^3 y + b_{12} x^2 y^2 + b_{13} xy^3 + b_{14} y^4. \end{aligned}$$

These Hamiltonians generate the following two Hamiltonian systems

$$\begin{aligned} \dot{x} = & \; a_2 + a_4 x + 2a_5 y + a_7 x^2 + 2a_8 xy + 3a_9 y^2 + a_{11} x^3 + 2a_{12} x^2 y \\ & + 3a_{13} xy^2 + 4a_{14} y^3, \\ \dot{y} = & \; 4a_{10} x^3 - a_1 - 2a_3 x - a_4 y - 3a_6 x^2 - 2a_7 xy - a_8 y^2 + 3a_{11} x^2 y \\ & + 2a_{12} xy^2 + a_{13} y^3, \end{aligned} \tag{9}$$

and

$$\begin{aligned} \dot{x} = \quad & b_2 + b_4x + 2b_5y + b_7x^2 + 2b_8xy + 3b_9y^2 + b_{11}x^3 + 2b_{12}x^2y \\ & +3b_{13}xy^2 + 4b_{14}y^3, \\ \dot{y} = \quad & 4b_{10}x^3 - b_1 - 2b_3x - b_4y - 3b_6x^2 - 2b_7xy - b_8y^2 + 3b_{11}x^2y \\ & +2b_{12}xy^2 + b_{13}y^3, \end{aligned} \tag{10}$$

respectively. The Hamiltonians $H_1(x,y)$ and $H_2(x,y)$ are first integrals of systems (9) and (10), respectively. Now, we look for the limit cycles that intersect the straight line $x = 0$ at the points $(0, y_1)$ and $(0, y_2)$, with $y_1 \neq y_2$. To complete that, we analyze how many solutions the system

$$\begin{aligned} e_1(y_1, y_2) &:= H_1(0, y_1) - H_1(0, y_2) = 0, \\ e_2(y_1, y_2) &:= H_2(0, y_1) - H_2(0, y_2) = 0, \end{aligned} \tag{11}$$

can have. Defining

$$E_1(y_1, y_2) := e_1(y_1, y_2)/(y_1 - y_2) \quad \text{and} \quad E_2(y_1, y_2) := e_2(y_1, y_2)/(y_1 - y_2).$$

Since we are interested in the solutions with $y_1 \neq y_2$, system (11) is equivalent to system $E_1(y_1, y_2) = E_2(y_1, y_2) = 0$, i.e.,

$$\begin{aligned} a_2 + a_5(y_1 + y_2) + a_9(y_1^2 + y_1y_2 + y_2^2) + a_{14}(y_1^3 + y_1^2y_2 + y_1y_2^2 + y_2^3) = 0, \\ b_2 + b_5(y_1 + y_2) + b_9(y_1^2 + y_1y_2 + y_2^2) + b_{14}(y_1^3 + y_1^2y_2 + y_1y_2^2 + y_2^3) = 0. \end{aligned}$$

Notice that

$$\begin{aligned} E_{12}(y_1, y_2) &= b_{14}E_1(y_1, y_2) - a_{14}E_2(y_1, y_2) \\ &= (b_{14}a_5 - a_{14}b_5)(y_1 + y_2) + (b_{14}a_9 - a_{14}b_9)(y_1^2 + y_1y_2 + y_2^2) \\ &\quad + b_{14}a_2 - a_{14}b_2 \end{aligned}$$

is a polynomial of degree two. Using the Bézout Theorem, the upper bound for the maximum number of solutions of system $E_1(y_1, y_2) = 0$ and $E_{12}(y_1, y_2) = 0$ is 6, when this system has finitely many solutions. Again, note that by the symmetry of these polynomial equations, if (y_1, y_2) is a solution, then (y_2, y_1) is also a solution, but these two solutions provide the same periodic orbit. This implies that the discontinuous piecewise differential systems has at most three limit cycles. This upper bound is reached, see Example 2 of Section 3. □

3. Examples

In this section, we provide in example 1 a discontinuous piecewise differential system separated by the straight line $x = 0$ is formed by two Hamiltonians systems of degree 2 with one limit cycle. Furthermore, in example 2, a discontinuous piecewise differential system separated by the straight line $x = 0$ is formed by two Hamiltonians systems of degree 3 with three limit cycles. Hence, these two examples complete the proof of Theorem 1.

Example 1. *Consider the following two Hamiltonians of degree three,*

$$\begin{aligned} H_1(x, y) &= x^3 - y^3 - y^2 + y, \\ H_2(x, y) &= -x^3 - xy - 8y^3 - y^2 + \frac{7y}{2}. \end{aligned} \tag{12}$$

These Hamiltonians generate the Hamiltonian systems

$$\dot{x} = 1 - 2y - 3y^2, \quad \dot{y} = -3x^2, \tag{13}$$

$$\dot{x} = -x - 24y^2 - 2y + \frac{7}{2}, \quad \dot{y} = 3x^2 + y, \tag{14}$$

respectively. Of course $H_1(x,y)$ and $H_2(x,y)$ are first integrals of systems (13) and (14), respectively. For this discontinuous piecewise differential system, system (11) provides the system

$$\begin{aligned} E_1(y_1,y_2) &= 1 - y_1 - y_1^2 - y_2 - y_1y_2 - y_2^2 = 0,\\ E_2(y_1,y_2) &= \frac{1}{2}(7 - 2y_1 - 16y_1^2 - 2y_2 - 16y_1y_2 - 16y_2^2) = 0. \end{aligned} \tag{15}$$

System (15) has the unique real solution

$$(\bar{y}_1,\bar{y}_2) = \left(\frac{1}{28}\left(9-\sqrt{37}\right), \frac{1}{28}\left(\sqrt{37}+9\right)\right). \tag{16}$$

Then, the two points of intersection with $x = 0$ of the limit cycle are $(0,\bar{y}_1)$ and $(0,\bar{y}_2)$, see this limit cycle in Figure 1.

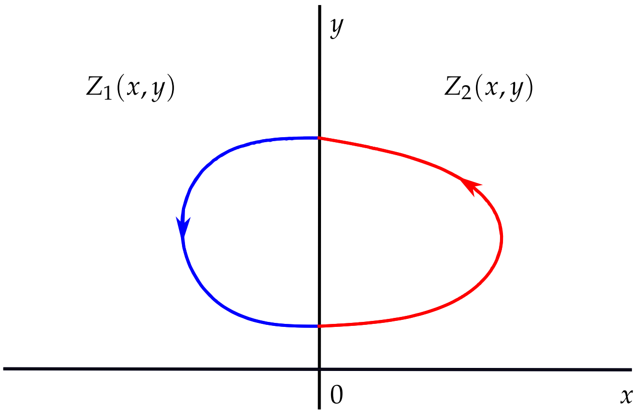

Figure 1. The limit cycle of the discontinuous piecewise differential system generated by Hamiltonian (12) passing through the points $(0,\bar{y}_1)$ and $(0,\bar{y}_2)$, where $\bar{y}_1$, $\bar{y}_2$ are given in (16). $H_1(x,y)$ define system in $x \le 0$, and $H_2(x,y)$ define the system in $x \ge 0$.

Example 2. *Consider the following two Hamiltonians of degree four*

$$\begin{aligned} H_1(x,y) &= 2x^3y + 2x^2 - \frac{4xy}{3} + y^4 - 4y^3 + \frac{51y^2}{10} - \frac{19y}{10},\\ H_2(x,y) &= 3x^4 + 2x^3 + xy^2 - 2xy + y^4 - \frac{31y^3}{12} + \frac{5y^2}{4} - \frac{y}{6}. \end{aligned} \tag{17}$$

These Hamiltonians generate the Hamiltonian systems

$$\dot{x} = 2x^3 - \frac{4x}{3} + 4y^3 - 12y^2 + \frac{51y}{5} - \frac{19}{10}, \quad \dot{y} = -6x^2y + 4x + \frac{4y}{3}, \tag{18}$$

$$\dot{x} = 2xy - 2x + 4y^3 - \frac{31y^2}{4} + \frac{5y}{2} - \frac{1}{6}, \quad \dot{y} = -12x^3 - 6x^2 - y^2 + 2y, \tag{19}$$

and $H_1(x,y)$ and $H_2(x,y)$ are first integrals of systems (18) and (19), respectively. For this discontinuous piecewise differential system, system (11) has only the following three real solutions

$$\begin{aligned} (\bar{y}_1^1,\bar{y}_2^1) &= (-0.206887, 2.01873),\\ (\bar{y}_1^2,\bar{y}_2^2) &= (0.141455, 0.393626),\\ (\bar{y}_1^3,\bar{y}_2^3) &= (1.41754, 1.67084). \end{aligned} \tag{20}$$

Then, the two points of intersection with $x = 0$ of each limit cycle are $(0,\bar{y}_1^i)$ and $(0,\bar{y}_2^i)$ for $i \in \{1,2,3\}$; see these three limit cycles in Figure 2.

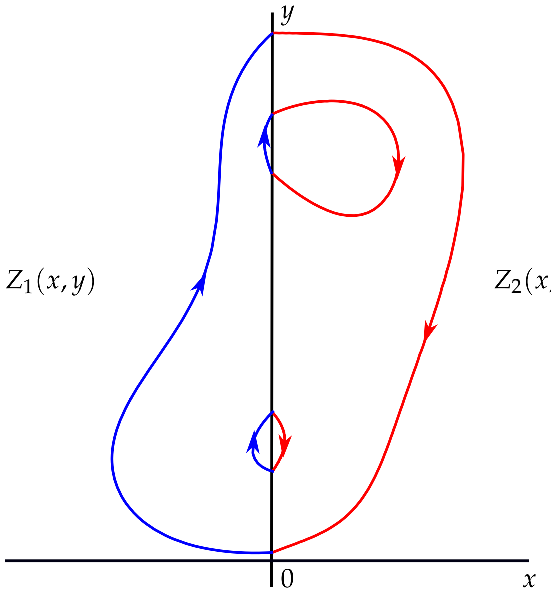

Figure 2. The three limit cycles of the discontinuous piecewise differential system generated by Hamiltonian (17) passing through the points $(0, \bar{y}_i^i)$, $i \in \{1, 2, 3\}$, where $\bar{y}_i^i$ are given in (20).

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